Robust Streaming Erasure Codes based on Deterministic Channel Approximations

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***Abstract*—We study near optimal error correction codes for real-time communication. In our setup the encoder must operate on an incoming source stream in a sequential manner, and the decoder must reconstruct each source packet within a fixed playback deadline of T packets. The underlying channel is a packet erasure channel that can introduce both burst and isolated losses.**

We first consider a class of channels that in any window of length $T+1$ introduce either a single erasure burst of a given maximum length B, or a certain maximum number N of isolated erasures. We demonstrate that for a fixed rate and delay, there exists a tradeoff between the achievable values of B and N, and propose a family of codes that is near optimal with respect to this tradeoff. We also consider another class of channels that introduce both a burst *and* an isolated loss in each window of interest and develop the associated streaming codes.

All our constructions are based on a layered design and provide significant improvements over baseline codes in simulations over the Gilbert-Elliott channel.

I. Introduction

Many emerging multimedia applications require error correction of streaming sources under strict latency constraints. The transmitter must encode a source stream sequentially and the receiver must decode each source packet within a fixed playback deadline. Naturally both the optimal structure and the fundamental limits of *streaming codes* are expected to be different from classical error correction codes. For example it is well known that the Shannon capacity of an erasure channel only depends on the fraction of erasures. However when delay constraints are imposed, the actual pattern of packet losses also becomes relevant. As such, the decoding delay over burst-erasure channels can be very different than over i.i.d. erasure channels. In practice channels such as the Gilbert-Elliott (GE) channel introduce both burst and isolated losses. The central question we address in this paper is how to construct streaming codes that outperform classical error correction codes over such channels.

We consider a class of channels that are simplifications of the GE channel. Such deterministic models are restricted to introduce only a certain class of erasure patterns, which correspond to the *dominant set* of error events associated with the original channels. We construct near optimal codes for such deterministic approximations and then demonstrate that the resulting codes also yield interesting performance gains over

Fig. 1. An Example of Channel I: In any sliding window of length $W=5$ there is either a single erasure burst of length no greater than $B=3$ or up-to $N=2$ erasures.

Fig. 2. An Example of Channel II: In any sliding window of length $W=5$ there is either a single erasure burst of length up to $B=3$ and possibly one isolated erasure, or $N=2$ isolated erasures.

the GE channel in simulations. In particular we propose two families of codes, MiDAS codes and PRC codes in this paper. The MiDAS codes attain a near optimal tradeoff between the burst error correction and isolated error correction capability in the streaming setup. The PRC codes are constructed for a more complex class of channels that involve some additional erasure patterns. One key feature in our construction is that they are based on a layered design. We first construct an optimal streaming code for the burst-erasure channel and then introduce another layer of parity checks for recovery from isolated erasures.

In related works, a class of optimal streaming codes for burst erasure channels is proposed in [1] . Unfortunately these constructions are sensitive to isolated packet losses. Reference [1] also presents some examples of robust codes using a computer search, but offers limited insights towards a general construction. In contrast the present paper proposes a systematic construction of robust streaming codes based on a layered design, establishes fundamental bounds and in the process verifies that some of the robust constructions proposed in [1] are also optimal. Recently connections between streaming codes and network coding have been studied in [2] but the focus is on different model than the present paper.

II. System Model

We consider a class of packet erasure channels where the erasure patterns are locally constrained. In any sliding window of length W, the channel can introduce only one of the following patterns:

- A single erasure burst of maximum length B plus a maximum of K isolated erasures or,

- A maximum of N erasures in arbitrary locations.

Note that $N \leq B+K$. We denote such a channel by $\mathcal{C}(N,B,K,W)$. We will focus on two special subclasses of such channels. The first class, Channel I is given by: $\mathcal{C}_{\text{I}}(N,B,W) \triangleq \mathcal{C}(N,B,0,W)$, i.e., it only introduces either a burst erasure or up to N arbitrary erasures. The second class, Channel II is given by: $\mathcal{C}_{\text{II}}(N,B,W) \triangleq \mathcal{C}(N,B,1,W)$. It allows for one burst erasure of maximum length B plus up to one isolated erasure, or N arbitrary erasures. These specific channels are inspired by the dominant erasure events associated with GE channel as discussed in section VI. Fig. 1 and 2 provide examples of channels $\mathcal{C}_{\text{I}}(2,3,5)$ and $\mathcal{C}_{\text{II}}(2,3,5)$ respectively.

