# Stirling's Formula

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In combinatorics, probability, information theory, and elsewhere, one often needs to have estimates of, or bounds on, the factorial function

$$n! = \prod_{i=1}^{n} i,$$

particularly for large $n$. When dealing with $n!$ it always occurs that it may be easier to take logarithms and deal with $\ln(n!) = \sum_{i=1}^{n} \ln(i)$ instead. Since $\ln(x)$ is an increasing function of $x > 0$, we have, for any $i \geq 1$,

$$\int_{i-1}^{i} \ln(x)\mathrm{d}x < \ln(i) < \int_{i}^{i+1} \ln(x)\mathrm{d}x.$$

Adding these inequalities with $i = 1, 2, \ldots, n$, we get

$$\int_{0}^{n} \ln(x)\mathrm{d}x < \ln(n!) < \int_{1}^{n+1} \ln(x)\mathrm{d}x,$$

Since, for $0 < a \leq b$, we have

$$\int_{a}^{b} \ln(x)\mathrm{d}x = \left. (x\ln(x) - x)\right|_{a}^{b} = b\ln b - b - a\ln a + a$$

and (remembering that $\lim_{a\to 0} a\ln a = 0$) we get

$$n\ln n - n < \ln(n!) < (n+1)\ln(n+1) - (n+1) + 1$$

or

$$\left(\frac{n}{e}\right)^{n} < n! < e\left(\frac{n+1}{e}\right)^{n+1}$$

Thus $n!$ grows more quickly than $(n/e)^n$, but not as quickly as $e((n+1)/e)^{n+1}$, i.e., $n!$ lies somewhere "in between".

This betweenness is captured in Stirling's Formula, which gives

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} = \sqrt{2\pi} n^{n+1/2} e^{-n},$$

where $\sim$ means that the ratio of the two sides approaches unity in the limit as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1. \tag{1}$$

True to Stigler's Law,[1] Stirling's formula was first discovered by Abraham de Moivre, who understood that

$$n! \sim K(n/e)^{n+1/2}$$

for some constant $K$. It was Stirling who first realized that $K = \sqrt{2\pi e}$, and both Stirling and de Moivre published proofs in 1730; see [1, Ch. 24] for a history.

In this brief note, inspired by [2], we equip ourselves with (1), and seek an *exact* expression for $n!$ in the form

$$n! = C(n/A)^{n+B} \prod_{i=n}^{\infty} f(i) \tag{2}$$

for some constants $A$, $B$, and $C$ (to be determined), and a suitably-defined function $f$. The infinite product expresses an $n$-dependent correction factor that provides the precise adjustment that must be made to $C(n/A)^{n+B}$ to get $n!$.

The convenience of (2) becomes evident when one considers the ratio $(n+1)!/n! = n+1$. We then get

$$n+1 = \frac{((n+1)/A)^{n+1+B}}{(n/A)^{n+B} f(n)}$$

from which we determine that

$$f(n) = \frac{1}{A}\left(1+\frac{1}{n}\right)^{n+B}.$$

Plugging into (2), the right-hand side becomes

$$g(n) = C(n/A)^{n+B} \prod_{i=n}^{\infty} \left(\frac{(1+\frac{1}{i})^{i+B}}{A}\right).$$

Of course to be useful, we need that the infinite product converges to a positive constant, which can only happen if

$$\lim_{i\to\infty} \frac{1}{A}\left(1+\frac{1}{i}\right)^{i+B} = 1$$

The limit is easily computed as $e/A$ (independent of $B$). Thus we find that the only possible choice for the constant $A$ is $A = e$, and we get that

$$g(n) = C(n/e)^{n+B} \prod_{i=n}^{\infty} \left(\frac{(1+\frac{1}{i})^{i+B}}{e}\right).$$

[1] Stigler's Law of Eponymy: "no scientific discovery is named after its original discoverer." This law is attributed to R. K. Merton.

Note that $g(n)$ now has the property that $g(n+1) = (n+1)g(n)$. Provided that we can choose constants $B$ and $C$ so that $g(1) = 1$, then we will have the desired equality. Thus we now require

$$1 = \lim_{m\to\infty} C(1/e)^{1+B} \prod_{i=1}^{m} \left( \frac{(1+\frac{1}{i})^{i+B}}{e} \right). \tag{3}$$

