Rational z-Transforms and Its Inverse

Professor Deepa Kundur

University of Toronto

Reference

Reference:

Sections 3.3 and 3.4 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Rational z-Transforms

Why Rational?

- $X(z)$ is a rational function iff it can be represented as the ratio of two polynomials in z^{-1} (or z):

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

- For LTI systems that are represented by LCCDEs, the z-Transform of the unit sample response $h(n)$, denoted $H(z) = \mathcal{Z}\{h(n)\}$, is rational

Poles and Zeros

- zeros of $X(z)$: values of z for which $X(z) = 0$
- poles of $X(z)$: values of z for which $X(z) = \infty$

Poles and Zeros of the Rational z-Transform

Let $a_0, b_0 \neq 0$:

$$
\begin{aligned}
X(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N/a_0} \\
&= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \\
&= G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}
\end{aligned}
$$

Poles and Zeros of the Rational z-Transform

$$
X(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)} \quad \text{where } G \equiv \frac{b_0}{a_0}
$$

- $X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$
- $X(z)$ has N finite poles at $z = p_1, p_2, \ldots, p_N$
- For $N - M \neq 0$
  - if $N - M > 0$, there are $|N - M|$ zeros at origin, $z = 0$
  - if $N - M < 0$, there are $|N - M|$ poles at origin, $z = 0$

Total number of zeros = Total number of poles

Poles and Zeros of the Rational z-Transform

Example:

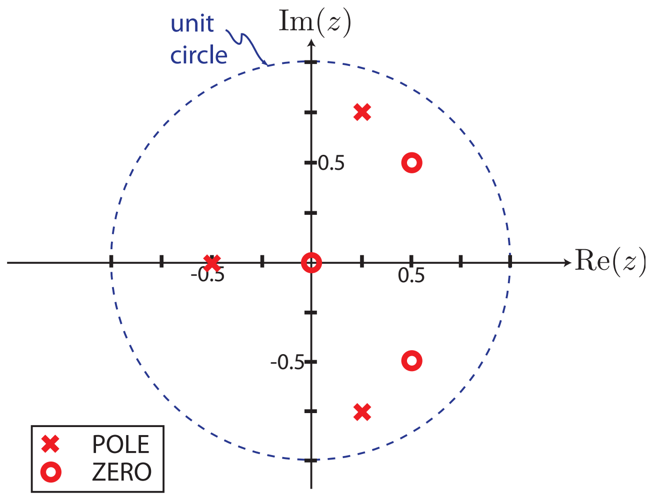
$$
\begin{aligned}
X(z) &= z \frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} \\
&= (z - 0) \frac{(z - (\frac{1}{2} + j\frac{1}{2}))(z - (\frac{1}{2} - j\frac{1}{2}))}{(z - (\frac{1}{4} + j\frac{3}{4}))(z - (\frac{1}{4} - j\frac{3}{4}))(z - (-\frac{1}{2}))}
\end{aligned}
$$

poles: $z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}$

zeros: $z = 0, \frac{1}{2} \pm j\frac{1}{2}$

Pole-Zero Plot

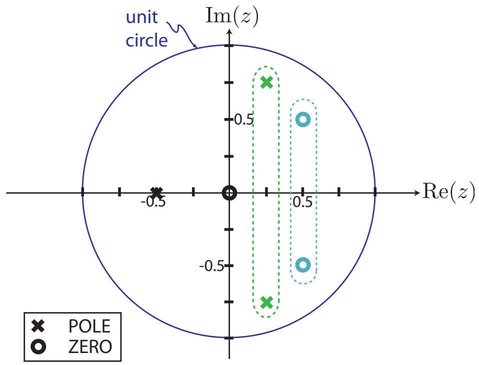
Example: poles: $z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}$, zeros: $z = 0, \frac{1}{2} \pm j\frac{1}{2}$



Pole-Zero Plot and Conjugate Pairs

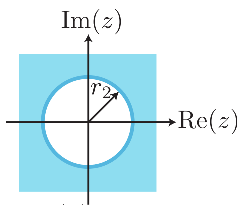
- For real time-domain signals, the coefficients of $X(z)$ are necessarily real
  - complex poles and zeros must occur in conjugate pairs
  - note: real poles and zeros do not have to be paired up

$$X(z) = z\frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} \Longrightarrow$$