Clearly the erasure patterns associated with channel $\mathcal{C}_{\text{II}}$ include those associated with $\mathcal{C}_{\text{I}}$. However we focus on channel $\mathcal{C}_{\text{I}}$ first as it is simpler to analyze and reveals some important insights which carry over to the codes for $\mathcal{C}_{\text{II}}$.

We next formally define a *streaming erasure code*. At each time $i \geq 0$, the encoder observes a source symbol $\mathbf{s}[i]$, drawn from a source alphabet $\mathcal{S}$ and generates a channel symbol $\mathbf{x}[i] = f_i(\mathbf{s}[0],\ldots,\mathbf{s}[i]) \in \mathcal{X}$. The channel output is either $\mathbf{y}[i] = \mathbf{x}[i]$ or $\mathbf{y}[i] = \star$, when the output is erased. The decoder is required to reconstruct each packet with a delay of T units i.e., for each $i \geq 0$ there exists a decoding function: $\mathbf{s}[i] = g_i(\mathbf{y}[0],\ldots,\mathbf{y}[i+T])$. A rate $R = \frac{H(\mathbf{s})}{\log_2 |\mathcal{X}|}$ is achievable if there exists a feasible code that recovers every erased symbol $\mathbf{s}[i]$ by time $i+T$.

III. Preliminaries

In this section we consider some previously studied error correction codes and characterize their performance in our proposed setup. We discuss the special cases under which these codes are optimal. Our new constructions use these codes as building blocks and therefore the review of these codes is useful.

A. Strongly-MDS Codes

Classical erasure codes are designed for maximizing the underlying distance properties. Roughly speaking, such codes will recover all the missing source symbols simultaneously once sufficiently many parity checks have been received at the decoder. Indeed a commonly used family of such codes, *random-linear codes*, see e.g., [3], are designed to guarantee that the underlying system of equations is nearly of a full rank. We discuss one particular class of *deterministic code* constructions with optimal distance properties below.

Consider a (n,k,m) convolutional code that maps an input source stream $\mathbf{s}[i] \in \mathbb{F}_q^k$ to an output $\mathbf{x}[i] \in \mathbb{F}_q^n$ using a memory m encoder i.e.,

$$\mathbf{x}^\dagger[i] = \sum_{t=0}^{m} \mathbf{s}^\dagger[i-t] \cdot \mathbf{G}_t, \tag{1}$$

where $\mathbf{G}_0,\ldots,\mathbf{G}_m$ are $k \times n$ matrices with elements in $\mathbb{F}_q$ and the notation † denotes the vector transpose. Furthermore the convolutional code is systematic if we can express each sub-generator matrix in the following form:

$$\mathbf{G}_0 = [\mathbf{I}_{k\times k}\ \mathbf{0}_{k\times n-k}], \qquad \mathbf{G}_i = [\mathbf{0}_{k\times k}\ \mathbf{H}_i],\ i=1,\ldots,T \tag{2}$$

where $\mathbf{I}_{k\times k}$ denotes the $k \times k$ identity matrix, $\mathbf{0}$ denotes the zero matrix, and each $\mathbf{H}_i \in \mathbb{F}_q^{k\times(n-k)}$. For a systematic convolutional code, Eq. (1) reduces to

$$\mathbf{x}[i] = \begin{bmatrix} \mathbf{s}[i] \\ \mathbf{p}[i] \end{bmatrix}, \qquad \mathbf{p}^\dagger[i] = \sum_{t=1}^{m} \mathbf{s}^\dagger[i-t] \cdot \mathbf{H}_t. \tag{3}$$

We are particularly interested in a class of Strongly-MDS codes [4]. In the streaming setup these codes have the following properties, which are established in [5].

