A $(\min, \times)$ Network Calculus for Multi-Hop Fading Channels

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***Abstract*—A fundamental problem for the delay and backlog analysis across multi-hop paths in wireless networks is how to account for the random properties of the wireless channel. Since the usual statistical models for radio signals in a propagation environment do not lend themselves easily to a description of the available service rate, the performance analysis of wireless networks has resorted to higher-layer abstractions, e.g., using Markov chain models. In this work, we propose a network calculus that can incorporate common statistical models of fading channels and obtain statistical bounds on delay and backlog across multiple nodes. We conduct the analysis in a transfer domain, which we refer to as the *SNR domain*, where the service process at a link is characterized by the instantaneous signal-to-noise ratio at the receiver. We discover that, in the transfer domain, the network model is governed by a dioid algebra, which we refer to as $(\min, \times)$ algebra. Using this algebra we derive the desired delay and backlog bounds. An application of the analysis is demonstrated for a simple multi-hop network with Rayleigh fading channels.**

I. Introduction

Network-layer performance analysis seeks to provide estimates on the delays experienced by traffic traversing the elements of a network, as well as the corresponding buffer requirements. For wireless networks, a question of interest is how the stochastic properties of wireless channels impact delay and backlog performance. Wireless channels are characterized by rapid variations of the channel quality caused by the mobility and location of communicating devices. This is due to *fading*, which is the deviation in the attenuation experienced by the transmitted signal when traversing a wireless channel. The term *fading channel* is used to refer to a channel that experiences such effects. In this paper we explore the network-layer performance of a multi-hop network where each link is represented by a fading channel.

We model the wireless network by tandem queues with variable capacity servers, where each server expresses the random capacity of a fading channel. We ignore the impact of coding by assuming that transmission rates over the fading channels are equal to their information-theoretic capacity limit, C, which is expressed as a function of the instantaneous signal-to-noise ratio (SNR) at the receiver, γ, by $C(\gamma) = W \log(1+\gamma)$, where W is the channel bandwidth (in *Hz*). Numerous models are available to describe the gain of fading channels depending on the type of fading (slow or fast), and the environment (e.g., urban or rural). The instantaneous, information-theoretic channel capacity of a fading channel can be represented as the logarithm of γ by (see Chp. 14.2 in [20])

$$C(\gamma) = c \log\big(g(\gamma)\big)\,, \tag{1}$$

where c is a constant and the function $g(\gamma)$ is used to characterize the fading channel. We are interested in finding bounds on the end-to-end delay and on buffer requirements for a cascade of fading channels, with store-and-forward processing at each channel.

The analysis in this paper takes a system-theoretic stochastic network calculus approach [15], which describes the network properties using a $(\min, +)$ dioid algebra. Arrival and departure processes at a network element are described by bivariate stochastic processes $A(\tau, t)$ and $D(\tau, t)$, respectively, denoting the cumulative arrivals and departures in the time interval $[\tau, t)$. A network element is characterized by the service process $S(\tau, t)$, denoting the available service in $[\tau, t)$. The input-output relationship at the network element is governed by

$$D(0, t) \geq A * S(0, t)\,, \tag{2}$$

where the $(\min, +)$ convolution operator '$*$' is defined as $f * g(\tau, t) = \inf_{\tau \leq u \leq t}\{f(\tau, u) + g(u, t)\}$. If network traffic passes through a tandem of N network elements with service processes $S_1, S_2, \ldots, S_N$, the service of the network as a whole can be expressed by the convolution $S_1 * S_2 * \ldots * S_N$.

Methodologies for network-layer performance analysis of wireless networks with fading channels include queueing theory [7], [13], [16], effective bandwidth [12], [18], [24]–[26] and, more recently, network calculus [5], [9], [10], [14], [19], [23]. A detailed discussion of these works is given in [1]. The stochastic properties of fading channels present a formidable challenge for a network-layer analysis since the service processes corresponding to the channel capacity of common fading channel models such as Rician, Rayleigh, or Nakagami-m, require to take a logarithm of their distributions. To avoid this complexity, researchers often turn to higher-layer abstractions to model fading channels. Widely used abstraction are the two-state Gilbert-Elliott model and its extensions to finite-state Markov channels (FSMC) [21]. FSMC models simplify the analysis to a degree that the network model becomes tractable, at least at a single node. Extensions to multi-hop settings encounter a rapidly growing state space. As of today, no general multihop analysis is available for fading channel models, such as Rician, Rayleigh, or Nakagami-m. We note that there is also a literature on physical-layer performance metrics of fading in multi-hop wireless relay network with independent, non-regenerative relays, so-called amplify-and-forward networks, e.g., [2], [11], [22]. Different from the store-and-forward architecture considered in a network-layer analysis, such networks do not involve buffers at intermediate nodes.

In this paper, we pursue a novel approach to the analysis of multi-hop wireless networks. We develop a calculus for wireless networks that can be applied to fading channel models from the wireless communication literature to provide network-layer performance bounds. We view the network-layer model with arrival, departure and service processes as residing in a *bit domain*, where traffic is measured in bits, and service availability is measured in bits per second. We view the fading channel models used in wireless communications as residing in an alternate domain, which we call the *SNR domain*, where channel properties are expressed in terms of the distribution of the signal-to-noise ratio at the receiver. We then derive a method to compute performance bounds from these traffic and service characterizations.

A key observation in our work is that service elements in the SNR domain obey the laws of a dioid algebra. We devise a suitable dioid, referred to as $(\min, \times)$ *algebra*, where the minimum takes the role of the standard addition, and the second operation is the usual multiplication, and use it for analysis in the SNR domain. In this domain multi-hop descriptions of fading channels become tractable. In particular, we find that a cascade of fading channels can be expressed in terms of a convolution in the new algebra of the constituting channels. The key to our analysis is that we derive performance bounds entirely in the SNR domain. Observing that the bit and SNR domains are linked by the exponential function, we transfer arrival and departure processes from the bit to the SNR domain. Then, we derive backlog and delay bounds in the transfer domain using the $(\min, \times)$ algebra. The results are mapped back to the original bit domain to finally give us the desired performance bounds. Our derivations in the SNR domain require the computation of products and quotients of random variables. Here, we take advantage of the *Mellin transform* to facilitate otherwise cumbersome calculations. Then, the computational problem is reduced to finding the Mellin transform for service and traffic processes.