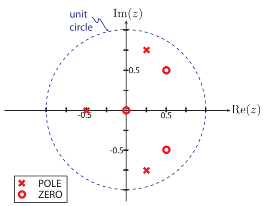
Pole-Zero Plot and the ROC

- Recall, for causal signals, the ROC will be the outer region of a disk



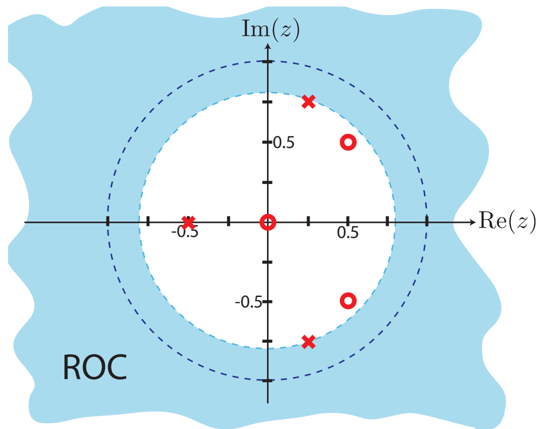
Causal: $x(n) = 0 \ \ n < 0$

- ROC cannot necessarily include poles ($\because X(p_k) = \infty$)



Pole-Zero Plot and the ROC

- Therefore, for a causal signal the ROC is the smallest (origin-centered) circle encompassing all the poles.



Causality and Stability

- Recall,

LTI system is stable $\iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty$

- Moreover,

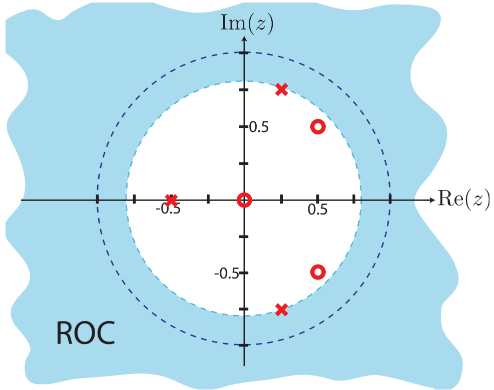
$$\begin{aligned} |H(z)| &= \left|\sum_{n=-\infty}^{\infty} h(n)z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |h(n)z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |h(n)| \quad \text{for } |z| = 1 \end{aligned}$$

- It can be shown:

LTI system is stable $\iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty \iff$ ROC of $H(z)$ contains unit circle

Pole-Zero Plot, Causality and Stability

- For stable systems, the ROC will include the unit circle.



- For stability of a causal system, the poles must lie inside the unit circle.

The System Function

$$\begin{aligned} h(n) &\overset{\mathcal{Z}}{\longleftrightarrow} H(z) \\ \text{time-domain} &\overset{\mathcal{Z}}{\longleftrightarrow} z\text{-domain} \\ \text{impulse response} &\overset{\mathcal{Z}}{\longleftrightarrow} \text{system function} \end{aligned}$$

$$y(n) = x(n) * h(n) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = X(z) \cdot H(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)}$$

The System Function of LCCDEs

$$\begin{aligned} y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \\ \mathcal{Z}\{y(n)\} &= \mathcal{Z}\{-\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)\} \\ \mathcal{Z}\{y(n)\} &= -\sum_{k=1}^{N} a_k \mathcal{Z}\{y(n-k)\} + \sum_{k=0}^{M} b_k \mathcal{Z}\{x(n-k)\} \\ Y(z) &= -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z) \end{aligned}$$

The System Function of LCCDEs

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]}$$

$$\text{LCCDE} \longleftrightarrow \text{Rational System Function}$$

Many signals of practical interest have a rational z-Transform.

Inversion of the z-Transform

Inversion of the z-Transform

Three popular methods:

1. Contour integration:
$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Expansion into a power series in z or z^{-1}:
$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$
and obtaining $x(k)$ for all k by inspection.