Lemma 1. *A (n,k,m) (systematic) Strongly-MDS code has the following properties for each $j = 0,1,\ldots,m$:*

P1. *Suppose that in the window $[0,j]$, there are no more than $(1-R)(j+1)$ erasures in arbitrary locations, then $\mathbf{s}[0]$ is recovered by time $t=j$.*

P2. *Suppose an erasure burst happens in the interval $[0,B-1]$, where $B \leq (1-R)(j+1)$, then all the symbols $\mathbf{s}[0],\ldots,\mathbf{s}[B-1]$ are simultaneously recovered at time $t=j$.*

As a direct consequence of Lemma 1, it can be seen that a rate R, Strongly-MDS code, can achieve any $N = B \leq (1-R)(T+1)$ over the $\mathcal{C}_{\text{I}}(N,B,T+1)$ channel. As will be shown subsequently (c.f. Theorem 1) this is in fact the maximum value of N that can be achieved. Nevertheless the largest value of B can be higher as discussed next.

B. Maximally Short Codes

Maximally Short codes, introduced in [1], are streaming codes that correct the longest possible erasure burst in any sliding window of length $T+1$. In particular the following result was established in [1].

Lemma 2 (Martinian and Sundberg [1]). *Consider any channel that in any window of length $T+1$ introduces a single erasure burst of length no more than B. For any (n,k,m) convolutional code which recovers every source packet $\mathbf{s}[i]$ by time $t=i+T$ we must have that*

$$B \leq T \cdot \min\left(1, \frac{1-R}{R}\right) \tag{4}$$

Furthermore the upper bound in (4) *can be attained by the Maximally Short (MS) Codes.*

We review an alternative construction presented in [5], [6] that also attains the optimal B in (4). In the proposed construction we split each source symbol $\mathbf{s}[i] \in \mathbb{F}_q^T$ into two groups $\mathbf{u}[i] \in \mathbb{F}_q^B$ and $\mathbf{v}[i] \in \mathbb{F}_q^{T-B}$ as follows:

$$\mathbf{s}[i] = (\underbrace{u_0[i],\ldots,u_{B-1}[i]}_{=\mathbf{u}[i]}, \underbrace{v_0[i],\ldots,v_{T-B-1}[i]}_{=\mathbf{v}[i]})^\dagger. \tag{5}$$

We apply a $(T,T-B,T)$ Strongly-MDS code on the symbols $\mathbf{v}[i]$ and generate parity check symbols

$$\mathbf{p}_v^\dagger[i] = \sum_{j=1}^{T} \mathbf{v}^\dagger[i-j] \cdot \mathbf{H}_j^v, \quad \mathbf{p}_v[i] \in \mathbb{F}_q^B, \tag{6}$$

where the matrices $\mathbf{H}_j^v$ are $(T-B)\times B$ matrices associated with the Strongly-MDS code (3). Next we superimpose the $\mathbf{u}[\cdot]$ symbols onto $\mathbf{p}_v[\cdot]$ and let

$$\mathbf{q}[i] = \mathbf{p}_v[i] + \mathbf{u}[i-T]. \tag{7}$$

The channel input at time i is given by $\mathbf{x}[i] = \left(\mathbf{u}^\dagger[i], \mathbf{v}^\dagger[i], \mathbf{q}^\dagger[i]\right)^\dagger \in \mathbb{F}_q^{T+B}$. Note that the rate of this code is $R = \frac{T}{T+B}$.

We omit the steps involved in the decoder and refer the reader to [5], [6]. Unfortunately the MS Codes are not robust against isolated erasures. In fact it can be easily seen that for the $\mathcal{C}_\text{I}(N, B, W)$ channel they only attain $N = 1$.

IV. Channel I: Code Constructions and Bounds

Throughout the analysis of Channel I, we select $W = T+1$ where recall that T is the decoding delay. Note that every source symbol $\mathbf{s}[i]$ remains active for a duration of $T+1$ symbols before its deadline expires. Therefore the erasure patterns observed in a window of length $T+1$ are naturally of interest. For such channels we will suppress the W parameter in the notation for $\mathcal{C}_\text{I}(\cdot)$ and simply use the notation $\mathcal{C}_\text{I}(N, B)$.

In this section we propose a new family of streaming codes that are near optimal for all rates for the channel $\mathcal{C}_\text{I}(N, B)$. Before stating our constructions, we use the following upper bound [6].