The main contribution of this paper is the development of a framework for studying the impact of channel gain models on the network-layer performance of wireless networks. For the purposes of this paper, the SNR domain is used solely as a transfer domain that enables us to solve an otherwise intractable mathematical problem. On the other hand, the ability to map quantities that appear in network-layer models and concepts found in a physical-layer analysis may prove useful in a broader context, e.g., for studying cross-layer performance issues in wireless communications. Moreover, the $(\min, \times)$ algebra and the Mellin transform form a tool set that can be applied more generally in wireless communications for studying the channel gain of cascades of fading channels. As the first paper on the $(\min, \times)$ network calculus algebra, our paper only considers simple network scenarios and makes numerous convenient assumptions (which are made explicit in Sec. II). There is room for significant future work on extensions of the model and a relaxation of the presented assumptions.

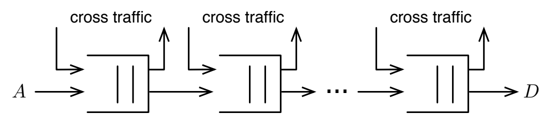

Fig. 1. Tandem network model.

The remainder of the paper is organized as follows. We describe the system model in Sec. II, where we also motivate the use of the SNR domain. In Sec. III we present the $(\min, \times)$ algebra and derive performance bounds. In Sec. IV we apply the analysis to a cascade of Rayleigh channels, and present numerical examples. We discuss brief conclusions in Sec. V.

II. Network Model in the Bit and SNR Domains

We consider a wireless N-node tandem network as shown in Fig. 1, where each node is modeled by a server with an infinite buffer. We are interested in the performance experienced by a (through) flow that traverses the entire network and may encounter cross traffic at each node. (Page limits prevent a discussion of cross-traffic in this paper. We refer to [1] for the analysis of a network with cross traffic.) One can think of the cross-traffic at a node as the aggregate of all traffic traversing the node that does not belong to the through flow. The service given to the through flow at a node is a random process, which is governed by the instantaneous channel capacity as well as the cross traffic at the node. We consider a fluid-flow traffic model where the flow is infinitely divisible. We will work in a discrete-time domain $\mathcal{T} = \{t_i : t_i = i\ \Delta t, i \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers and Δt is length of the time unit. Setting $\Delta t = 1$ allows us to replace t_i by i, which we interpret as the index of a time slot. We assume that the system is started with empty queues at time $t = 0$.

Different nodes and different traffic flows will be distinguished by subscripts. The cumulative arrivals to, the service offered by, and the departures from the node are represented by random processes A_n, S_n, and D_n that will be described more precisely below, with $A_n = D_{n-1}$ for $n = 1, \ldots, N-1$. We denote by $A = A_1$ and $D = D_N$, respectively, the arrivals to and the departures from the tandem network. Throughout, we assume that arrival and service processes satisfy stationary bounds.

A. Traffic and Service in the Bit Domain

Consider for the moment a single node. Dropping subscripts, we write

$$A(\tau, t) = \sum_{i=\tau}^{t-1} a_i \,, \quad \text{and} \quad D(\tau, t) = \sum_{i=\tau}^{t-1} d_i \,,$$

for the cumulative arrivals and departures, respectively, at the node in the time interval $[\tau, t)$, where a_i denotes the arrivals and d_i the departures in the i-th time slot. Due to causality, we have $D(0, t) \leq A(0, t)$. The processes lie in the set $\mathcal{F}$ of bivariate functions $f(\tau, t)$ that are increasing in their second argument and satisfy $f(t, t) = 0$ for all t. The backlog at time $t > 0$ is given by

$$B(t) = A(0, t) - D(0, t) \,, \tag{3}$$

and the delay at the node is given by

$$W(t) = \inf \{u \geq 0 : A(0,t) \leq D(0,t+u)\} \ . \tag{4}$$

The service of the node in the time interval $[\tau, t)$ is given by a random process $S(\tau, t)$, such that Eq. (2) holds for every arrival process A and the corresponding departure process D. This service description with bivariate functions is referred to as *dynamic server*. Initially defined for non-random service [4], dynamic servers have been extended to random processes in [3], [8].

The above model is a typical network-layer model, where traffic and service are measured in bits. We thus refer to this model of arrivals, departures, and service as residing in a *bit domain*.

The network calculus exploits that networks that satisfy the input-output relation of Eq. (2) with equality can be viewed as linear systems in a $(\min,+)$ dioid algebra [17]. In the $(\mathbb{R}\cup\{+\infty\}, \min, +)$ dioid, the minimum and addition take the place of the standard addition and multiplication operations. The network calculus is based on the fact that $(\mathcal{F}, \min, *)$ is again a dioid [3]. Note that the min-plus convolution, which provides the second operation in the dioid, is not commutative in $\mathcal{F}$.

B. Service Model for Fading Channels

To compute a service model for a wireless channel, we assume that the channel state information is sampled at equal time intervals Δt. With $\Delta t = 1$, let γ_i denote the instantaneous signal-to-noise ratio observed at the receiver in the i-th sampling epoch. Then, γ_i is a nonnegative random variable that has the probability distribution of the underlying fading model. We assume that the random variables γ_i are independent and identically distributed. This assumption is justified when Δt is longer than the channel coherence time. Otherwise, the assumption will result in optimistic bounds. We emphasize that the network calculus in this paper applies to settings without independence, however, the derivation of performance bounds will proceed differently. Using Eq. (1), the instantaneous service offered by the channel in the i-th slot is given by $\log g(\gamma_i)$, and the corresponding service process is given by

$$S(\tau, t) = \sum_{i=\tau}^{t-1} \log g(\gamma_i)\,, \tag{5}$$

where we have chosen units such that the constant in Eq. (1) takes the value $c = 1$.