Let

$$h(m) = \sum_{i=1}^{m} \left( (i+B)\ln\left(\frac{i+1}{i}\right) - 1 \right),$$

which is the logarithm of the product in (3). We require $h(m)$ to approach a finite limit as $m \to \infty$. Expanding $h(m)$, we get

$$\begin{aligned}
h(m) =& - m + B\sum_{i=1}^{m} (-\ln(i) + \ln(i+1)) + \sum_{i=1}^{m} i\,(-\ln(i) + \ln(i+1)) \\
=& - m + B\,(-\ln(1) + \ln(2) - \ln(2) + \ln(3) + \cdots - \ln(m) + \ln(m+1)) \\
& + (-\ln(1) + \ln(2) - 2\ln(2) + 2\ln(3) - 3\ln(3) + 3\ln(4) + \cdots - m\ln(m) + m\ln(m+1)) \\
=& - m + B\ln(m+1) - \ln(m!) + m\ln(m+1) \\
=&(m+B)\ln(m+1) - m - \ln(m!).
\end{aligned}$$

Now, using the fact from (1) that $\ln(m!) \to \frac{1}{2}\ln m + \frac{1}{2}\ln(2\pi) + m\ln(m) - m$ as $m \to \infty$, we get that

$$\begin{aligned}
\lim_{m\to\infty} h(m) =& \lim_{m\to\infty} \left( -m + B\ln(m+1) + m\ln(m+1) - \frac{1}{2}\ln m - \frac{1}{2}\ln(2\pi) - m\ln(m) + m \right) \\
=& -\frac{1}{2}\ln(2\pi) + \lim_{m\to\infty} \left( (m+B)\ln(m+1) - \left(m+\frac{1}{2}\right)\ln(m) \right) \\
=& -\frac{1}{2}\ln(2\pi) + \lim_{m\to\infty} \left( \left(m+\frac{1}{2}\right)\ln\left(\frac{m+1}{m}\right) + \left(B-\frac{1}{2}\right)\ln(m+1) \right) \\
=&1 - \frac{1}{2}\ln(2\pi) + \lim_{m\to\infty} \left(B-\frac{1}{2}\right)\ln(m+1) \\
=& \begin{cases} \infty, & \text{if } B > 1/2; \\ 1 - \frac{1}{2}\ln(2\pi), & \text{if } B = 1/2; \\ -\infty, & \text{if } B < 1/2. \end{cases}
\end{aligned}$$

Thus we see that we have no choice but to set $B = 1/2$, and, for this choice of $B$ we can determine the value of $C$ to arrive at Mermin's equality [2]

$$n! = (2\pi n)^{1/2} \left(\frac{n}{e}\right)^{n} \prod_{i=n}^{\infty} \left( \frac{(1+1/i)^{i+1/2}}{e} \right). \tag{4}$$

Now to find bounds on $n!$, one approach is to simply find bounds on

$$\prod_{i=n}^{\infty} \left( \frac{(1+1/i)^{i+1/2}}{e} \right).$$

Since $\frac{(1+1/i)^{i+1/2}}{e} > 1$ for all $i \geq 1$, an obvious lower bound is

$$n! > (2\pi n)^{1/2} \left(\frac{n}{e}\right)^n$$

An upper bound can be obtained from the Taylor series expansion for $\frac{1}{2}\ln((1+x)/(1-x))$, namely

$$\frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Substituting $x = 1/(2i+1)$, we get

$$\frac{1}{2} \ln \left( \frac{1+i}{i} \right) = \frac{1}{2i+1} + \frac{1}{3(2i+1)^3} + \frac{1}{5(2i+1)^5} + \cdots,$$

from which it follows that

$$\left( \frac{2i+1}{2} \right) \ln \left( \frac{1+i}{i} \right) = 1 + \frac{1}{3(2i+1)^2} + \frac{1}{5(2i+1)^4} + \cdots.$$

Subtracting one from both sides we get the upper bound

$$\begin{aligned} \left( \frac{2i+1}{2} \right) \ln \left( \frac{1+i}{i} \right) - 1 &< \frac{1}{3} \left( \frac{1}{(2i+1)^2} + \frac{1}{(2i+1)^4} + \cdots \right) \\ &= \frac{1}{12} \left( \frac{1}{i} - \frac{1}{i+1} \right). \end{aligned}$$

Exponentiating and substituting into (4) we get a telescoping exponent that yields the bound

$$n! < (2\pi n)^{1/2} \left(\frac{n}{e}\right)^n e^{1/(12n)}.$$

More elaborate bounds are possible. Following the same Taylor series approach, we see that

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \prod_{i=n}^{\infty} \exp \left( \frac{1}{3(2i+1)^2} + \frac{1}{5(2i+1)^4} + \cdots \right)$$

# References


[1] Anders Hald, *A History of Probability and Statistics and their Applications Before 1750.* New York: Wiley, 1990.

[2] N. David Mermin, "Stirling's formula!," Am. J. Phys., vol. 52, pp. 362–365, April 1984. Reprinted as Ch. 23 of N. David Mermin, *Boojums All the Way Through: Communicating Science in a Prosaic Age.* New York: Cambridge University Press, 1990.