Theorem 1 (Badr et al. [6])**.** *Any achievable rate for $\mathcal{C}_I(N, B)$, satisfies*

$$\left(\frac{R}{1-R}\right) B + N \leq T + 1. \tag{8}$$

and furthermore $N \leq B$ and $B \leq T$.

Theorem 1 shows that when the rate R and delay T are fixed there exists a tradeoff between the achievable values of B and N. We cannot have streaming codes that simultaneously correct long erasure bursts and many isolated erasures. The upper bound (8) also shows that the $R = 1/2$ codes found via a computer search in [1, Section V-B] are indeed optimal.

We propose a class of codes, *Maximum Distance And Span Tradeoff (MiDAS) codes*, that achieve near-optimal tradeoff.

Theorem 2. *For the channel $\mathcal{C}_I(N, B)$ and delay T with $N \leq B$ and $B \leq T$, there exists a code of rate R that satisfies*

$$\left(\frac{R}{1-R}\right) B + N \geq T. \tag{9}$$

Remark 1. *Comparing* (9) *with the upper bound in* (8)*, the only difference is the additional constant,* 1*, in the right hand side of the inequalities.*

Our proposed construction is discussed below. We split the source symbol $\mathbf{s}[i] \in \mathbb{F}_q^T$ into two groups $\mathbf{u}[i]$ and $\mathbf{v}[i]$ as in (5) and generate the parity checks $\mathbf{q}[i]$ as in (7). The resulting code up to this point is just a MS code that can correct an erasure burst of length B with a delay of T.

We further apply another $(B, B+K, T)$ Strongly-MDS code to the $\mathbf{u}[i]$ symbols and generate the set of parity check symbols,

$$\mathbf{p}_u^\dagger[i] = \sum_{j=1}^{T} \mathbf{u}^\dagger[i-j] \cdot \mathbf{H}_j^u, \qquad \mathbf{p}_u[i] \in \mathbb{F}_q^K, \tag{10}$$

where $\mathbf{H}_j^u$ are $B \times K$ matrices associated with a Strongly-MDS code (2). We finally concatenate the parity checks $\mathbf{q}[i]$ and $\mathbf{p}_u[i]$ with the source symbols i.e., $\mathbf{x}[i] = \left(\mathbf{u}^\dagger[i], \mathbf{v}^\dagger[i], \mathbf{q}^\dagger[i], \mathbf{p}_u^\dagger[i]\right)^\dagger$.

Note that $\mathbf{x}[i] \in \mathbb{F}_q^{(T+B+K)}$ and the associated rate is given by $R = \frac{T}{T+B+K}$. In our setup K is a free parameter which can be used to vary N. For any $\mathcal{C}_\text{I}(N, B)$ channel and delay T it suffices [5] to select

$$K = \frac{N}{T+1-N} B. \tag{11}$$

Substituting this value of K into the rate expression and through some straightforward manipulations [5] the relation (9) follows.

Table I illustrates a MiDAS code of rate $\frac{7}{11}$ designed for $\mathcal{C}_\text{I}(2, 3)$ with $T = 7$. Note that we split each source symbol $\mathbf{s}_i$ into $\mathbf{u}_i$ and $\mathbf{v}_i$ as in (5). We let $\mathbf{u}_i \in \mathbb{F}_q^3$, $\mathbf{v}_i \in \mathbb{F}_q^4$, $\mathbf{p}_v(\cdot) \in \mathbb{F}_q^3$ and $\mathbf{p}_u(\cdot) \in \mathbb{F}_q$. For convenience, the subscript of $\mathbf{u}$ and $\mathbf{v}$ denotes the time index, whereas the notation $\mathbf{v}^t$ denotes a sequence of T consecutive symbols $\mathbf{v}_t, \mathbf{v}_{t-1}, \ldots, \mathbf{v}_{t-T+1}$. Let $\mathbf{q}_i = \mathbf{p}_v(\mathbf{v}^{i-1}) + \mathbf{u}_{i-T}$. The input at time i corresponds to the i-th column in Table I.