The service description in Eq. (5) requires us to work with the logarithm of fading distributions, which presents a non-trivial technical difficulty via the usual network calculus or queueing theory. On the other hand, observe that the exponential $\mathcal{S}(\tau, t) = e^{S(\tau,t)}$ is described more simply by

$$\mathcal{S}(\tau, t) = \prod_{i=\tau}^{t-1} g(\gamma_i) \ . \tag{6}$$

This motivates the development of a system model that allows us to exploit the more tractable service representation in Eq. (6). In this alternative model, arrivals, departures, and service reside in a different domain, where we can work directly with the distribution functions of the fading channel gain and the corresponding SNR at the receiver.

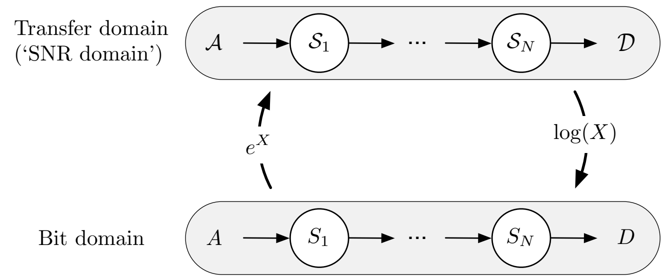

Fig. 2. Transfer Domain of Network Model.

C. Network Model in the SNR Domain

We now proceed by mapping the network model from Fig. 1 into a transfer domain, which we refer to as *SNR domain*. We will seek to derive performance bounds in the transfer domain, and then map the results to the bit domain to obtain network-layer bounds for backlog and delays. The relationship of the network models in bit domain and SNR domain is illustrated in Fig. 2.

In the previous subsection, we constructed the service process for a wireless link in the SNR domain in Eq. (6) as

$$\mathcal{S}(\tau, t) = e^{S(\tau,t)}\,.$$

By analogy, we describe the arrivals and departures in the SNR domain by

$$\mathcal{A}(\tau, t) \triangleq e^{A(\tau,t)} \quad \text{and} \quad \mathcal{D}(\tau, t) \triangleq e^{D(\tau,t)}\,.$$

Throughout this paper, we use calligraphic upper-case letters to represent processes that characterize traffic or service as a function of the instantaneous SNR in the sense of Eq. (6). Due to the monotonicity of the exponential function, $\mathcal{D}(0, t)$ and $\mathcal{A}(0, t)$ are increasing in t, and satisfy the causality property $\mathcal{D}(0, t) \leq \mathcal{A}(0, t)$. The backlog process is accordingly described by

$$\mathcal{B}(t) \triangleq e^{B(t)} = \mathcal{A}(t)/\mathcal{D}(t)\,.$$

Since time is not affected by this transformation, the delay is given by

$$\mathcal{W}(t) \triangleq W(t) = \inf\{u \geq 0 : \mathcal{A}(t) \leq \mathcal{D}(t+u)\}\,. \tag{7}$$

To interpret these processes in the transfer domain, let $\gamma_{a,i} \triangleq g^{-1}(e^{a_i})$ be the instantaneous channel SNR required to transmit a_i in a single time slot, assuming transmission at the rate of the capacity limit. The arrival process in the SNR domain can then be expressed in terms of these variables as

$$\mathcal{A}(\tau, t) = \prod_{i=\tau}^{t-1} g(\gamma_{a,i})\,. \tag{8}$$

Here, we are treating channel quality expressed in terms of the instantaneous SNR as a commodity. An arrival in a time unit represents a workload, where $\gamma_{a,i}$ expresses the amount of resources that will be consumed by the workload. The backlog can similarly be expressed in terms of the instantaneous SNR as

$$\mathcal{B}(t) = \prod_{i=t}^{t+\tau_B-1} g(\gamma_i) \ ,$$

with the interpretation that a node with backlog $B(t)$ at time t requires full use of the channel capacity for τ_B time units to clear the backlog.

Most importantly, the concept of the dynamic server translates to the SNR domain. In a network system, the service process in the bit domain satisfies Eq. (2) if and only if the process $\mathcal{S}(\tau,t) = e^{S(\tau,t)}$ in the SNR domain satisfies

$$\mathcal{D}(0,t) \geq \inf_{0\leq u\leq t} \{\mathcal{A}(0,u) \cdot \mathcal{S}(u,t)\} \ . \tag{9}$$

We refer to a network element that satisfies Eq. (9) for any sample path as *dynamic SNR server*. In this general setting, we do not require that $\mathcal{S}$ takes the form in Eq. (6), in particular, $\mathcal{S}(\tau,t) = \mathcal{S}(\tau,u) \cdot \mathcal{S}(u,t)$ does not need to hold.

Traffic aggregation in the SNR domain is expressed in terms of a product. When M flows have arrivals at a node with arrival processes denoted by $A_k, k = 1,\ldots,M$, then the total arrival, A_{agg}, are given for any $0 \leq \tau \leq t$ by

$$A_{\text{agg}}(\tau,t) = \sum_{k=1}^{M} A_k(\tau,t) \ .$$

If we let $\mathcal{A}_j$ and $\mathcal{A}_{\text{agg}}$ denote the corresponding processes in the SNR domain, we see that

$$\mathcal{A}_{\text{agg}}(\tau,t) = \prod_{k=1}^{M} \mathcal{A}_k(\tau,t) \ .$$

With the above definitions, the usual network description by a $(\min,+)$ dioid algebra in the bit domain can be expressed in the SNR domain by a dioid algebra on $\mathcal{F}$ where the second operator is a multiplication. This enables the development of the $(\min,\times)$ network calculus in Sec. III. We observe that the exponential function defines a one-to-one correspondence between arrival and departure processes in the bit and SNR domains. The physical arrival, departure, service, and backlog processes can be recovered from their counterparts in the SNR domain by taking a logarithm (see Fig. 2).

III. Stochastic $(\min,\times)$ Network Calculus

This section contains our main contribution: an analytical framework for statistical end-to-end performance bounds for a network, where service is expressed in terms of fading distributions residing in the SNR domain.

