Layered Error-Correction Codes for Real-Time Streaming over Erasure Channels

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University of Toronto

Joint Work:
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Wai-Tian Tan (Cisco),
John Apostolopoulos (Cisco).

Sequential and Adaptive Information Theory Workshop
McGill University
Nov 7, 2013

Real-Time Communication System

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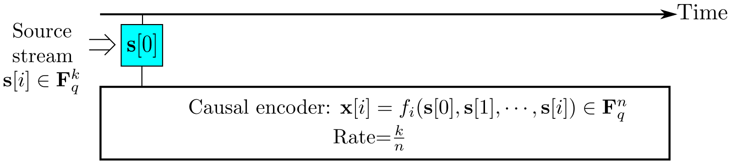

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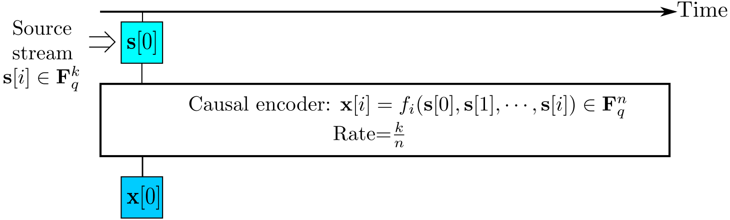

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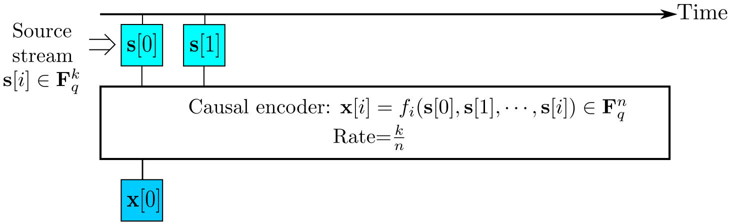

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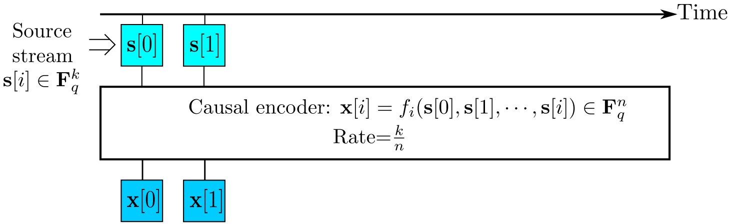

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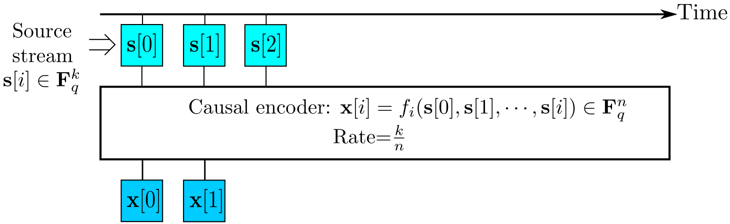

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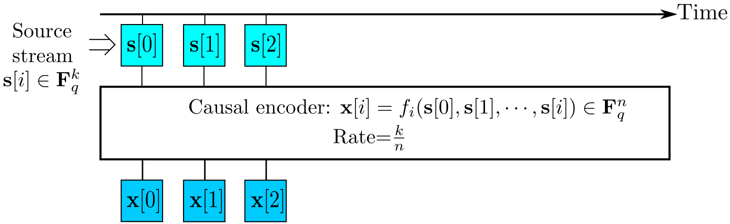

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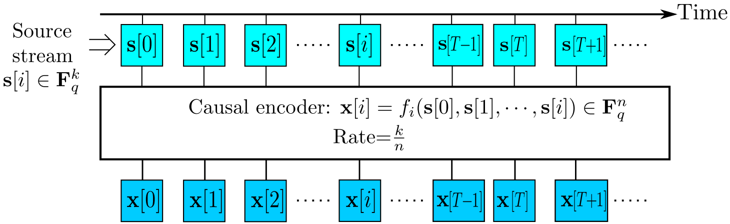

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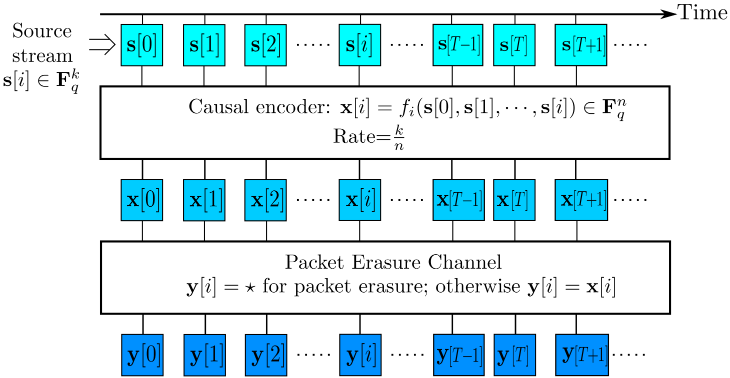

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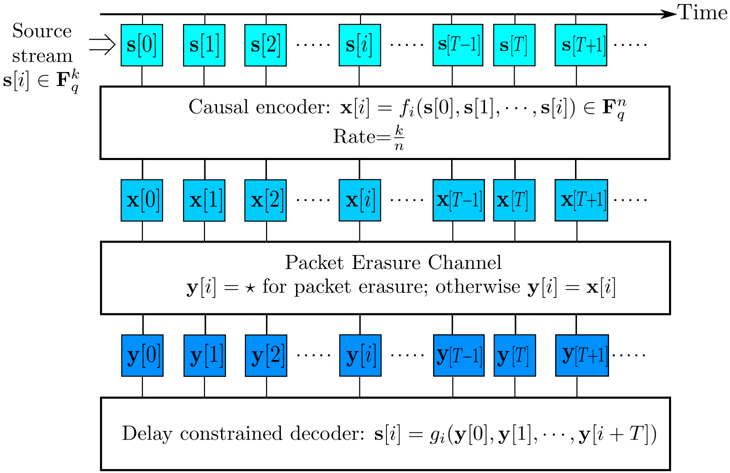

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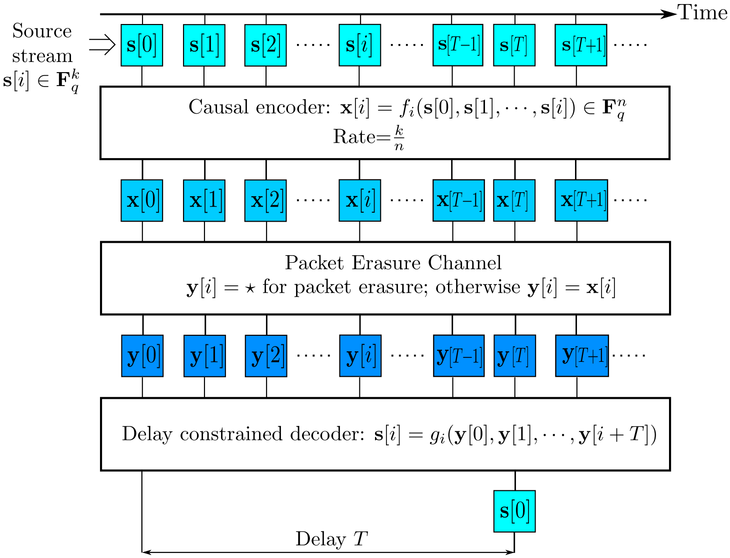

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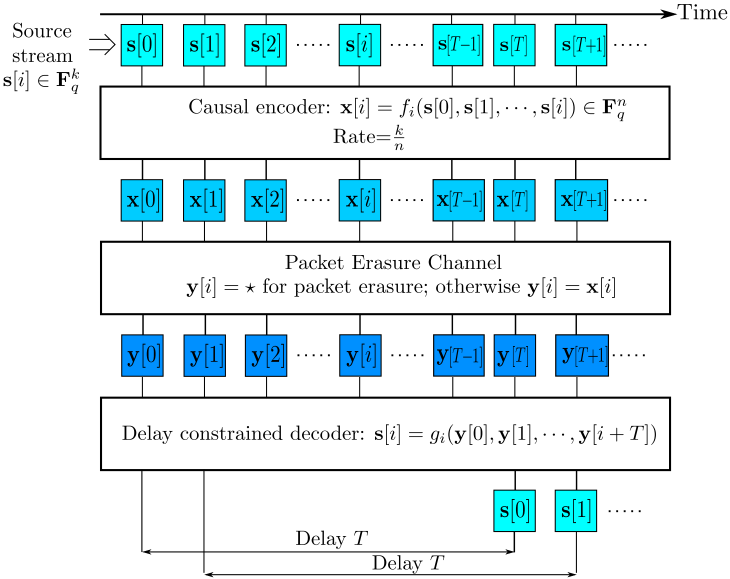
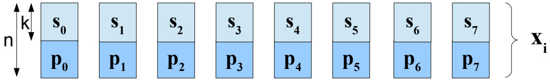


Motivating Example



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Erasure Codes:

- Random Linear Codes
- Strongly-MDS Codes (Gabidulin'88, Gluesing-Luerssen'06)

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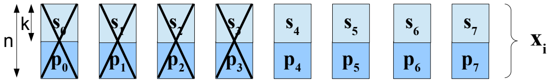