$\lvert . \rvert$	0	1	2	3	…	7	8
3	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	…	$\mathbf{u}_7$	$\mathbf{u}_8$
4	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	…	$\mathbf{v}_7$	$\mathbf{v}_8$
3	$\mathbf{p}_v(\mathbf{v}^{-1})$ $+\mathbf{u}_{-7}$	$\mathbf{p}_v(\mathbf{v}^{0})$ $+\mathbf{u}_{-6}$	$\mathbf{p}_v(\mathbf{v}^{1})$ $+\mathbf{u}_{-5}$	$\mathbf{p}_v(\mathbf{v}^{2})$ $+\mathbf{u}_{-4}$	…	$\mathbf{p}_v(\mathbf{v}^{6})$ $+\mathbf{u}_{0}$	$\mathbf{p}_v(\mathbf{v}^{7})$ $+\mathbf{u}_{1}$
1	$\mathbf{p}_u(\mathbf{u}^{-1})$	$\mathbf{p}_u(\mathbf{u}^{0})$	$\mathbf{p}_u(\mathbf{u}^{1})$	$\mathbf{p}_u(\mathbf{u}^{2})$	…	$\mathbf{p}_u(\mathbf{u}^{6})$	$\mathbf{p}_u(\mathbf{u}^{7})$

TABLE I

A MiDAS Code with $(N, B) = (2, 3)$ and rate $\frac{7}{11}$ for a delay of $T = 7$. The number of sub-symbols in each group is denoted in the first column.

To illustrate the decoding, assume an erasure burst of length $B = 3$ happens in the interval $[0, 2]$. In Table I, $\mathbf{u}_{-4}, \ldots, \mathbf{u}_{-1}$ are not erased and thus can be subtracted to recover the parity checks $\mathbf{p}_v$ in the interval $[3, 6]$. These parities belong to the $(7, 4, 7)$ Strongly-MDS Code and suffice to recover $\mathbf{v}_0, \ldots, \mathbf{v}_2$ simultaneously at time $T - 1 = 6$. In particular note that $(1 - R_v)T = (1 - \frac{4}{7})(7) = 3$ and thus property P2 of Lemma 1 can be immediately applied. At this point all the erased $\mathbf{v}[\cdot]$ symbols have been recovered. Next, $\mathbf{u}_0$ is recovered at time 7 by subtracting out $\mathbf{p}_v(\mathbf{v}^8)$. Likewise $\mathbf{u}_1$ and $\mathbf{u}_2$ are sequentially recovered at time 8 and 9.

To compute the achievable N, it suffices to compute N^v for the $(\mathbf{v}, \mathbf{p}_v(\cdot))$ code in the interval $[0, 6]$ and N^u for the $(\mathbf{u}, \mathbf{p}_u(\cdot))$ code in the interval $[0, 7]$. Using that $R_v = 4/7$ and $R_u = 3/4$ it follows from Lemma 1 that $N^v = (1 - R_v)(T) = 3$ and similarly $N^u = (1 - R_u)(T+1) = 2$ and hence $N = \min(N^v, N^u) = 2$.

Table II summarizes the achievable N and B for different codes. The first three rows correspond to Strongly-MDS, MS and MiDAS codes discussed in sections III-A, III-B and IV respectively. The last row correspond to a family of codes – Embedded Random Linear Codes (E-RLC) – proposed in [6]. For a given rate R, these codes can achieve any (N, B) that satisfy:

$$\left(\frac{R}{1-R}\right)(B+N-1) \geq T. \tag{12}$$

While such constructions are optimal for $R = 1/2$, they are sub-optimal in general. In contrast, the MiDAS codes achieve a tradeoff, very close to the upper bound for all rates.

Code	N	B
Strongly-MDS Codes	$(1-R)(T+1)$	$(1-R)(T+1)$
Maximally Short Codes	1	$T \cdot \min\left(\frac{1}{R}-1, 1\right)$
MiDAS Codes	$\min\left(B, T-\frac{R}{1-R}B\right)$	$B \in [1, T]$
E-RLC Codes [6] $\Delta \in [R(T+1), T-1]$, $R \geq 1/2$	$\frac{1-R}{R}(T-\Delta)$	$\frac{1-R}{R}\Delta$

TABLE II
ACHIEVABLE (N, B) FOR CHANNEL $\mathcal{C}_{\mathrm{I}}(\cdot)$ FOR DIFFERENT FAMILIES OF STREAMING CODES. (IF ANY OF THE ENTRIES IS NOT AN INTEGER WE SHOULD TAKE THE FLOOR OF ITS VALUE.)