By an *SNR process* we mean a bivariate process $\mathcal{X}(\tau,t)$ taking values in $\mathbb{R}^+$ that is increasing in the second argument, with $\mathcal{X}(t,t) = 1$ for all t. The space of SNR processes will be denoted by $\mathcal{F}^+$. For any pair of SNR processes $\mathcal{X}(\tau,t)$ and $\mathcal{Y}(\tau,t)$, set

$$\mathcal{X} \otimes \mathcal{Y}(\tau,t) \stackrel{\triangle}{=} \inf_{\tau\leq u\leq t} \left\{\mathcal{X}(\tau,u) \cdot \mathcal{Y}(u,t)\right\} \ , \tag{10}$$

and

$$\mathcal{X} \oslash \mathcal{Y}(\tau,t) \stackrel{\triangle}{=} \sup_{u\leq \tau} \left\{\frac{\mathcal{X}(u,t)}{\mathcal{Y}(u,\tau)}\right\} . \tag{11}$$

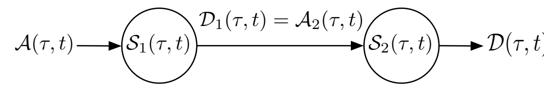

Fig. 3. Tandem of dynamic SNR servers.

We refer to '$\otimes$' and '$\oslash$' as the $(\min,\times)$ *convolution* and $(\min,\times)$ *deconvolution* operators, respectively.

The arrival, departure, and service processes constructed in the previous section are SNR processes. With the $(\min,\times)$ convolution, we can express the defining property of a dynamic SNR server from Eq. (9) as

$$\mathcal{D}(0,t) \geq \mathcal{A} \otimes \mathcal{S}(0,t) \tag{12}$$

for every pair of SNR arrival and departure processes $\mathcal{A}(\tau,t)$ and $\mathcal{D}(\tau,t)$.

We note that, in fact, for any system description in the bit domain by the $(\mathbb{R} \cup \{+\infty\}, \min, +)$ and the $(\mathcal{F}, \min, *)$ dioid algebras there exists a corresponding characterization in the SNR domain using the $(\mathbb{R}^+ \cup \{+\infty\}, \min, \times)$ and $(\mathcal{F}^+, \min, \otimes)$ dioids.

A. Server Concatenation and Performance Bounds

The existing network calculus in the bit domain allows for the concatenation of tandem service elements using the $(\min,+)$ convolution (see page 1). As an immediate consequence, single node performance bounds are extended to a multi-hop setting. We now establish the corresponding result in the $(\min,\times)$ network calculus. Specifically, the concatenation of dynamic SNR servers is again a dynamic SNR server. We will prove the result for a tandem network of two nodes, as shown in Fig. 3.

Lemma 1. *Let $\mathcal{S}_1(\tau,t)$ and $\mathcal{S}_2(\tau,t)$ be two dynamic SNR servers in tandem as shown in Fig. 3. Then, the service offered by the tandem of nodes is given by the dynamic SNR server $\mathcal{S}_{\text{net}}(\tau,t)$ with*

$$\mathcal{S}_{\text{net}}(\tau,t) = \mathcal{S}_1 \otimes \mathcal{S}_2(\tau,t) \ .$$

Proof: Using Eq. (9), the departure process $\mathcal{D}(0,t)$ can be written as

$$\begin{aligned}
\mathcal{D}(0,t) &\geq \inf_{0\leq u\leq t} \{\mathcal{A}_2(0,u) \cdot \mathcal{S}_2(u,t)\} \\
&\geq \inf_{0\leq u\leq t} \Big\{ \inf_{0\leq \tau\leq u} \{\mathcal{A}(0,\tau) \cdot \mathcal{S}_1(\tau,u)\} \cdot \mathcal{S}_2(u,t)\Big\} \\
&= \inf_{0\leq \tau\leq t} \Big\{\mathcal{A}(0,\tau) \cdot \inf_{\tau\leq u\leq t} \{\mathcal{S}_1(\tau,u) \cdot \mathcal{S}_2(u,t)\}\Big\} \\
&= \inf_{0\leq \tau\leq t} \big\{\mathcal{A}(0,\tau) \cdot (\mathcal{S}_1 \otimes \mathcal{S}_2)(\tau,t)\big\} .
\end{aligned}$$

■

The extension to networks with more than two nodes follows by iteratively applying Lemma 1. Hence, the dynamic network SNR server with N nodes in tandem is given by

$$\mathcal{S}_{\text{net}}(\tau,t) = \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \cdots \otimes \mathcal{S}_N(\tau,t) \ . \tag{13}$$

Performance bounds in the $(\min,\times)$ network calculus are computed with the $(\min,\times)$ deconvolution operator. This is analogous to the role of the $(\min,+)$ deconvolution in the

$(\min,+)$ network calculus. The bounds are expressed in the following lemma.

Lemma 2. *Given a system with SNR arrival process $\mathcal{A}(\tau,t)$ and dynamic SNR server $\mathcal{S}(\tau,t)$.*

- OUTPUT BURSTINESS. *The SNR departure process is bounded by $\mathcal{D}(\tau,t) \le \mathcal{A} \oslash \mathcal{S}(\tau,t)$.*
- BACKLOG BOUND. *The SNR backlog process is bounded by $\mathcal{B}(t) \le \mathcal{A} \oslash \mathcal{S}(t,t)$.*
- DELAY BOUND. *The delay process is bounded by $\mathcal{W}(t) \le \inf\Big\{d \ge 0 : \mathcal{A} \oslash \mathcal{S}(t+d,t) \le 1\Big\}$.*

Proof: For the output bound, we fix τ and t with $0 \le \tau \le t$ and derive

$$\begin{aligned}\mathcal{D}(\tau,t) &= \frac{\mathcal{D}(0,t)}{\mathcal{D}(0,\tau)} \le \frac{\mathcal{A}(0,t)}{\mathcal{D}(0,\tau)} \\ &\le \sup_{0\le u\le\tau}\Big\{\frac{\mathcal{A}(0,t)}{\mathcal{A}(0,u)\cdot\mathcal{S}(u,\tau)}\Big\} \\ &= \sup_{0\le u\le\tau}\Big\{\frac{\mathcal{A}(u,t)}{\mathcal{S}(u,\tau)}\Big\},\end{aligned}$$

where we used the inequality $\mathcal{D}(0,\tau) \ge \mathcal{A} \otimes \mathcal{S}(0,\tau)$ in the second line.