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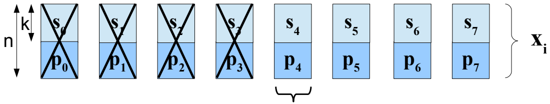

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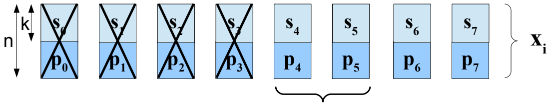

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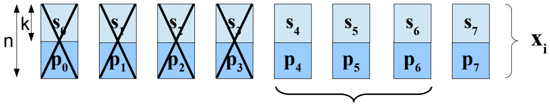

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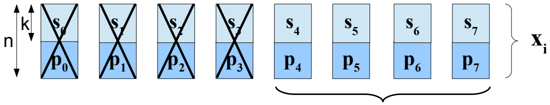

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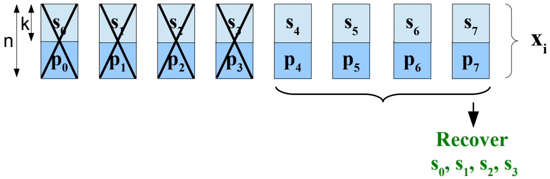

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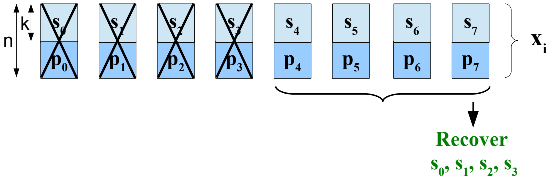

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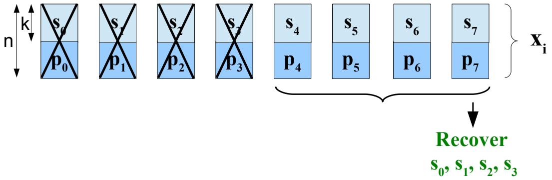

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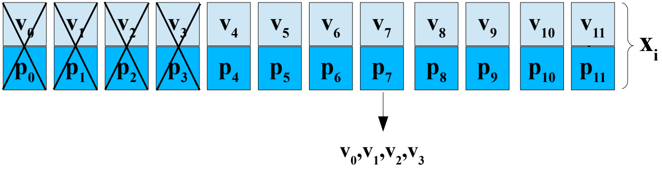
$$\begin{bmatrix} \mathbf{p}_4 \\ \mathbf{p}_5 \\ \mathbf{p}_6 \\ \mathbf{p}_7 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_5 & \mathbf{H}_4 & \mathbf{H}_3 & \mathbf{H}_2 \\ 0 & \mathbf{H}_5 & \mathbf{H}_4 & \mathbf{H}_3 \\ 0 & 0 & \mathbf{H}_5 & \mathbf{H}_4 \end{bmatrix}}_{\text{full rank}} \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{bmatrix}$$

Motivating Example- II

$B = 4$, $T = 8$

Rate $1/2$ Baseline Erasure Codes, $T = 7$



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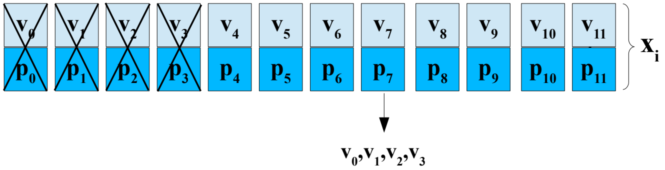
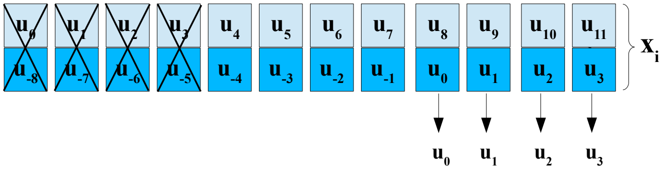


Rate $1/2$ Repetition Code, $T = 8$



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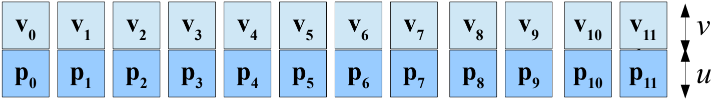

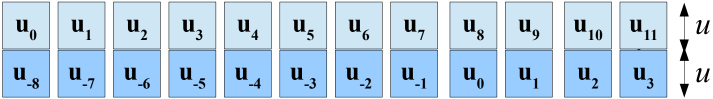

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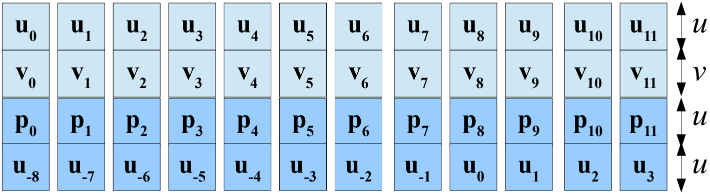

$$R = \frac{u+v}{3u+v} = \frac{1}{2}$$

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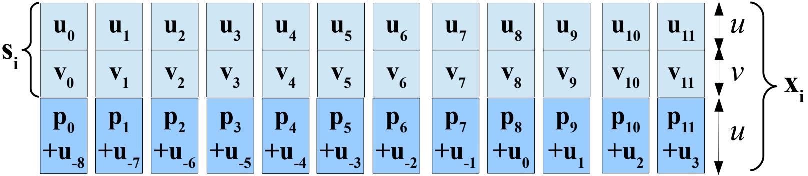

$$R = \frac{u+v}{2u+v} = \frac{2}{3}$$

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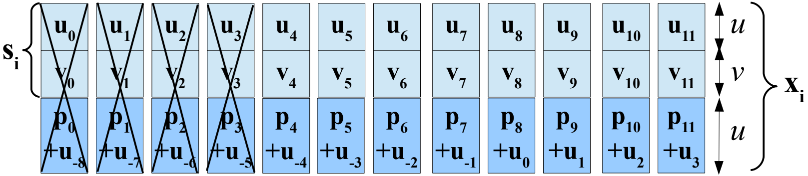

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Encoding:

1. Split each source symbol into 2 groups $\mathbf{s}_i = (\mathbf{u}_i, \mathbf{v}_i)$
2. Apply Erasure code to the $\mathbf{v}_i$ stream generating $\mathbf{p}_i$ parities
3. Repeat the $\mathbf{u}_i$ symbols with a shift of T
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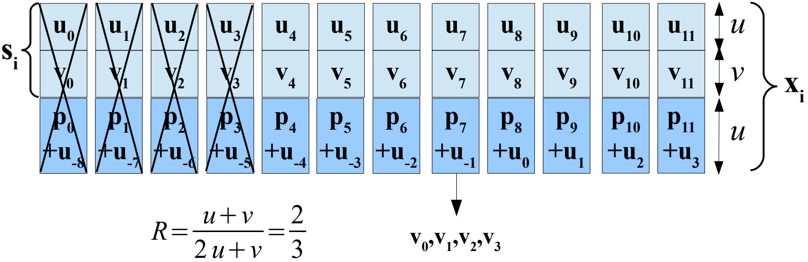

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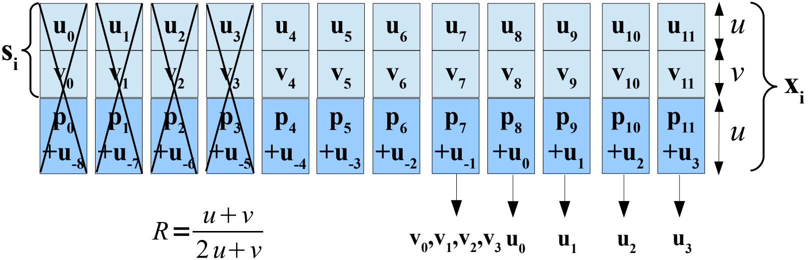

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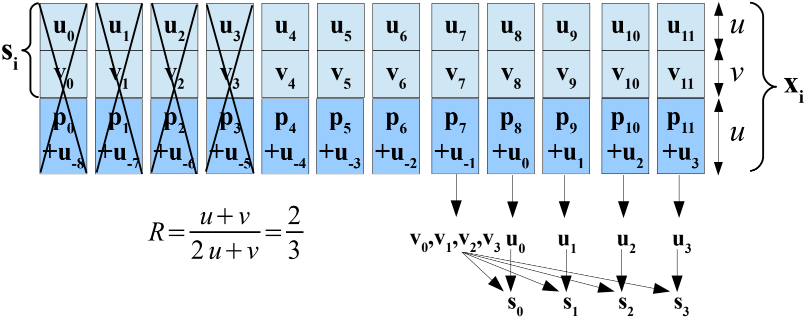

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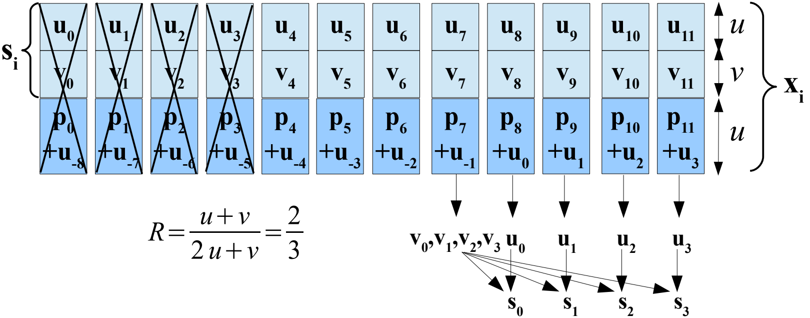

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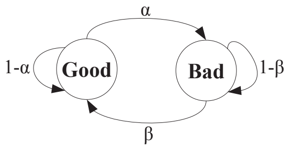
5. Maximally Short Codes (Martinian and Sundberg '04, Martinian and Trott '07)

Key Questions

- Can we construct streaming codes for realistic channels?