V. PARTIAL RECOVERY CODES FOR CHANNEL II

In this section we study streaming erasure codes for channel $\mathcal{C}_{\mathrm{II}}$. Recall that in any sliding window of length W, such channels permit erasure patterns consisting of (i) one erasure burst plus one isolated erasure either before or after the burst, or (ii) up to N erasures in arbitrary locations. The burst plus isolated erasure pattern captures the transition between the bad and good states on the GE channel which appear to be dominant when the delay T is large. Motivated by the MiDAS code construction we focus on codes that correct only these patterns and refer to the channel as $\mathcal{C}_{\mathrm{II}}(B, W)$. Thereafter, one can extend the construction to achieve any desired N, by suitably concatenating additional parity checks.

An isolated erasure is defined to be *associated* with the erasure burst, if it occurs within the T symbols before or after this burst. Throughout this section, we let $W = 2T + B$ since we are interested in an interval of length T before or after the erasure burst. Note that every erasure burst has at most one associated isolated erasure. Conversely every isolated erasure can be associated with no more than one erasure burst.

It turns out that codes that achieve perfect recovery over $\mathcal{C}_{\mathrm{II}}$ require a significant overhead, particularly when T is close to B. Therefore we consider partial recovery codes as discussed next.

Definition 1. *A* Partial Recovery Code (PRC) *for* $\mathcal{C}_{\mathrm{II}}(B, W = 2T + B)$ *recovers all but at-most one source symbol with a delay of* T *in each pattern consisting of an erasure burst and its associated isolated erasure.*

Theorem 3. *There exists a partial recovery code for* $\mathcal{C}_{\mathrm{II}}(B, 2T + B)$ *of rate,*

$$R = \max_{B<\Delta<T} \frac{\Delta(T-\Delta)+(B+1)}{\Delta(T-\Delta)+(B+1)(T-\Delta+2)}, \tag{13}$$

that satisfy Definition 1.

The main steps in our proposed construction are as follows. Let u, v, s and Δ be integers that will be specified in the sequel. We let $\mathbf{s}[i] \in \mathbb{F}_q^{u+v}$.

1) **Source Splitting**: As in (5) we split $\mathbf{s}[i]$ into two groups $\mathbf{u}[i] \in \mathbb{F}_q^u$ and $\mathbf{v}[i] \in \mathbb{F}_q^v$.
2) **Construction of $\mathcal{C}_{12}$**: We apply a $(v + u + s, v, T)$ Strongly-MDS code $\mathcal{C}_{12} : (\mathbf{v}[i], \mathbf{p}[i])$ to the $\mathbf{v}[\cdot]$ symbols of rate $R_{12} = \frac{v}{v+u+s}$ to generate parity check symbols $\mathbf{p}[\cdot] \in \mathbb{F}_q^{u+s}$,
$$\mathbf{p}^\dagger[i] = \sum_{j=1}^{T} \mathbf{v}^\dagger[i-j] \cdot \mathbf{H}_j, \tag{14}$$
where $\mathbf{H}_1, \ldots, \mathbf{H}_T \in \mathbb{F}_q^{v\times(u+s)}$ are the matrices associated with the Strongly-MDS code (3).
3) **Parity Check Splitting**: We split each $\mathbf{p}[i]$ into two groups $\mathbf{p}_1[i] \in \mathbb{F}_q^u$ and $\mathbf{p}_2[i] \in \mathbb{F}_q^s$ by assigning the first u sub-symbols in $\mathbf{p}[i]$ to $\mathbf{p}_1[i]$ and the remaining s sub-symbols of $\mathbf{p}[i]$ to $\mathbf{p}_2[i]$. We can express:
$$\mathbf{p}_k^\dagger[i] = \sum_{j=1}^{T} \mathbf{v}^\dagger[i-j] \cdot \mathbf{H}_j^k, \quad k = 1, 2 \tag{15}$$
where the matrices $\mathbf{H}_j^k$ are given by $\mathbf{H}_j = [\mathbf{H}_j^1 \mid \mathbf{H}_j^2]$. It can be shown that both $\mathbf{H}_j^1$ and $\mathbf{H}_j^2$ satisfy the Strongly-MDS property [4, Theorem 2.4] and therefore the codes $\mathcal{C}_1 : (\mathbf{v}[i], \mathbf{p}_1[i])$ and $\mathcal{C}_2 : (\mathbf{v}[i], \mathbf{p}_2[i])$ are both Strongly-MDS codes.
4) **Repetition Code**: We combine a shifted copy of $\mathbf{u}[\cdot]$ with the $\mathbf{p}_1[\cdot]$ parity checks to generate $\mathbf{q}[i] = \mathbf{p}_1[i] + \mathbf{u}[i-\Delta]$. Here $\Delta \in \{B+1, \ldots, T\}$ denotes the shift applied to the $\mathbf{u}[\cdot]$ stream before embedding it onto the $\mathbf{p}_1[\cdot]$ stream.
5) **Channel Symbol**: We concatenate the generated layers of parity check symbols to the source symbol to construct the channel symbol,
$$\mathbf{x}[i] = (\mathbf{s}^\dagger[i], \mathbf{q}^\dagger[i], \mathbf{p}_2^\dagger[i])^\dagger. \tag{16}$$