For any fixed sample path, fix an arbitrary $t \ge 0$. The bound on the backlog is derived by

$$\begin{aligned}\mathcal{B}(t) &= \frac{\mathcal{A}(0,t)}{\mathcal{D}(0,t)} \\ &\le \sup_{0\le u\le t}\Big\{\frac{\mathcal{A}(0,t)}{\mathcal{A}(0,u)\cdot\mathcal{S}(u,t)}\Big\} \\ &= \sup_{0\le u\le t}\Big\{\frac{\mathcal{A}(u,t)}{\mathcal{S}(u,t)}\Big\},\end{aligned}$$

where we used $\mathcal{D}(0,t) \ge \mathcal{A} \otimes \mathcal{S}(0,t)$ in the second step.

Recall that the delay is invariant under the transform of domains, that is, $\mathcal{W}(t) = W(t)$. By definition of the delay in Eq. (7), a delay bound w satisfies

$$\begin{aligned}\mathcal{W}(t) &= \inf\Big\{w \ge 0 : \frac{\mathcal{A}(0,t)}{\mathcal{D}(0,t+w)} \le 1\Big\} \\ &\le \inf\Big\{w \ge 0 : \sup_{0\le u\le t}\Big\{\frac{\mathcal{A}(0,t)}{\mathcal{A}(0,u)\cdot\mathcal{S}(u,t+w)}\Big\} \le 1\Big\} \\ &= \inf\Big\{w \ge 0 : \sup_{0\le u\le t}\Big\{\frac{\mathcal{A}(u,t)}{\mathcal{S}(u,t+w)}\Big\} \le 1\Big\}. \quad (14)\end{aligned}$$

where we used the inequality $\mathcal{D}(0,t+w) \ge \mathcal{A} \otimes \mathcal{S}(0,t+w)$ in the second line. ∎

With an algebraic description for network performance bounds in the SNR domain in hand, we now turn to the problem of computing the bounds.

B. The Mellin Transform in the SNR domain

The concise (and familiar) expressions from the previous section for the network service and performance bounds in the SNR domain hide the difficulty of computing the expressions. In fact, all expressions of the $(\min,\times)$ network calculus contain products or quotients of random variables. The *Mellin transform* [6] facilitates such computations, particularly when the arrival and service processes are independent.

The Mellin transform of a nonnegative random variable X is defined by

$$\mathcal{M}_X(s) = E[X^{s-1}] \quad (15)$$

for any complex number s for which the expectation on the right hand side exists.

We will exploit that the Mellin transform of a product of two independent random variables X and Y equals the product of their Mellin transforms,

$$\mathcal{M}_{X\cdot Y}(s) = \mathcal{M}_X(s)\cdot\mathcal{M}_Y(s). \quad (16)$$

Similarly, the Mellin transform of the quotient of independent random variables is given by

$$\mathcal{M}_{X/Y}(s) = \mathcal{M}_X(s)\cdot\mathcal{M}_Y(2-s). \quad (17)$$

We will evaluate the Mellin transform only for real-valued s, where it is always well-defined (though it may take the value $+\infty$). When $s > 1$, the Mellin transform is order-preserving, i.e., for any pair of random variables X, Y with $Pr(X > Y) = 0$ we have $\mathcal{M}_X(s) \le \mathcal{M}_Y(s)$ for all s. When $s < 1$, the order is reversed. Hence bounds on the distribution of a random variable X generally imply bounds on its Mellin transform.

A more subtle question is how to obtain bounds on the distribution of a random variable from its Mellin transform. Here, the complex inversion formula is not helpful. Instead, we will use the moment bound

$$Pr(X \ge a) \le a^{-s}\mathcal{M}_X(1+s) \quad (18)$$

for all $a > 0$ and $s > 0$. For bivariate random processes $\mathcal{X}(\tau,t)$, we will write $\mathcal{M}_\mathcal{X}(s,\tau,t) \stackrel{\triangle}{=} \mathcal{M}_{\mathcal{X}(\tau,t)}(s)$.

In our calculus, we work with the Mellin transform of $(\min,\times)$ convolutions and deconvolutions, which not only involves products and quotients, but also requires to compute infimums and supremums. The computation of the exact Mellin transform for these operations is generally not feasible. We therefore resort to bounds, as specified in the next lemma.

Lemma 3. *Let $\mathcal{X}(\tau,t)$ and $\mathcal{Y}(\tau,t)$ be two independent non-negative bivariate random processes. For $s < 1$, the Mellin transform of the $(\min,\times)$ convolution $\mathcal{X} \otimes \mathcal{Y}(\tau,t)$ is bounded by*

$$\mathcal{M}_{\mathcal{X}\otimes\mathcal{Y}}(s,\tau,t) \le \sum_{u=\tau}^{t}\mathcal{M}_\mathcal{X}(s,\tau,u)\cdot\mathcal{M}_\mathcal{Y}(s,u,t)\,. \quad (19)$$

For $s > 1$, the Mellin transform of the $(\min,\times)$ deconvolution $\mathcal{X} \oslash \mathcal{Y}(\tau,t)$ is bounded by

$$\mathcal{M}_{\mathcal{X}\oslash\mathcal{Y}}(s,\tau,t) \le \sum_{u=0}^{\tau}\mathcal{M}_\mathcal{X}(s,u,t)\cdot\mathcal{M}_\mathcal{Y}(2-s,u,\tau)\,. \quad (20)$$