Gilbert-Elliott Model

- How much performance gains can we obtain?
- What are the fundamental metrics for low-delay error correction codes?

Problem Setup

System Model

- Source Model : i.i.d. sequence $s[t] \sim p_s(\cdot) = \mathrm{Unif}\{(\mathbb{F}_q)^k\}$
- Streaming Encoder: $x[t] = f_t\,(s[1], \ldots, s[t])$, $x[t] \in (\mathbb{F}_q)^n$
- Erasure Channel
- Delay-Constrained Decoder: $\hat{s}[t] = g_t(y[1], \ldots, y[t+T])$
- Rate $R = \frac{k}{n}$

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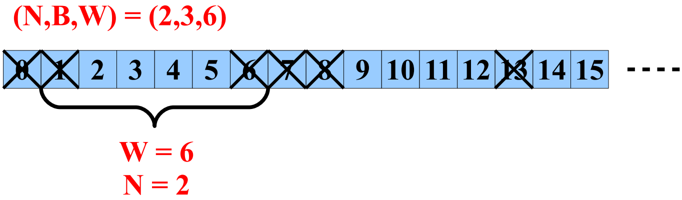

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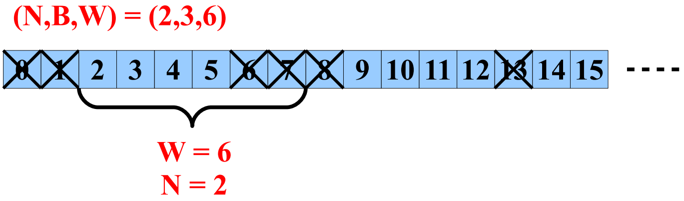

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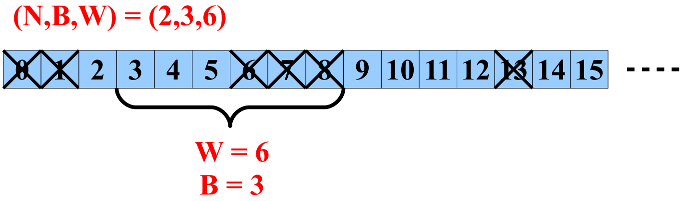

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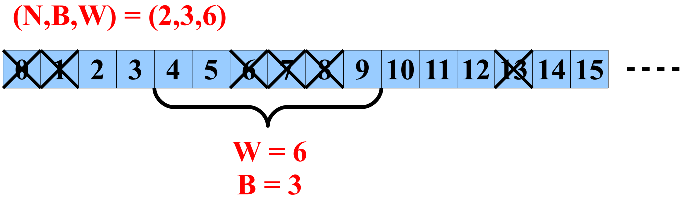


Capacity: $C(N, B, W, T)$

Streaming Codes — Capacity

Badr-Khisti-Tan-Apostolopoulos (2013)

Theorem

Consider the $\mathcal{C}(N, B, W)$ *channel, with* $W \geq B + 1$*, and let the delay be* T.

Upper-Bound *For any rate* R *code, we have:*

$$\left(\frac{R}{1-R}\right) B + N \leq \min(W, T+1)$$

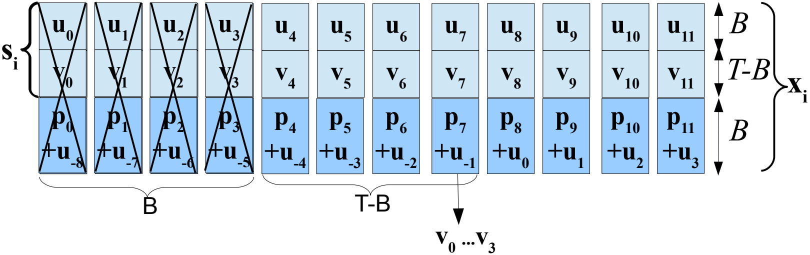
Lower-Bound: *There exists a rate* R *code that satisfies:*

$$\left(\frac{R}{1-R}\right) B + N \geq \min(W, T+1) - 1.$$

The gap between the upper and lower bound is 1 *unit of delay.*

Streaming Codes - Burst Erasure Channels

$$C(N=1,B,W\geq T+1)=\frac{T}{T+B}$$



- Assume $\mathbf{s}_i \in \mathbb{F}_q^T$. Let $\mathbf{v}_i \in \mathbb{F}_q^{T-B}$, $\mathbf{u}_i, \mathbf{p}_i \in \mathbb{F}_q^B$
- Recovery of $\{\mathbf{v}_i\}$ by time $T-1$
  - Number of Unknowns: B symbols over $\mathbb{F}_q^{T-B}$
  - Number of Equations: $T-B$ symbols over $\mathbb{F}_q^B$
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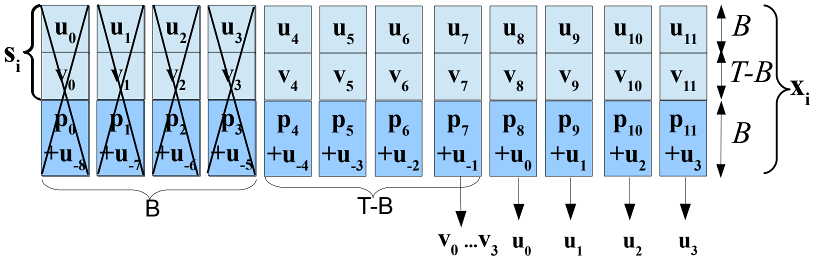

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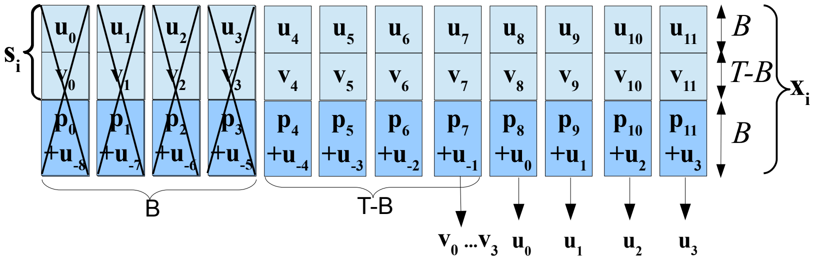
$C(N=1, B, W \geq T+1) = \frac{T}{T+B}$

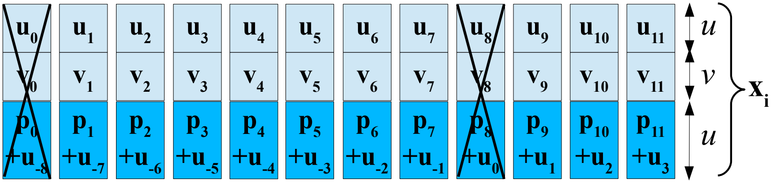


- Assume $\mathbf{s}_i \in \mathbb{F}_q^T$. Let $\mathbf{v}_i \in \mathbb{F}_q^{T-B}$, $\mathbf{u}_i, \mathbf{p}_i \in \mathbb{F}_q^B$
- Recovery of $\{\mathbf{v}_i\}$ by time $T-1$
  - Number of Unknowns: B symbols over $\mathbb{F}_q^{T-B}$
  - Number of Equations: $T-B$ symbols over $\mathbb{F}_q^B$
- Recovery of $\mathbf{u}_i$ at time $i+T$

Streaming Codes - Isolated Erasures

$C(N \geq 2$, B, W)

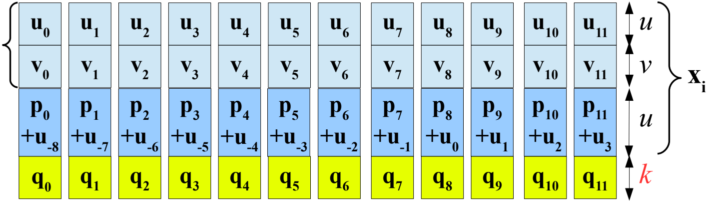
$T = 8$



- Erasures at time $t = 0$ and $t = 8$
- $\mathbf{u}_0$ cannot be recovered due to a repetition code

Proposed Approach: Layering

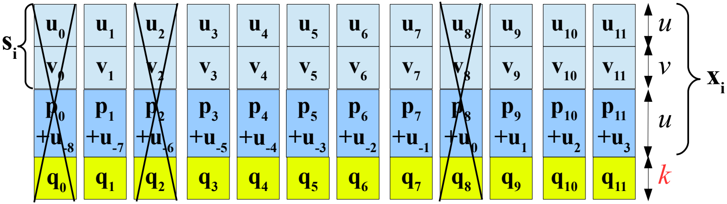
$C(N \geq 2$, B, W)