The rate of the code in (16) is clearly $R = \frac{u+v}{2u+v+s}$. We further select, for any given $\Delta \in [B+1, T]$,

$$\begin{aligned} u &= (B+1)(T-\Delta+1) - (\Delta-B-1) \\ v &= (T-\Delta+1)(\Delta-B-1) \\ s &= \Delta - B - 1, \end{aligned} \tag{17}$$

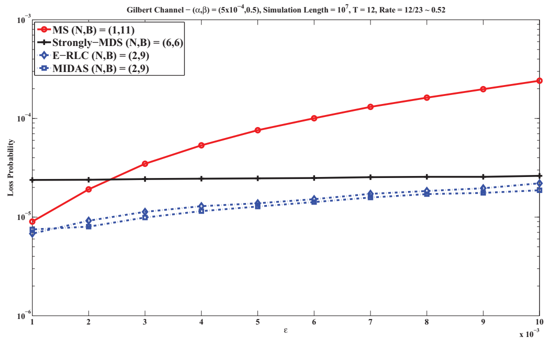
which upon direct substitution yields (13). The feasibility of these parameters, the decoding steps, and an example are presented in [5].

Remark 2. *If we ignore the integer constraints, the optimal* Δ *in* (13) *is given by* $\Delta^\star = T + 1 - \sqrt{T-B}$ *and the associated rate is given by:*

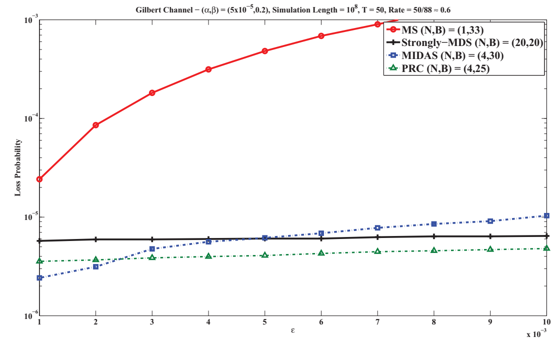
$$R^\star = \frac{(T+2)\sqrt{T-B} - 2(T-B)}{(T+B+3)\sqrt{T-B} - 2(T-B)}. \tag{18}$$

VI. SIMULATIONS RESULTS

We consider a two-state Gilbert-Elliott channel model. In the "good state" each channel packet is lost with a probability of ε whereas in the "bad state" each channel packet is lost with



(a) Simulation over a Gilbert-Elliott Channel with $(\alpha, \beta) = (5 \times 10^{-4}, 0.5)$. All codes are evaluated using a decoding delay of $T = 12$ symbols and a rate of $R = 12/23 \approx 0.52$.



(b) Simulation over a Gilbert-Elliott Channel with $(\alpha, \beta) = (5 \times 10^{-5}, 0.2)$. All codes are evaluated using a decoding delay of $T = 50$ symbols and a rate of $R = 50/88 \approx 0.6$.