Proof: Note that the function $f(z) = z^{s-1}$ is increasing for $s > 1$ and decreasing for $s < 1$. For $s < 1$, the convolution is estimated by

$$\begin{aligned}\mathcal{M}_{\mathcal{X}\otimes\mathcal{Y}}(s,\tau,t) &= E\Big[\Big(\inf_{\tau\le u\le t}\{\mathcal{X}(\tau,u)\cdot\mathcal{Y}(u,t)\}\Big)^{s-1}\Big] \\ &= E\Big[\sup_{\tau\le u\le t}\{(\mathcal{X}(\tau,u))^{s-1}\cdot(\mathcal{Y}(u,t))^{s-1}\}\Big] \\ &\le \sum_{u=\tau}^{t} E\big[(\mathcal{X}(\tau,u))^{s-1}\big]\cdot E\big[(\mathcal{Y}(u,t))^{s-1}\big]\,.\end{aligned}$$

In the last line, we have used the non-negativity of $\mathcal{X}$ and $\mathcal{Y}$ and the union bound to replace the supremum with a sum, and their independence to evaluate the expectation of the products. Eq. (19) follows by inserting the definition of the Mellin transform. The deconvolution is similarly estimated for $s>1$ by

$$\begin{aligned}\mathcal{M}_{\mathcal{X}\oslash\mathcal{Y}}(s,\tau,t) &= E\Big[\Big(\sup_{u\le\tau}\big\{\mathcal{X}(u,t)/\mathcal{Y}(u,\tau)\big\}\Big)^{s-1}\Big] \\ &= E\Big[\sup_{0\le u\le\tau}\big\{\big(\mathcal{X}(u,t)\big)^{s-1}\cdot\big(\mathcal{Y}(u,\tau)\big)^{1-s}\big\}\Big] \\ &\le \sum_{u=0}^{\tau} E\big[\big(\mathcal{X}(u,t)\big)^{s-1}\big]\cdot E\big[\big(\mathcal{Y}(u,\tau)\big)^{1-s}\big]\,,\end{aligned}$$

and Eq. (20) follows from the definition of the Mellin transform. ■

As an application of Lemmas 1 and 3, we compute a bound on the Mellin transform of the service process for a cascade of fading channels. We make the idealizing assumption that the channels are independent.

Corollary 1. *Consider a cascade of N independent, identically distributed fading channels, where the service process for each channel is given by Eq. (6), with i.i.d. random variables γ_i. Let $\mathcal{S}_{\rm net}(\tau,t)$ denote the SNR service process for the entire cascade. Then, the Mellin transform for this process satisfies*

$$\mathcal{M}_{\mathcal{S}_{\rm net}}(s,\tau,t) \le \binom{N-1+t-\tau}{t-\tau}\cdot\big(\mathcal{M}_{g(\gamma)}(s)\big)^{t-\tau}$$

for all $s<1$.

Proof: We use the server concatenation formula in Eq. (13) to represent the service of the cascade as $\mathcal{S}_{\rm net}(\tau,t)=\mathcal{S}_1\otimes\mathcal{S}_2\otimes\ldots\otimes\mathcal{S}_N(\tau,t)$. By Lemma 3, its Mellin transform satisfies for $s<1$

$$\mathcal{M}_{\mathcal{S}_{\rm net}}(s,\tau,t) \le \sum_{u_1,\ldots,u_{N-1}}\prod_{n=1}^{N}\mathcal{M}_{\mathcal{S}}(s,u_{n-1},u_n)\,, \qquad (21)$$

where the sum runs over all sequences $u_0\le u_1\le\cdots\le u_N$ with $u_0=\tau$ and $u_N=t$. The assumptions on the service processes of the individual channels imply that each product evaluates to the same function

$$\begin{aligned}\prod_{n=1}^{N}\mathcal{M}_{\mathcal{S}}(s,u_{n-1},u_n) &= \prod_{n=1}^{N}\big(\mathcal{M}_{g(\gamma)}(s)\big)^{u_n-u_{n-1}} \\ &= \big(\mathcal{M}_{g(\gamma)}(s)\big)^{t-\tau}\,,\end{aligned}$$

where γ is a random variable that has the same distribution as the γ_i. Since the number of summands in Eq. (21) is given by $\binom{N-1+t-\tau}{t-\tau}$, the claim follows. ■

C. Performance Bounds for the Bit Domain

We next obtain network-level performance bounds for the bit domain. This involves a transformation from the SNR domain to the bit domain via the relationship in Fig. 2, which provides the translation of the abstract metrics $\mathcal{D}$ and $\mathcal{B}$ into processes D and B, which, along with W, are concrete measures for traffic burstiness, buffer requirements, and delay.

Theorem 1. *Given a system where arrivals are described by a bivariate process $A(\tau,t)$, and the available service is given by a dynamic server $S(\tau,t)$. Let $\mathcal{A}(\tau,t)$ and $\mathcal{S}(\tau,t)$ be the corresponding SNR processes. Fix $\varepsilon>0$ and define, for $s>0$,*

$$\mathsf{M}(s,\tau,t) = \sum_{u=0}^{\min(\tau,t)}\mathcal{M}_{\mathcal{A}}(1+s,u,t)\cdot\mathcal{M}_{\mathcal{S}}(1-s,u,\tau)\,.$$

Then, we have the following probabilistic performance bounds.

- OUTPUT BURSTINESS: *$Pr\big(D(\tau,t)>d^\varepsilon\big)\le\varepsilon$, where*

$$d^\varepsilon(\tau,t) = \inf_{s>0}\Big\{\frac{1}{s}\big(\log\mathsf{M}(s,\tau,t)-\log\varepsilon\big)\Big\}\,;$$

- BACKLOG: *$Pr\big(B(t)>b^\varepsilon\big)\le\varepsilon$, where*

$$b^\varepsilon = \inf_{s>0}\Big\{\frac{1}{s}\big(\log\mathsf{M}(s,t,t)-\log\varepsilon\big)\Big\}\,;$$

- DELAY: *$Pr\big(W(t)>w^\varepsilon\big)\le\varepsilon$, where w^ε is the smallest number satisfying*

$$\inf_{s>0}\Big\{\mathsf{M}(s,t+w^\varepsilon,t)\Big\}\le\varepsilon\,.$$

Proof: For the bound on the distribution of the output burstiness, we start from the inequality $\mathcal{D}(\tau,t)\le\mathcal{A}\otimes\mathcal{S}(\tau,t)$. It follows from the moment bound and Lemma 3 that, for any choice of $d>0$ and all $s>0$

$$\begin{aligned}Pr(D(\tau,t)>d) &= Pr(\mathcal{D}(t)>e^d) \\ &\le Pr(\mathcal{A}\oslash\mathcal{S}(\tau,t)>e^d) \\ &\le (e^d)^{-s}\mathcal{M}_{\mathcal{A}\oslash\mathcal{S}}(1+s,\tau,t) \\ &= e^{-sd}\mathsf{M}(s,\tau,t)\,.\end{aligned}$$

To obtain the claim, we set the right hand side equal to ε, solve for d, and optimize over the value of $s>0$ to obtain $d^\varepsilon(\tau,t)$. The proof of the backlog bound proceeds in the same way, starting from the inequality $\mathcal{B}(t)\le\mathcal{A}(0,t)/\mathcal{D}(0,t)$.