Layered Code Design

- Burst-Erasure Streaming Code $\mathcal{C}_1 : (\mathbf{u}_i, \mathbf{v}_i, \mathbf{p}_i + \mathbf{u}_{i-T})$
- Erasure Code: $\mathbf{q}_i = \sum_{t=1}^{M} \mathbf{u}_{i-t} \cdot \mathbf{H}_t^u$, $\mathbf{q}_i \in \mathbb{F}_q^k$
- $\mathcal{C}_2$: $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{p}_i + \mathbf{u}_{i-T}, \mathbf{q}_i)$

$$R = \frac{T}{T+B+k}, \qquad k = \frac{N}{T-N+1} B$$



Layered Code Design

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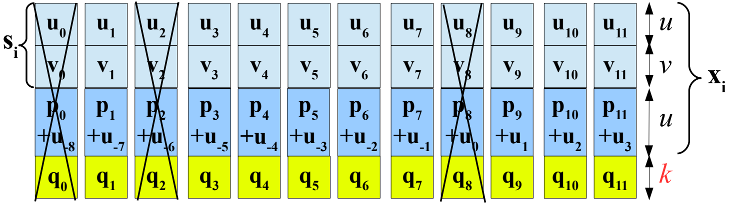

Layered Code Design

- Burst-Erasure Streaming Code $\mathcal{C}_1 : (\mathbf{u}_i, \mathbf{v}_i, \mathbf{p}_i + \mathbf{u}_{i-T})$
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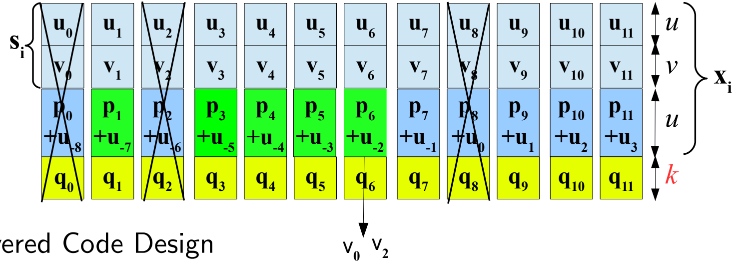

Layered Code Design

- Burst-Erasure Streaming Code $\mathcal{C}_1 : (\mathbf{u}_i, \mathbf{v}_i, \mathbf{p}_i + \mathbf{u}_{i-T})$
- Erasure Code: $\mathbf{q}_i = \sum_{t=1}^{M} \mathbf{u}_{i-t} \cdot \mathbf{H}_t^u$, $\mathbf{q}_i \in \mathbb{F}_q^k$
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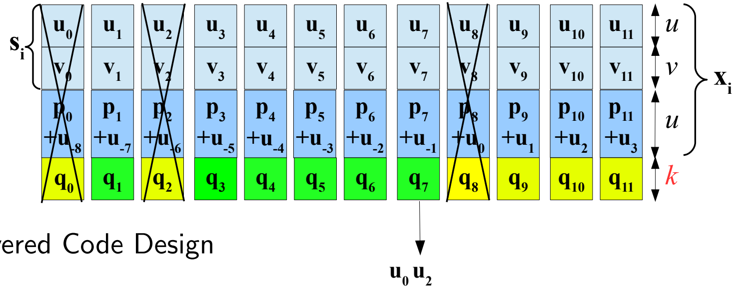
Proposed Approach: Layering

$C(N \geq 2$, B, W)



Layered Code Design

- Burst-Erasure Streaming Code $\mathcal{C}_1$: $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{p}_i + \mathbf{u}_{i-T})$
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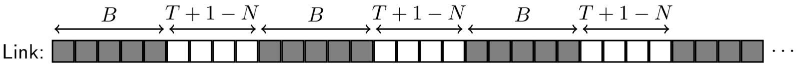
$$R = \frac{T}{T+B+k}, \qquad k = \frac{N}{T-N+1}B$$

Upper Bound

$W \geq T+1$

- Periodic Erasure Channel with $P = T + 1 - N + B$.
- B erasures in each burst.
- Guard Interval $= T + 1 - N$.



Show that any low-delay code recovers every symbol on this erasure channel.

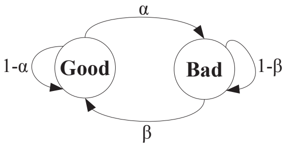
$$R \leq \frac{T+1-N}{T+1-N+B}$$

Simulation Results

Gilbert-Eliott Channel $(\alpha, \beta) = (5 \times 10^{-4}, 0.5)$, $T = 12$ and $R = 12/23$

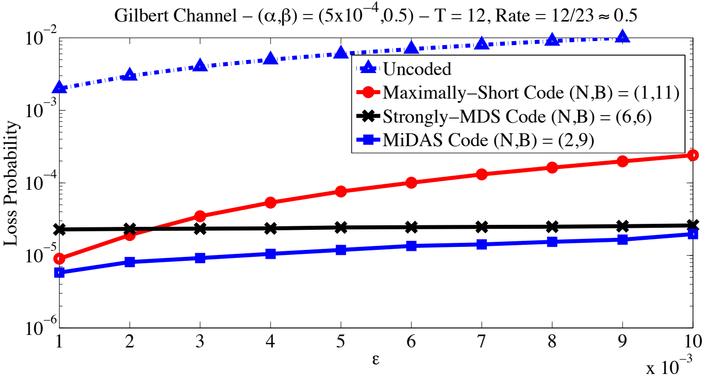
Gilbert Elliott Channel

- Good State: $\Pr(\text{loss}) = \varepsilon$
- Bad State: $\Pr(\text{loss}) = 1$



Simulation Results

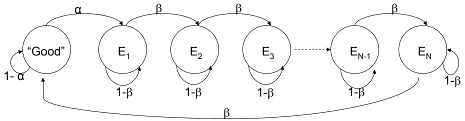
Gilbert-Eliott Channel $(\alpha, \beta) = (5 \times 10^{-4}, 0.5)$, $T = 12$ and $R = 12/23$



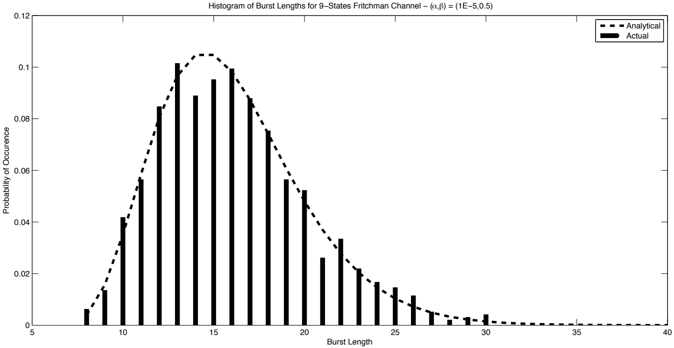
Code	N	B	Code	N	B
Strongly MDS	6	6	MiDAS	2	9
Burst-Erasure	1	11			

Simulation Results-II

Fritchman Channel $(\alpha, \beta) = (1e-5, 0.5)$ and $T = 40$ and $R = 40/79$, 9 states

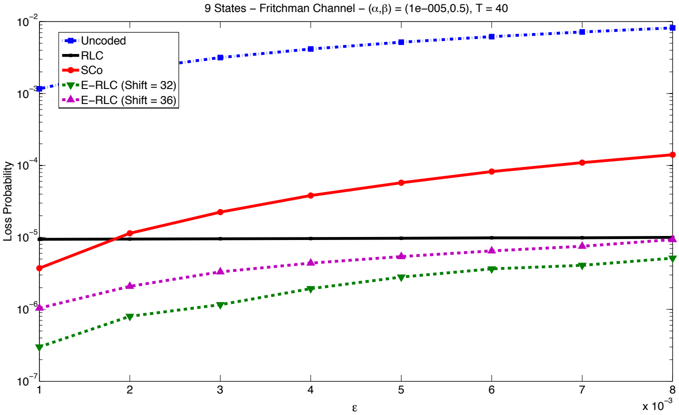


- $\alpha = 1e-5$
- $\beta = 0.5$



Simulation Results-II

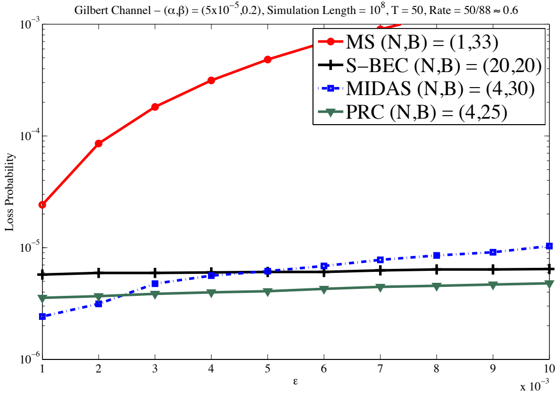
Fritchman Channel $(\alpha, \beta) = (1e - 5, 0.5)$ and $T = 40$ and $R = 40/79$, 9 states



Code	N	B	Code	N	B
Strongly MDS	20	20	MiDAS-I	8	31
Burst Erasure	1	39	MiDAS-II	4	35

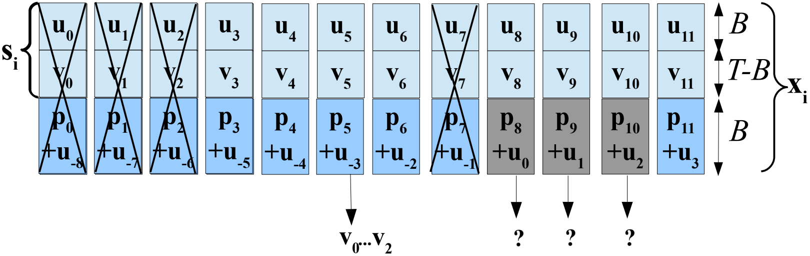
Simulation Results - III

Gilbert-Eliott Channel $(\alpha, \beta) = (5 \times 10^{-5}, 0.2)$, $T = 50$ and $R \approx 0.6$



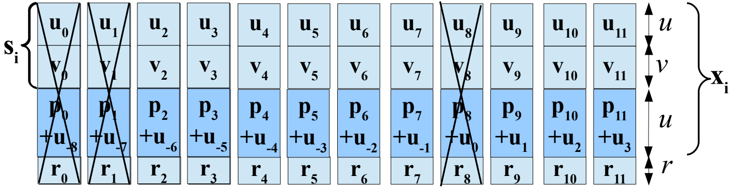
Code	N	B	Code	N	B
Strongly MDS	20	20	MiDAS	4	30
Burst-Erasure	1	33	PRC	4	25

When do Earlier Constructions Fail?