Fig. 3. Simulation Experiments for Gilbert-Elliott Channel Model with different parameters.

a probability of 1. The average loss rate of the Gilbert-Elliott channel is given by

$$\Pr(\mathcal{E}) = \frac{\beta}{\alpha + \beta}\varepsilon + \frac{\alpha}{\alpha + \beta}. \quad (19)$$

where α and β denote the transition probability from the good state to the bad state and vice versa.

Fig. 3(a) and Fig. 3(b) show the simulation performance over a Gilbert-Elliott Channel. The parameters chosen in the two plots are as shown in Table III.

	Fig. 3(a)			Fig. 3(b)		
(α, β)	$(5 \times 10^{-4}, 0.5)$			$(5 \times 10^{-5}, 0.2)$		
Channel Length	10^7			10^8		
Rate R	$12/23 \approx 0.52$			$50/88 \approx 0.6$		
Delay T	12			50		
ε	10^{-3}	5×10^{-3}	10^{-2}	10^{-3}	5×10^{-3}	10^{-2}
Burst Only	0.9642	0.8796	0.7869	0.9005	0.5988	0.3563
Burst + Isolated	0.0268	0.1065	0.1851	0.0923	0.3065	0.3698
Burst + Multiple Isolated	0.0032	0.0081	0.0222	0.0062	0.0937	0.2729
Burst Gaps $< T$	0.0058	0.0058	0.0058	0.0010	0.0010	0.0010

TABLE III
PARAMETERS OF GILBERT-ELLIOTT CHANNEL USED IN SIMULATIONS. WE ALSO PRESENT THE EMPIRICAL FRACTION OF DIFFERENT ERASURE PATTERNS OBSERVED ACROSS A SAMPLE PATH.

The channel parameters for the $T = 12$ case are the same as those used in [1, Section 4-B, Fig. 5]. For each of the channels we generate a sample realization and compute the residual loss rate for each of the codes in sections III, IV and V. We also categorize the empirical fraction of different erasure patterns associated with each erasure burst for the two channels for $\varepsilon \in \{10^{-3}, 5\times10^{-3}, 10^{-2}\}$ in Table III. The first row "Burst-Only" denotes those erasure bursts where there are no isolated erasures in a window of length T before and after the burst. The second row counts those patterns where only one isolated loss occurs in such a window. The third row allows for multiple losses in this window. The fourth row counts those patterns where the inter-burst gap is less than T. Note that the contribution of burst plus isolated losses is significant particularly for the second channel when the delay $T = 50$.

All codes in Fig. 3(a) are selected to have a rate of $R = 12/23 \approx 0.52$ and the delay is $T = 12$. The black horizontal line is the loss rate of the Strongly-MDS code. It achieves $B = N = 6$. Thus its performance is limited by its burst-correction capability. The red-curve which deteriorates rapidly as we increase ε is the Maximally Short codes (MS). It achieves $B = 11$ and $N = 1$. Thus in general it cannot recover from even two losses occurring in a window of length $T + 1$. The remaining two plots correspond to the E-RLC codes and MiDAS codes, both of which achieve $B = 9$ and $N = 2$. The loss probability also deteriorates with ε for both codes, although at a much lower rate. Thus a slight decrease in B, while improving N from 1 to 2 exhibits noticeable gains over both MS and Strongly-MDS codes.

In Fig. 3(b) the rate of all codes except PRC is set to $R = 50/88 \approx 0.57$ while the rate of the PRC codes is 0.6. The delay is set to $T = 50$. The Strongly-MDS codes achieve $B = N = 20$ whereas the MS codes achieve $N = 1$ and $B = 33$. Both codes suffer from the same phenomenon discussed in the previous case. We also consider the MiDAS code with $N = 4$ and $B = 30$. We observe that its performance deteriorates as ε is increased, mainly because it fails on burst plus isolated loss patterns, which are dominant in this regime (see Table III). The PRC code which achieves $N = 4$ and $B = 25$ and can also handle burst plus one isolated loss exhibits the best performance in this plot and we refer to [5] for further discussion.

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