The Wiener-Khinchin Theorem

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For any signal $x(t)$, let

$$x_T(t) = x(t)\,\mathrm{rect}\left(\frac{t}{T}\right)$$

denote a "time-windowed" projection of $x(t)$ taking value zero outside of the interval $[-T/2, T/2)$, where $T > 0$. Assume, for each T, that the Fourier transform of $x_T(t)$ exists, and is given by $X_T(f)$. To a power signal x we may associate the *power spectral density* given by

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2,$$

measured in units of W/Hz. Roughly speaking, $S_x(f)$ measures the contribution to the power of x made by complex-exponential signal components at frequency f. The total power associated with x is then given by

$$P_x = \int_{-\infty}^{\infty} S_x(f)\,\mathrm{d}f.$$

If x is passed through an LTI system with frequency response $H(f)$, then the power spectral density of the output y is given by $S_y(f) = S_x(f)|H(f)|^2$, having total power

$$P_y = \int_{-\infty}^{\infty} S_x(f)|H(f)|^2\,\mathrm{d}f.$$

In particular, note that if $H(f)$ is a very narrow bandpass filter centered at some frequency f_0, then the power of the output is approximately proportional to $S_x(f_0)$; thus, the function $S_x(f)$ does indeed serve as a density function for power.

We would like to extend this notion of power spectral density to wide-sense stationary random processes. For any fixed power signal x, at any given frequency f, observe that $S_x(f)$ is some fixed non-negative real value that depends on x. For a wide-sense stationary random process X having power signals as sample functions, it makes sense to define the power spectral density via the expected value of that real value, i.e., to define

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} E[|X_T(f)|^2], \tag{1}$$

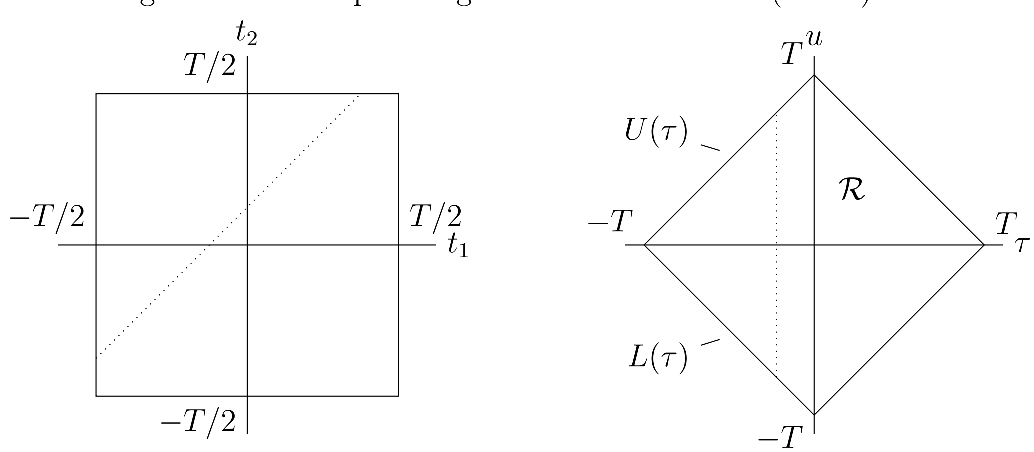
whenever the limit exists. We take (1) as the definition of the power spectral density.

Suppose now that the (complex-valued) random process has autocorrelation function $R_X(\tau) = E[X(t)X^*(t-\tau)]$, and that the Fourier transform of $R_X(\tau)$ exists and is denoted $\hat{R}_X(f)$. The Wiener-Khinchin theorem states that, under mild conditions, $S_X(f) = \hat{R}_X(f)$, i.e., that the power spectral density associated with a wide-sense stationary random process is equal to the Fourier transform of the autocorrelation function associated with that process.

To see this, we follow [1, Sec. 11.2], and note that according to our definition (1), we have

$$\begin{aligned}
S_X(f) &= \lim_{T\to\infty} \frac{1}{T} E[|X_T(f)|^2] \\
&= \lim_{T\to\infty} \frac{1}{T} E\left[\int_{-T/2}^{T/2} X(t_1)e^{-\mathrm{j}\,2\pi f t_1}\,\mathrm{d}t_1 \int_{-T/2}^{T/2} X^*(t_2)e^{\mathrm{j}\,2\pi f t_2}\,\mathrm{d}t_2\right] \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} E[X(t_1)X^*(t_2)]e^{-\mathrm{j}\,2\pi f(t_1-t_2)}\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_X(t_1-t_2)e^{-\mathrm{j}\,2\pi f(t_1-t_2)}\,\mathrm{d}t_2\,\mathrm{d}t_1.
\end{aligned}$$

The region of integration is the square region shown on the left (below).