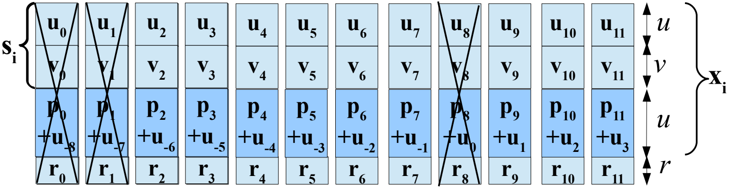
- Burst spans $t \in [0, 2]$
- Isolated Erasure $i \in [7, 10]$
- Interference from $\mathbf{v}_i$ during the recovery of $\mathbf{u}_0, \ldots, \mathbf{u}_2$

Layered Construction for Partial Recovery



- $\mathbf{r}_i = \sum_{t=1}^{M} \mathbf{v}_{i-t} \cdot \mathbf{H}_t^v$
- Shift in Repetition Code: $\mathbf{p}_i + \mathbf{u}_{i-(T-\Delta)}$
- Isolated Loss: $v \leq (\Delta + 1)r$
- Burst Loss: $v(B+1) \leq (T - \Delta - B)(u + r)$

Layered Construction for Partial Recovery



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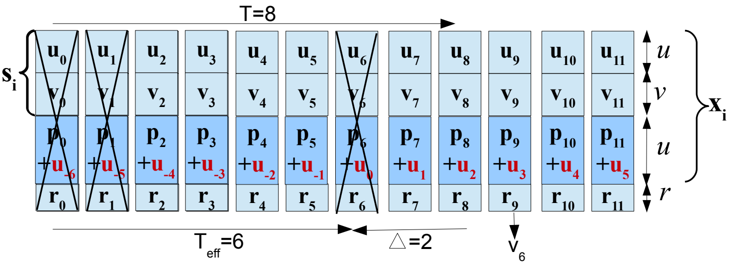
Streaming Codes — Burst plus Isolated Erasures

Badr-Khisti-Tan-Apostolopoulos (2013)

Layered Construction for Partial Recovery



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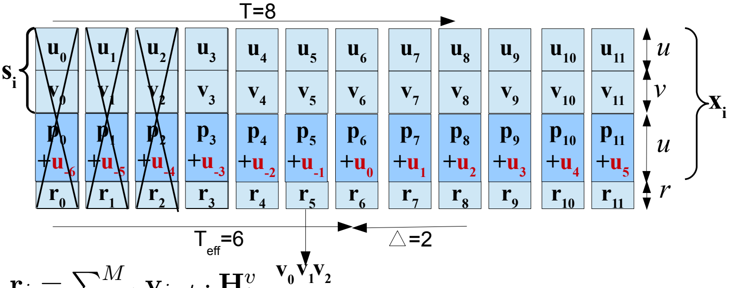

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Streaming Codes — Burst plus Isolated Erasures

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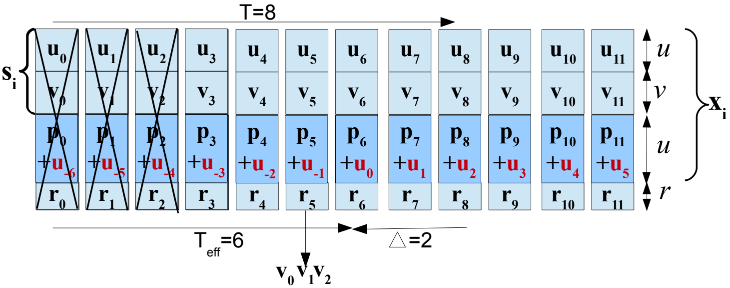


Code-Params

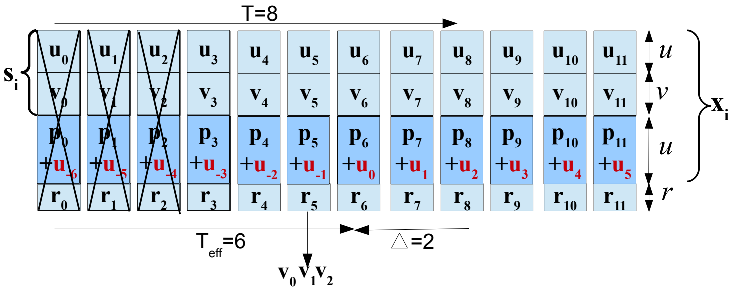
$$
\begin{aligned}
v &= (T-\Delta+1)(\Delta-B-1)\\
u &=(B+1)(T-\Delta+1)\\
&\quad -(\Delta-B-1)\\
r &= (\Delta-B-1)
\end{aligned}
$$

Rate

$$
\begin{aligned}
R &= \frac{u+v}{2u+v+r}\\
&= \frac{\Delta(T-\Delta)+(B+1)}{\Delta(T-\Delta)+(B+1)(T-\Delta+2)}\\
\Delta^* &= T+1-\sqrt{T-B}
\end{aligned}
$$

Streaming Codes — Burst plus Isolated Erasures

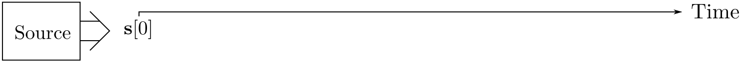
Badr-Khisti-Tan-Apostolopoulos (2013)



Layering Principle

- Start with a code for $\mathcal{C}(N, B, W)$
- Add additional parity check symbols for more sophisticated patterns.

Unequal Source/Channel Rates



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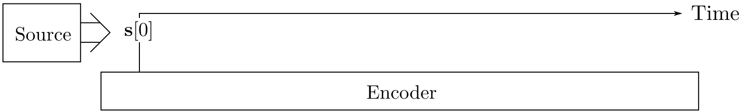

- Source model: i.i.d. process with $\mathbf{s}[i] \sim$ uniform over $\mathbf{F}_q^k$

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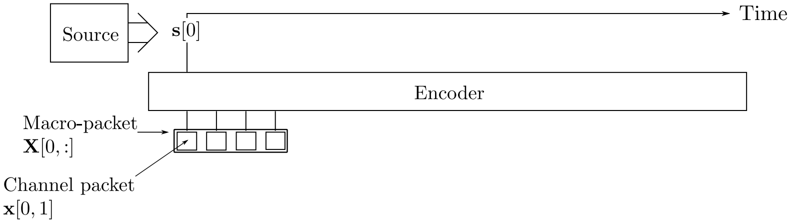

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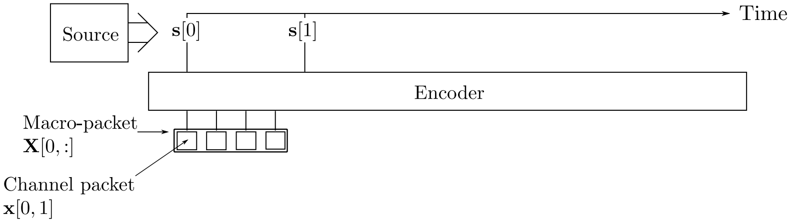

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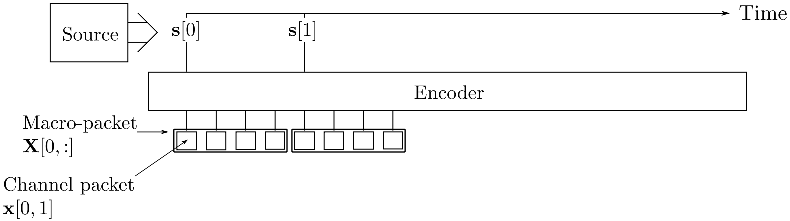
- Streaming encoder: $\mathbf{x}[i,j] = f_{i,j}(\mathbf{s}[0], \mathbf{s}[1], \cdots, \mathbf{s}[i]) \in \mathbf{F}_q^n$
  Macro-packet: $\mathbf{X}[i,:] = [\mathbf{x}[i,1] \mid \ \ldots \ \mid \mathbf{x}[i,M]]$
  Rate: $R = \frac{\mathsf{H}(\mathbf{s})}{n \times M} = \frac{k}{n \times M}$