3. Partial-fraction expansion and table look-up.

Expansion into Power Series

Example:

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n}{n} z^{-n}$$

By inspection:

$$x(n) = \begin{cases} \frac{(-1)^{n+1} a^n}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Partial-Fraction Expansion

1. Find the distinct poles of $X(z)$: $p_1, p_2, \ldots, p_K$ and their corresponding multiplicities $m_1, m_2, \ldots, m_K$.

2. The partial-fraction expansion is of the form:

$$z^{-R}X(z) = \sum_{k=1}^{K} \left(\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z-p_k)^2} + \cdots + \frac{A_{mk}}{(z-p_k)^{m_k}} \right)$$

where p_k is an m_kth order pole (i.e., has multiplicity m_k) and R is selected to make $z^{-R}X(z)$ a strictly proper rational function.

3. Use an appropriate approach to compute $\{A_{ik}\}$

Partial-Fraction Expansion

Example: Find $x(n)$ given poles of $X(z)$ at $p_1 = -2$ and a double pole at $p_2 = p_3 = 1$; specifically,

$$\begin{aligned} X(z) &= \frac{1}{(1+2z^{-1})(1-z^{-1})^2} \\ \frac{X(z)}{z} &= \frac{z^2}{(z+2)(z-1)^2} \\ \frac{z^2}{(z+2)(z-1)^2} &= \frac{A_1}{z+2} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \end{aligned}$$

Note: we need a strictly proper rational function.
DO NOT FORGET TO MULTIPLY BY z IN THE END.

Partial-Fraction Expansion

$$\begin{aligned} \frac{z^2(z+2)}{(z+2)(z-1)^2} &= \frac{A_1(z+2)}{z+2} + \frac{A_2(z+2)}{z-1} + \frac{A_3(z+2)}{(z-1)^2} \\ \frac{z^2}{(z-1)^2} &= A_1 + \frac{A_2(z+2)}{z-1} + \left.\frac{A_3(z+2)}{(z-1)^2}\right|_{z=-2} \\ A_1 &= \frac{4}{9} \end{aligned}$$

Partial-Fraction Expansion

$$\begin{aligned} \frac{z^2(z-1)^2}{(z+2)(z-1)^2} &= \frac{A_1(z-1)^2}{z+2} + \frac{A_2(z-1)^2}{z-1} + \frac{A_3(z-1)^2}{(z-1)^2} \\ \frac{z^2}{(z+2)} &= \frac{A_1(z-1)^2}{z+2} + A_2(z-1) + \left. A_3 \right|_{z=1} \\ A_3 &= \frac{1}{3} \end{aligned}$$

Partial-Fraction Expansion

$$
\begin{aligned}
\frac{z^2(z-1)^2}{(z+2)(z-1)^2} &= \frac{A_1(z-1)^2}{z+2} + \frac{A_2(z-1)^2}{z-1} + \frac{A_3(z-1)^2}{(z-1)^2} \\
\frac{z^2}{(z+2)} &= \frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3 \\
\frac{d}{dz}\left[\frac{z^2}{(z+2)}\right] &= \frac{d}{dz}\left[\frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3\right]\Bigg|_{z=1} \\
A_2 &= \frac{5}{9}
\end{aligned}
$$

Partial-Fraction Expansion

Therefore, assuming causality, and using the following pairs:

$$
\begin{aligned}
a^n u(n) &\stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \\
na^n u(n) &\stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}
\end{aligned}
$$

$$
\begin{aligned}
X(z) &= \frac{4}{9}\frac{1}{1+2z^{-1}} + \frac{5}{9}\frac{1}{1-z^{-1}} + \frac{1}{3}\frac{z^{-1}}{(1-z^{-1})^2} \\
x(n) &= \frac{4}{9}(-2)^n u(n) + \frac{5}{9}u(n) + \frac{1}{3}nu(n) \\
&= \left[\frac{(-2)^{n+2}}{9} + \frac{5}{9} + \frac{n}{3}\right] u(n)
\end{aligned}
$$

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