Note however that the integrand is constant along contours where $t_1 - t_2$ is a constant (e.g., along the dotted line shown). This motivates us to apply a change of variables, defining $\tau = t_1 - t_2$ and $u = t_1 + t_2$. Under this change of variables, after noting that

$$\begin{bmatrix} \tau \\ u \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix},$$

so that we have a Jacobian matrix of determinant 2, we get

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} \int_{\mathcal{R}} R_X(\tau) e^{-\mathrm{j}\,2\pi f\tau} \frac{\mathrm{d}u\,\mathrm{d}\tau}{2},$$

where $\mathcal{R}$ is the rotated-and-scaled region shown on the right (in the figure above). Note that other changes of variables are possible; for example one might take $\tau = t_1 - t_2$, $u = t_1$. The region of integration will be different, but the final answer will be the same.

For each value of τ, let us denote the value of u along lower boundary of the region $\mathcal{R}$ as $L(\tau)$, and let us denote the value of u along the upper boundary as $U(\tau)$, as indicated in the figure. This gives us

$$\begin{aligned} S_X(f) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{L(\tau)}^{U(\tau)} R_X(\tau) e^{-\mathrm{j}\,2\pi f\tau}\,\mathrm{d}u\,\mathrm{d}\tau \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_X(\tau) e^{-\mathrm{j}\,2\pi f\tau} \left(\int_{L(\tau)}^{U(\tau)} \mathrm{d}u \right) \mathrm{d}\tau \\ &\overset{(a)}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_X(\tau) e^{-\mathrm{j}\,2\pi f\tau} (2T - 2|\tau|)\,\mathrm{d}\tau \\ &= \lim_{T\to\infty} \int_{-T}^{T} R_X(\tau) \left(1 - \frac{|\tau|}{T}\right) e^{-\mathrm{j}\,2\pi f\tau}\,\mathrm{d}\tau, \end{aligned} \tag{2}$$

where (a) follows from the fact that, when $|\tau| \le T$, we have $U(\tau) - L(\tau) = 2T - 2|\tau|$.

We recognize that the integral in (2) is simply taking the Fourier transform of the product of $R_X(\tau)$ and the triangle function $\frac{1}{T}\operatorname{rect}(\tau/T) \star \operatorname{rect}(\tau/T)$. Applying the modulation theorem (i.e., the Fourier transform of a product is the convolution of the Fourier transforms), we get

$$\begin{aligned} S_X(f) &= \lim_{T\to\infty} \hat{R}_X(f) \star T\operatorname{sinc}^2(fT) \\ &= \lim_{T\to\infty} \int_{-\infty}^{\infty} \hat{R}_X(f - x) T \operatorname{sinc}^2(xT)\,\mathrm{d}x. \end{aligned}$$

For any fixed value of f, observe that the convolution is carrying out a "weighted average" of the values of $\hat{R}_X$, with values near f (corresponding to small $|x|$) given larger weight and with values farther from f (corresponding to large $|x|$) given smaller weight. Indeed, the function $T\operatorname{sinc}^2(fT)$ has unit area, is everywhere non-negative, and, for $f \neq 0$, converges to zero as $T \to \infty$. In other words, $T\operatorname{sinc}^2(fT)$ seems to have, when T is large, the properties of a Dirac delta. Thus, we are tempted to write

$$S_X(f) = \lim_{T\to\infty} \hat{R}_X(f) \star T\operatorname{sinc}^2(fT) = \hat{R}_X(f) \star \delta(f) = \hat{R}_X(f).$$

To justify this, let us abstract the situation. Suppose that we have a family of real-valued functions $s_T(x)$, parameterized by a positive real number T, with the properties that

1. $s_T(x) \geq 0$ for all x and all $T > 0$,
2. $\int_{-\infty}^{\infty} s_T(x)\,\mathrm{d}x = 1$ for all $T > 0$, and
3. for every $\epsilon > 0$ and for every $\delta > 0$, we can find a $T_0 > 0$ such that $\int_{-\delta}^{\delta} s_T(x)\,\mathrm{d}x \geq 1 - \epsilon$ for every $T \geq T_0$.

Such a family is called an "approximate identity under convolution." It is not hard to show that $\{T \operatorname{sinc}^2(xT) : T > 0\}$ is indeed such a family.

Lemma 1. *Let $g(x)$ be a given real-valued function that is continuous at $x = 0$, and is bounded by some $B \geq 1$, i.e., for all x, $|g(x)| \leq B$. Let $\{s_T(x) : T > 0\}$ be an approximate identity under convolution, and furthermore suppose that*

$$\int_{-\infty}^{\infty} g(x) s_T(x)\, dx$$

exists for all $T > 0$. We then have that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} g(x) s_T(x)\, dx = g(0). \tag{3}$$

Proof. We need to show that for every $\epsilon > 0$ there exists $T_0 > 0$ such that $T \geq T_0$ implies

$$\left| \int_{-\infty}^{\infty} g(x) s_T(x)\,\mathrm{d}x - g(0) \right| \leq \epsilon.$$