Unequal Source/Channel Rates



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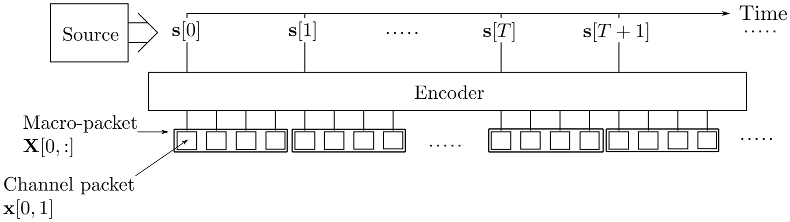

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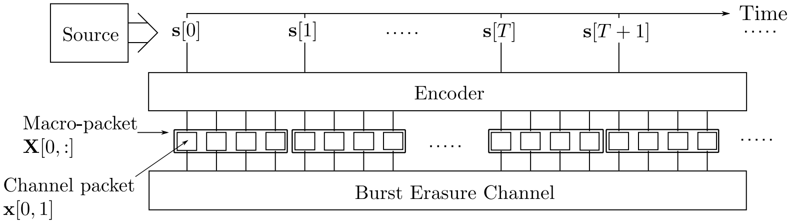

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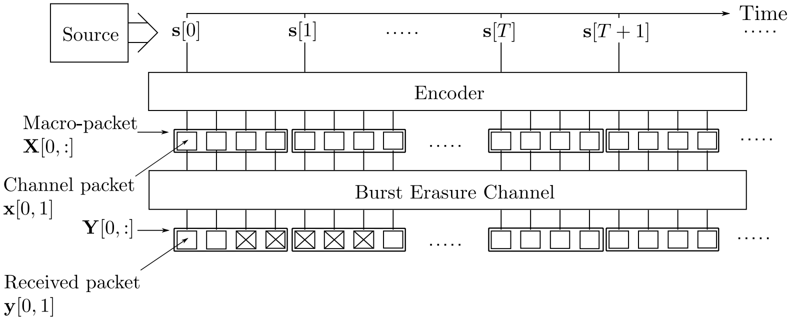

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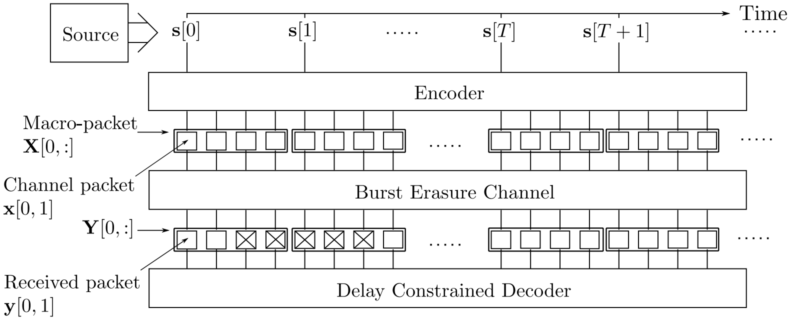
Unequal Source/Channel Rates



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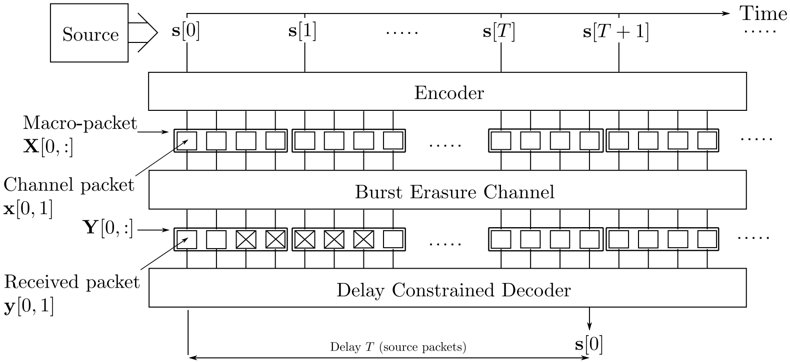

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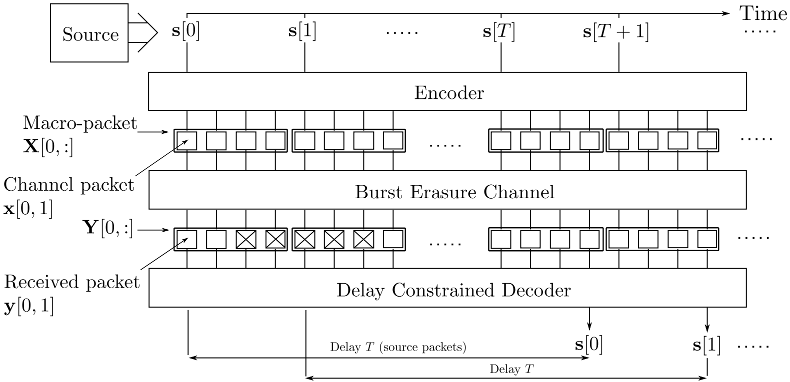
- Packet erasure channel: erasure burst of maximum B channel packets
- Delay-constrained decoder: $\mathbf{s}[i]$ recovered by macro-packet $i + T$

Unequal Source/Channel Rates



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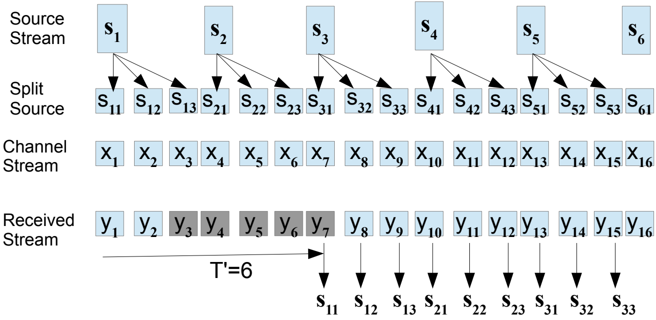
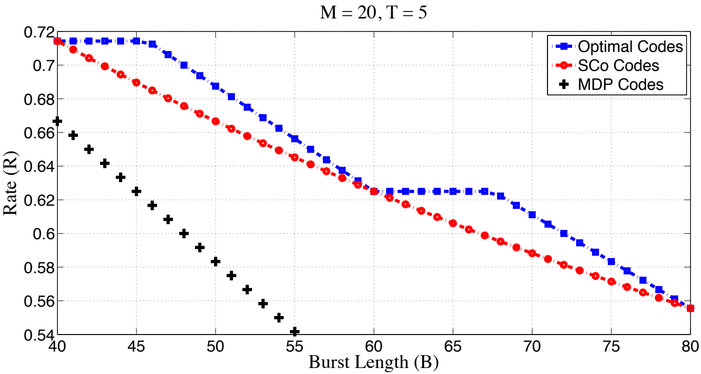
Source-splitting based scheme



- Split $\mathbf{s}[i]$ into sub-symbols $(\mathbf{s}[i,1], \mathbf{s}[i,2], \ldots, \mathbf{s}[i,M])$.
- Apply a streaming code for $M = 1$.
- Decoding Delay $T' = M \cdot T$.
- $R = \frac{MT}{MT+B}$

Unequal Source/Channel Rates

Patil-Badr-Khisti-Tan 2013

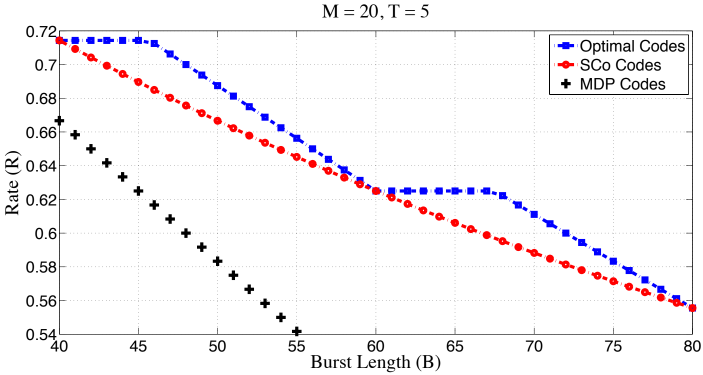


- Source Splitting:

  $R = \frac{MT}{MT+B}$

- Strongly MDS:

  $R = 1 - \frac{B}{M(T+1)}$



- Source Splitting:

  $R = \frac{MT}{MT+B}$

- Strongly MDS:

  $R = 1 - \frac{B}{M(T+1)}$

Capacity: Let $B = bM + B'$, where $B' \in \{0, \ldots, M-1\}$

$$C = \begin{cases} \frac{T}{T+b}, & 0 \leq B' \leq \frac{b}{T+b}M \\ \frac{M(T+b+1)-B}{M(T+b+1)}, & \frac{b}{T+b}M \leq B' \leq M-1 \end{cases}$$

Optimal Code Construction I

Patil-Badr-Khisti-Tan 2013

Key Idea:

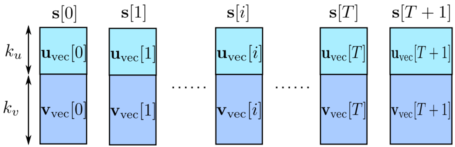
1. Apply code for $M = 1$ on complete source-packet to generate the entire macro-packet.
2. Map/reshape the generated macro-packet into M individual channel-packets.