The delay bound is slightly more subtle. Fix $t\ge0$. Using Lemma 2 and the moment bound with $a=1$, we obtain that

$$\begin{aligned}Pr(\mathcal{W}(t)>w) &\le Pr(\mathcal{A}\oslash\mathcal{S}(t+w,t)>1) \\ &\le \mathcal{M}_{A\oslash\mathcal{S}}(s+1,t+w,t)\end{aligned}$$

for every $s>0$. By Lemma 3, the Mellin transform $\mathcal{M}_{A\oslash\mathcal{S}}(s+1,t+w,t)$ satisfies a bound that agrees with the function $\mathsf{M}(s,t+w,t)$, except that the upper limit in the summation that defines $\mathsf{M}(s,t+w,t)$ would have to be replaced by $\tau=t+w$. To obtain a sharper estimate, we use instead Eq. (14) from the proof of Lemma 2. The resulting bound is that

$$\mathcal{Z}(t)\triangleq\sup_{0\le u\le t}\Big\{\frac{\mathcal{A}(u,t)}{\mathcal{S}(u,t+w)}\Big\}$$

satisfies

$$\begin{aligned} Pr(\mathcal{W}(t) > w) &\leq Pr(\mathcal{Z}(t) > 1) \\ &\leq \mathcal{M}_{\mathcal{Z}(t)}(s+1) \\ &\leq \mathsf{M}(s, t+w, t)\,, \end{aligned} \tag{22}$$

where we have used that the supremum in the definition of $\mathcal{Z}$ extends only up to $u = t$, and then repeated the proof of Eq. (20). The claim follows by optimizing over s. ■

Corresponding bounds as in Theorem 1 can be obtained using the $(\min, +)$ algebra and the network calculus with moment-generating functions [8]. The significance of Theorem 1 is that it permits the application of the network calculus, where traffic is characterized in the bit domain, and service is naturally expressed in the SNR domain. This will become evident in the next section, where the theorem gives us concise bounds for delays and backlog of multi-hop networks with Rayleigh fading channels.

IV. Network Performance of Rayleigh Channels

We now apply the techniques developed in the two previous sections to a network of Rayleigh channels. Consider the dynamic SNR server description for a Rayleigh fading channel, as constructed in Sec. II.B. We use Eq. (6), with the function $g(\gamma)$ given by

$$g(\gamma) = 1 + \gamma = 1 + \bar{\gamma}|h|^2, \tag{23}$$

where $\bar{\gamma}$ is the average SNR of the channel and $|h|$ is the fading gain. For Rayleigh fading, $|h|$ is a Rayleigh random variable with probability density $f(x) = 2xe^{-x^2}$. In a physical system, $\bar{\gamma} = \bar{P}_r/\sigma^2$, where $\bar{P}_r$ and σ^2 are the received signal power and the (additive white Gaussian) noise power at the receiver, respectively. Then, $|h|^2$ is exponentially distributed, and the Mellin transform of $g(\gamma)$ is given by

$$\mathcal{M}_{g(\gamma)}(s) = e^{1/\bar{\gamma}}\bar{\gamma}^{s-1}\Gamma(s, \bar{\gamma}^{-1})\,,$$

where $\Gamma(s, y) = \int_y^\infty x^{s-1}e^{-x}\,dx$ is the upper incomplete Gamma function. Using the assumption that the γ_i are independent and identically distributed, we obtain for the Mellin transform of the dynamic server

$$\mathcal{M}_{\mathcal{S}}(s, \tau, t) = \left(e^{1/\bar{\gamma}}\bar{\gamma}^{s-1}\Gamma(s, \bar{\gamma}^{-1})\right)^{t-\tau}. \tag{24}$$

For the arrival process, we use a characterization due to Chang [3], where the moment-generating function of the cumulative arrival process in the bit domain is bounded by

$$\frac{1}{s}\log E[e^{sA(\tau,t)}] \leq \rho(s)\cdot(t-\tau) + \sigma(s)$$

for some $s > 0$. In general, $\rho(s)$ and $\sigma(s)$ are nonnegative increasing functions of s that may become infinite when s is large. This traffic class, referred to as $(\sigma(s), \rho(s))$-bounded arrivals, is broad enough to include Markov-modulated arrival processes. The corresponding class of SNR arrival processes is defined by the condition that

$$\mathcal{M}_{\mathcal{A}}(s, \tau, t) \leq e^{(s-1)\cdot(\rho(s-1)\cdot(t-\tau)+\sigma(s-1))} \tag{25}$$

for some $s > 1$.