Thus, let us fix $\epsilon > 0$. As a first step, let us choose a value $\delta > 0$ so that whenever $|x| \leq \delta$, we have that $|g(x) - g(0)| \leq \epsilon/3$; since $g(x)$ is continuous at $x = 0$, such a δ certainly exists. Next, let us choose T_0 such that for all $T \geq T_0$ we have $\int_{-\delta}^{\delta} s_T(x)\,\mathrm{d}x \geq 1 - \epsilon/(3B)$; by the third property of approximate identities, such a T_0 certainly exists. Now, let us write

$$\int_{-\infty}^{\infty} g(x)s_T(x)\,\mathrm{d}x = \int_{-\delta}^{\delta} g(x)s_T(x)\,\mathrm{d}x + \int_{-\infty}^{-\delta} g(x)s_T(x)\,\mathrm{d}x + \int_{\delta}^{\infty} g(x)s_T(x)\,\mathrm{d}x$$

and bound the contributions made to the left-hand integral by the terms on the right-hand side. With the choices for δ and T_0 just made, we observe that

$$\int_{-\delta}^{\delta} g(x)s_T(x)\,\mathrm{d}x \leq \int_{-\delta}^{\delta} (g(0) + \epsilon/3)s_T(x)\,\mathrm{d}x \leq \int_{-\infty}^{\infty} (g(0) + \epsilon/3)s_T(x)\,\mathrm{d}x = g(0) + \epsilon/3.$$

Furthermore

$$\begin{aligned}\int_{-\delta}^{\delta} g(x)s_T(x)\,\mathrm{d}x &\geq \int_{-\delta}^{\delta} (g(0) - \epsilon/3)s_T(x)\,\mathrm{d}x \geq (g(0) - \epsilon/3)(1 - \epsilon/(3B)) \\ &\overset{(a)}{\geq} (g(0) - \epsilon/3)(1 - \epsilon/3) \geq g(0) - 2\epsilon/3,\end{aligned}$$

where inequality (a) follows from the fact that $B \geq 1$. Next let

$$I_2 = \int_{-\infty}^{-\delta} g(x) s_T(x)\,\mathrm{d}x + \int_{\delta}^{\infty} g(x) s_T(x)\,\mathrm{d}x,$$

and observe that

$$|I_2| \leq \int_{-\infty}^{-\delta} B s_T(x)\,\mathrm{d}x + \int_{\delta}^{\infty} B s_T(x)\,\mathrm{d}x \leq B\epsilon/(3B) = \epsilon/3.$$

In sum, we see that for all $T \geq T_0$ we have

$$g(0) - \epsilon \leq \int_{-\infty}^{\infty} g(x) s_T(x)\,\mathrm{d}x \leq g(0) + 2\epsilon/3 \leq g(0) + \epsilon,$$

which is what we set out to show. □

Observe that Lemma 1 extends to complex-valued functions $g(x)$ that are bounded in magnitude and continuous at $x = 0$, as the real and imaginary parts of the integrand can be treated separately.

Note that, even though $s_T(t)$ does not approach a well-defined *function* as $T \to \infty$, the limit in (3) nevertheless converges to a well-defined linear *functional*: namely, the one that maps $g(x)$ to its value $g(0)$ at $x = 0$. This is the essence of the definition of the Dirac delta as a distribution; see, e.g., [2].

Now suppose that $R_X(\tau)$ is absolutely integrable, i.e., that $B = \int_{-\infty}^{\infty} |R_X(\tau)|\,\mathrm{d}\tau$ exists. We then have, for any f, that

$$|\hat{R}_X(f)| = \left| \int_{-\infty}^{\infty} R_X(\tau) e^{-\mathrm{j}\,2\pi f \tau}\,\mathrm{d}\tau \right| \leq \int_{-\infty}^{\infty} |R_X(\tau) e^{-\mathrm{j}\,2\pi f \tau}|\,\mathrm{d}\tau = \int_{-\infty}^{\infty} |R_X(\tau)|\,\mathrm{d}\tau = B.$$

Thus, if $R_X(\tau)$ is absolutely integrable, its Fourier transform $\hat{R}_X(f)$ (if it exists) is bounded.

Now let us put the pieces together. First, we already have that

$$S_X(f) = \lim_{T\to\infty} \int_{-\infty}^{\infty} \hat{R}_X(f - x) T \operatorname{sinc}^2(xT)\,\mathrm{d}x$$

and we know that $\{T \operatorname{sinc}^2 xT) : T > 0\}$ is an approximate identity for convolution. Furthermore, assuming that $R_X(\tau)$ is absolutely integrable implies that $\hat{R}_X(f)$ is bounded. Thus, Lemma 1 applies, and we get that

$$S_X(f) = \hat{R}_X(f)$$

at all points f where $\hat{R}_X(f)$ is continuous. This is the Wiener-Khinchin theorem.

Acknowledgment

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References

[1] B. P. Lathi, *Modern Digital and Analog Communication Systems*, 3rd Edition. Oxford University Press, 1998.

[2] R. S. Strichartz, *A Guide to Distribution Theory and Fourier Transforms*. CRC Press, 1994.