Optimal Code Construction II

Patil-Badr-Khisti-Tan 2013

1. Source splitting
   - split $\mathbf{s}[i]$ into into two groups

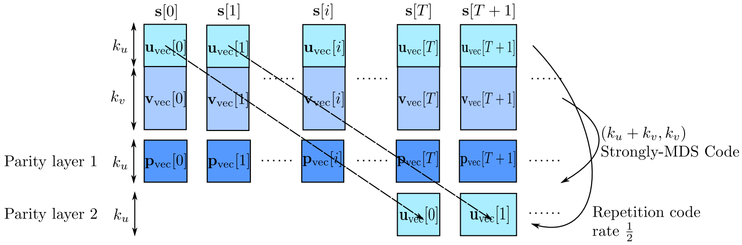
$$\begin{aligned}\mathbf{s}[i] &= (s_1[i], \ldots, s_k[i]) \\ &= (\underbrace{u_1[i], \ldots, u_{k_u}[i]}_{\mathbf{u}_{\text{vec}}[i]}, \underbrace{v_1[i], \ldots, v_{k_v}[i]}_{\mathbf{v}_{\text{vec}}[i]})\end{aligned}$$



Optimal Code Construction III

Patil-Badr-Khisti-Tan 2013

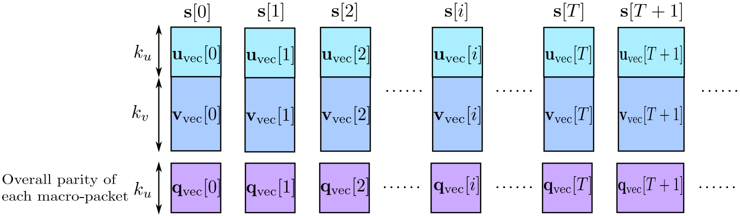
2. Parity generation
   - layer 1: $(k_v + k_u, k_v, T)$ Strongly-MDS code applied to $\mathbf{v}_{\text{vec}}[\cdot]$ generating $\mathbf{p}_{\text{vec}}[i]$
   - layer 2: repetition code on $\mathbf{u}_{\text{vec}}$ with a shift of T



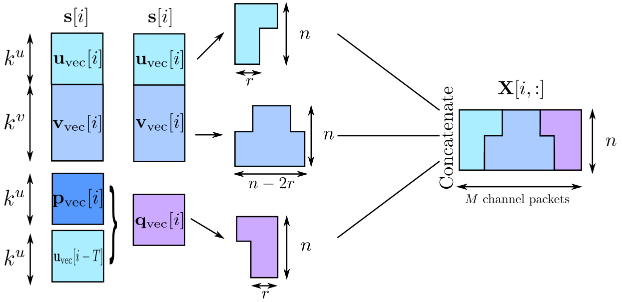
Optimal Code Construction IV

Patil-Badr-Khisti-Tan 2013

- Overall combined parity: $\mathbf{q}_{\text{vec}}[i] = \mathbf{p}_{\text{vec}}[i] + \mathbf{u}_{\text{vec}}[i-T]$

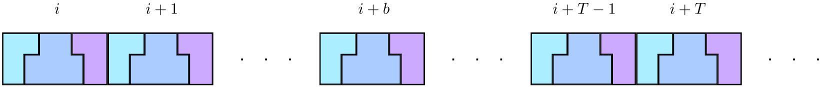


3. Mapping of macro-packet to individual channel-packets
   - Map the generated macro-packet into M individual channel-packets



Rate of the code$=\frac{k_u+k_v}{2k_u+k_v}$

Overall macro-packet structure:

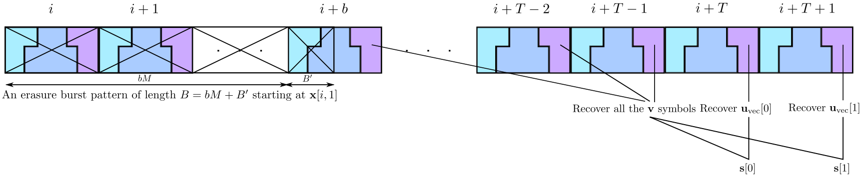


Overall macro-packet arrangment

Decoding Analysis

Patil-Badr-Khisti-Tan 2013

Key Fact: The worst case burst pattern starts at the beginning of the macro-packet.



Let $B = bM + B'$. Two cases depending on whether $B' \lessgtr (1-R)M$:

Burst only erases symbols from $\mathbf{u}_{\text{vec}}[i+b]$

- Recovery of $\mathbf{v}$ symbols: $(k_u + k_v)b = k_u T$
- Optimal spitting ratio: $\frac{k_u}{k_v} = \frac{b}{T-b}$
- $R = \frac{T}{T+b}$

Burst erases symbols from $\mathbf{v}_{\text{vec}}[i+b]$

- Recovery of $\mathbf{v}$ symbols: $(k_u + k_v)b + (B'n - k_u) = k_u T$
- Optimal spitting ratio: $\frac{k_u}{k_v} = \frac{B}{M(T+b+1)-2B}$
- $R = \frac{M(T+b+1)-B}{M(T+b+1)}$

Capacity Result

Theorem

For the streaming setup considered, with any M, T and B, the streaming capacity C is given by the following expression:

$$C = \begin{cases} \frac{T}{T+b}, & B' \leq \frac{b}{T+b}M,\ T \geq b, \\ \frac{M(T+b+1)-B}{M(T+b+1)}, & B' > \frac{b}{T+b}M,\ T > b, \\ \frac{M-B'}{M}, & B' > \frac{M}{2},\ T = b, \\ 0, & T < b. \end{cases}$$

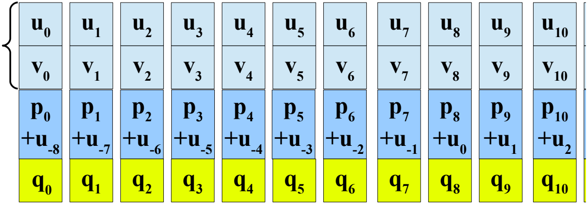
where the constants b and B' are defined via

$$B = bM + B', \quad B' \in \{0, 1, \ldots, M-1\}, \quad b \in \mathbb{N}^0.$$

□

Robust Extension for $M = 1$

- Append an additional layer parity checks containing Strongly-MDS code on $\mathbf{u}$



- $R = \frac{u+v}{2u+v+k}$.
- Very close to being optimal for $k = \frac{N}{T-N+1}B$

Robust Extension for any M

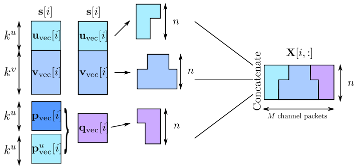
Many possibilities for the placement of Strongly-MDS parities for u!

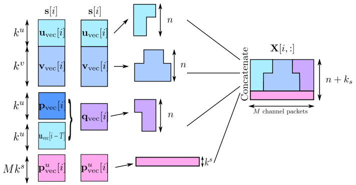
Approach I:

- Repetition code replaced by Strongly-MDS code for $\mathbf{u}$
- Rate of the code unchanged.
- $N = r$ is achievable.

Approach II

- Append additional layer of Strongly-MDS code for $\mathbf{u}$
- $R = \frac{k_u + k_v}{2k_u + k_v + k_s}$.
- k_s chosen according to given N.





Related Works

- Burst Erasure Channel: Maximally Short Codes (Block Code + Interleaving), Martinian and Sundberg (IT-2004), Martinian and Trott (ISIT-2007)

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  - Multicast Extension (Khisti-Singh 2009, Badr-Lui-Khisti Allerton 2010)
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- Tree Codes: Schulman (IT 1996), Sahai (2001), Martinian and Wornell (Allerton 2004), Sukhavasi and Hassibi (2011)

Distance and Span Properties

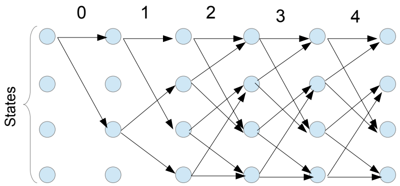
- Append an additional layer parity checks containing Strongly-MDS code on $\mathbf{u}$

$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$	$\mathbf{u}_8$	$\mathbf{u}_9$	$\mathbf{u}_{10}$
$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{p}_0 + \mathbf{u}_{-8}$	$\mathbf{p}_1 + \mathbf{u}_{-7}$	$\mathbf{p}_2 + \mathbf{u}_{-6}$	$\mathbf{p}_3 + \mathbf{u}_{-5}$	$\mathbf{p}_4 + \mathbf{u}_{-4}$	$\mathbf{p}_5 + \mathbf{u}_{-3}$	$\mathbf{p}_6 + \mathbf{u}_{-2}$	$\mathbf{p}_7 + \mathbf{u}_{-1}$	$\mathbf{p}_8 + \mathbf{u}_0$	$\mathbf{p}_9 + \mathbf{u}_1$	$\mathbf{p}_{10} + \mathbf{u}_2$
$\mathbf{q}_0$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{q}_4$	$\mathbf{q}_5$	$\mathbf{q}_6$	$\mathbf{q}_7$	$\mathbf{q}_8$	$\mathbf{q}_9$	$\mathbf{q}_{10}$

- $R = \frac{u+v}{2u+v+k}$.
- Very close to being optimal for $k = \frac{N}{T-N+1}B$
- MiDAS $\rightarrow$ (Near) Maximum Distance And Span tradeoff

Distance and Span Properties

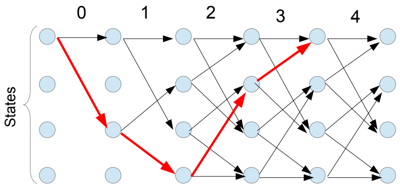
Consider (n, k, m) Convolutional code: $\mathbf{x}_i = \sum_{j=0}^{m} \mathbf{s}_{i-j}\mathbf{G}_j$



Trellis Diagram

Distance and Span Properties

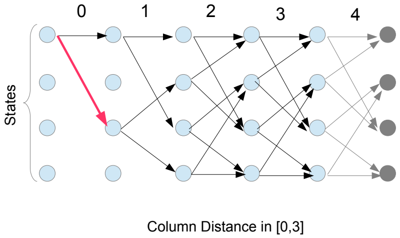
Consider (n, k, m) Convolutional code: $\mathbf{x}_i = \sum_{j=0}^{m} \mathbf{s}_{i-j}\mathbf{G}_j$



Trellis Diagram – Free Distance

Distance and Span Properties

Consider (n, k, m) Convolutional code: $\mathbf{x}_i = \sum_{j=0}^{m} \mathbf{s}_{i-j} \mathbf{G}_j$



Column Distance: d_T

$$d_T = \min_{\substack{[\mathbf{s}_0, \ldots, \mathbf{s}_T] \\ \mathbf{s}_0 \neq 0}} \mathrm{wt} \left(\begin{bmatrix} \mathbf{s}_0 & \ldots & \mathbf{s}_T \end{bmatrix} \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \ldots & \mathbf{G}_T \\ 0 & \mathbf{G}_0 & \ldots & \mathbf{G}_{T-1} \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \mathbf{G}_0 \end{bmatrix} \right)$$

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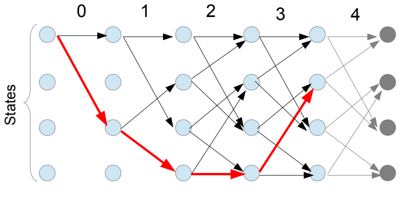

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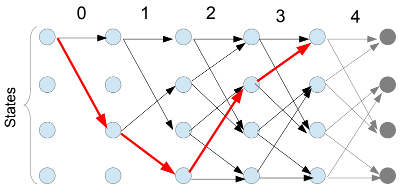

Column Span in [0,3]

Column Span: c_T

$$c_T = \min_{\substack{[\mathbf{s}_0,\ldots,\mathbf{s}_T] \\ \mathbf{s}_0 \neq 0}} \text{span} \left(\begin{bmatrix} \mathbf{s}_0 & \ldots & \mathbf{s}_T \end{bmatrix} \begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & \ldots & \mathbf{G}_T \\ 0 & \mathbf{G}_0 & \ldots & \mathbf{G}_{T-1} \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \mathbf{G}_0 \end{bmatrix} \right)$$

Column-Distance & Column Span Tradeoff

Badr-Khisti-Tan-Apostolopoulos (2013)

Theorem

Consider a $\mathcal{C}(N, B, W)$ channel with delay T and $W \geq T + 1$. A streaming code is feasible over this channel if and only if it satisfies: $d_T \geq N + 1$ and $c_T \geq B + 1$

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Theorem

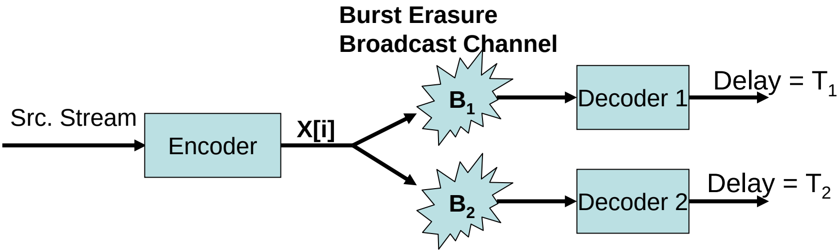
For any rate R convolutional code and any $T \geq 0$ the Column-Distance d_T and Column-Span c_T satisfy the following:

$$\left(\frac{R}{1-R}\right) c_T + d_T \leq T + 1 + \frac{1}{1-R}$$

There exists a rate R code (MiDAS Code) over a sufficiently large field that satisfies:

$$\left(\frac{R}{1-R}\right) c_T + d_T \geq T + \frac{1}{1-R}$$

Multicast (Low-Delay) Codes

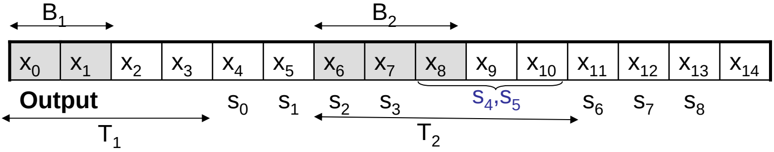


Motivation

- $B_1 < B_2$
- Receiver 1 : Good Channel State
- Receiver 2: Weaker Channel State
- Delay adapts to Channel State

Channel-Adaptive Delay

Example: $B_1 = 2$, $T_1 = 4$, $B_2 = 3$ and $T_2 = 5$.

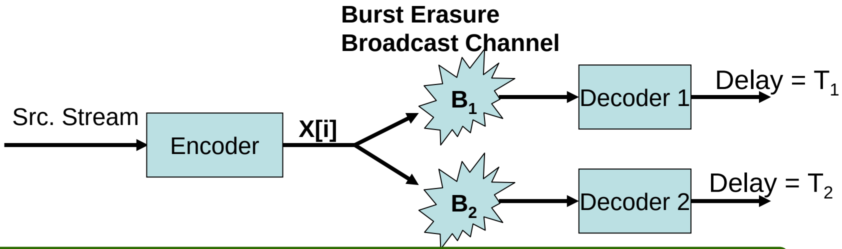


- A burst of length B_1 results in a delay of T_1.
- A burst of length B_2 results in a delay of T_2.
- Stretch-Factor: $s = \frac{T_2 - B_1}{T_1 - B_1}$

Tradeoff between s and $\Pr(\text{loss})$.

Diversity Embedded Streaming Codes: DE-SCo

Badr-Khisti-Martinian 2011



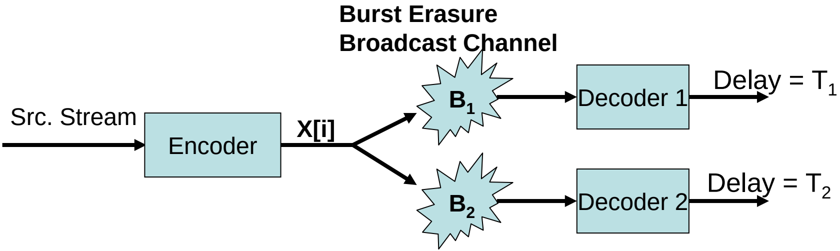
Theorem

There exists a $\{(B_1, T_1), (B_2, T_2)\}$ *DE-SCo construction of rate* $R = \frac{T_1}{T_1+B_1}$ *provided*

$$T_2 \geq T_2^\star \triangleq \frac{B_2}{B_1} T_1 + B_1.$$

This code has polynomial-time encoding and decoding complexity.

Multicast (Low Delay) Capacity



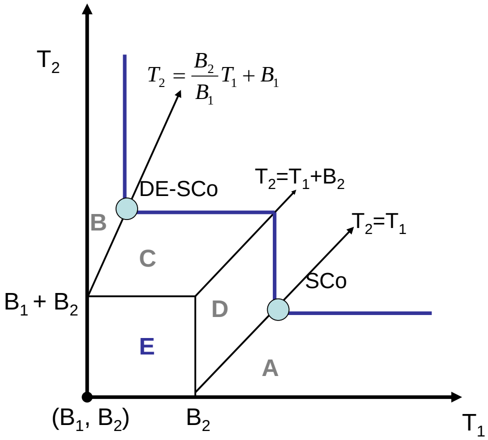
Capacity Function

- Capacity function $C(T_1, T_2, B_1, B_2)$
- Single User Upper Bound: $C \leq \min\left(\frac{T_1}{T_1+B_1}, \frac{T_2}{T_2+B_2}\right)$
- Concatenation Lower Bound: $C \geq \frac{1}{1+\frac{B_1}{T_1}+\frac{B_2}{T_2}}$

Multicast (Low Delay) Capacity

Badr-Khisti-Lui 2011

Assume w.l.o.g. $B_2 \geq B_1$



Region	Capacity
A	$\frac{T_2}{T_2 + B_2}$
B	$\frac{T_1}{T_1 + B_1}$
C	$\frac{T_2 - B_1}{T_2 - B_1 + B_2}$
D	$\frac{T_1}{T_1 + B_2}$
E	??

Other Extensions

- Multiple Erasure Bursts (Li-Khisti-Girod 2011) - Interleaved Low-Delay Codes
- Multiple Links (Lui-Badr-Khisti 2011) - Layered coding for burst erasure channels
- Multiple Source Streams with Different Decoding Delays (Lui 2011) - Embedded Codes

Conclusions

- Error Correction Codes for Real-Time Streaming
- Deterministic Channel Models $\mathcal{C}(N, B, W)$
- Tradeoff between achievable N and B
- MiDAS Constructions
- Column-Distance and Column-Span Tradeoff
- Partial Recovery Codes for Burst $+$ Isolated Erasures
- Unequal Source-Channel Rates