A. Performance Bounds of Rayleigh Fading Channels

We consider the transmission of $(\sigma(s), \rho(s))$-bounded arrivals on a Rayleigh fading channel. To obtain single-hop performance bounds, we apply Theorem 1 with the expressions for the Mellin transforms of the SNR service and arrival processes from Eqs. (24) and (25). For the function $\mathsf{M}(s, \tau, t)$ from the statement of the theorem, we compute for $s > 0$

$$\mathsf{M}(s,\tau,t) \leq e^{s\cdot(\rho(s)(t-\tau)+\sigma(s))} \times \sum_{u=[\tau-t]_+}^{\infty} \big(\underbrace{e^{s\cdot\rho(s)}e^{1/\bar{\gamma}}\bar{\gamma}^{-s}\Gamma(1-s, \bar{\gamma}^{-1})}_{\triangleq V(s)}\big)^u\,,$$

where $[\tau - t]_+$ is the maximum of $\tau - t$ and 0. The sum converges when $V(s) < 1$, which can be interpreted as a stability condition. Inserting the result into Theorem 1, we obtain for the output burstiness the probabilistic bound

$$d^\varepsilon(\tau, t) = \inf_{s>0}\Big\{\rho(s)(t-\tau) + \sigma(s) - \frac{1}{s}\big(\log(1-V(s)) + \log\varepsilon\big)\Big\}\,.$$

The backlog bound is obtained by setting $\tau = t$,

$$b^\varepsilon = \inf_{s>0}\left\{\sigma(s) - \frac{1}{s}\big(\log(1-V(s)) + \log\varepsilon\big)\right\}\,.$$

The delay bound is the smallest number w^ε such that

$$\inf_{s>0}\Big\{\frac{e^{s\cdot(-\rho(s)w^\varepsilon+\sigma(s))}}{1-V(s)}\cdot (V(s))^{w^\varepsilon}\Big\} \leq \varepsilon\,.$$

We also derive end-to-end bounds for a cascade of N Rayleigh channels with $(\sigma(s), \rho(s))$-bounded arrivals, using the same parameters as before. Let $\mathcal{S}_{\rm net}(\tau, t)$ be the service process for the entire cascade. By Corollary 1, its Mellin transform satisfies for $0 \leq \tau \leq t$ and $s < 1$

$$\mathcal{M}_{\mathcal{S}_{\rm net}}(s,\tau,t) \leq \binom{N-1+t-\tau}{t-\tau}\cdot\left(e^{1/\bar{\gamma}}\bar{\gamma}^{s-1}\Gamma(s,\bar{\gamma}^{-1})\right)^{t-\tau}.$$

We will use again Theorem 1. For $0 \leq \tau \leq t$ and $s > 0$, we compute

$$\mathsf{M}_{\rm net}(s,\tau,t) \leq \frac{e^{s\cdot(\rho(s)(t-\tau)+\sigma(s))}}{(1-V(s))^N}\,,$$

where $V(s)$ is as defined above, and where we applied the combinatorial identity

$$\sum_{u=0}^{\infty} x^u\binom{N-1+u}{u} = \frac{1}{(1-x)^N}\,, \tag{26}$$

for any $N > 1$ and for $0 < x < 1$. Inserting $\mathsf{M}_{\rm net}(s,\tau,t)$ into Theorem 1 gives for the end-to-end output bound, denoted by $d^\varepsilon_{\rm net}(\tau, t)$, the value

$$d^\varepsilon_{\rm net}(\tau,t) = \inf_{s>0}\Big\{\rho(s)(t-\tau) + \sigma(s) - \frac{1}{s}\big(N\log(1-V(s)) + \log\varepsilon\big)\Big\}\,.$$

Note that for $N = 1$, this bound agrees with the previous bound for a single node. In the same way, we derive the probabilistic end-to-end backlog bound

$$b_{\text{net}}^{\varepsilon} = \inf_{s>0} \left\{ \sigma(s) - \frac{1}{s}\big(N \log(1 - V(s)) + \log \varepsilon\big) \right\} . \quad (27)$$

For the delay bound, we estimate for $w \geq 0$

$$\begin{aligned} &\mathsf{M}_{\text{net}}(s, t+w, t) \\ &\leq \; e^{s(-\rho(s)w + \sigma(s))} \sum_{u=w}^{\infty} \binom{N-1+u}{u} (V(s))^u \\ &\leq \; \frac{e^{s(-\rho(s)w+\sigma(s))}}{(1 - V(s))^N} \cdot \min\left\{1, (V(s))^w w^{N-1}\right\} , \quad (28) \end{aligned}$$

so long as $V(s) < 1$. Here, the first term in the minimum is obtained by extending the summation down to $u = 0$, and the second term results from the inequality

$$\binom{N-1+u}{u} \leq w^{N-1} \cdot \binom{N-1+u-w}{u-w}$$

for $u \geq w$. In both cases, the resulting sum can be evaluated with Eq. (26). The delay bound w^{ε} is determined according to Theorem 1 by setting the right hand side of Eq. (28) equal to ε, solving for w, and minimizing over s. Because of the complexity of the bound in Eq. (28), the last two steps can only be performed numerically.

It is apparent that the complexity of computing end-to-end bounds is no different than bounds for a single channel. More importantly, we observe that the end-to-end bounds scale linearly in the number of nodes N. Relaxing the strong independence assumptions on the channel properties would give different scaling properties.

B. Numerical Examples

We next present numerical results, where we assume cascades of Rayleigh channels with a transmission bandwidth of $W = 20$ kHz. For traffic, we use $(\sigma(s), \rho(s))$-bounded arrivals with default values $\sigma(s) \equiv 50$ kb and $\rho(s) \equiv 30$ kbps, i.e., the bounds on rate and bursts are deterministic. Hence, the only source of randomness in the examples is the randomness of the channels. We use a violation probability of $\varepsilon = 10^{-4}$.

In Fig. 4 we show the end-to-end backlog for a cascade of N Rayleigh channels, as a function of the average SNR of each channel. Even though the backlog bounds increase only linearly in the number of nodes, it cannot be assumed that the backlog is equally distributed across the nodes. Therefore the per-node requirements – at least for the last node of the cascade – must satisfy the end-to-end bounds. When the SNR of the nodes is sufficiently high, the backlog remains low even for a large number of hops. We observe that the channel becomes saturated for $\bar{\gamma} = 5$ dB. When the number of channels is small, the backlog increases sharply in the vicinity of $\bar{\gamma} = 5$ dB, but remains low everywhere else.

In Fig. 5 we present, for a fixed average SNR of $\bar{\gamma} = 20$ dB, how the end-to-end backlog increases as a function of the transmission rate, for different network sizes. Here, the maximum achievable rate that does not result in a 'blow-up' of the backlog decreases as the number of nodes is increased.

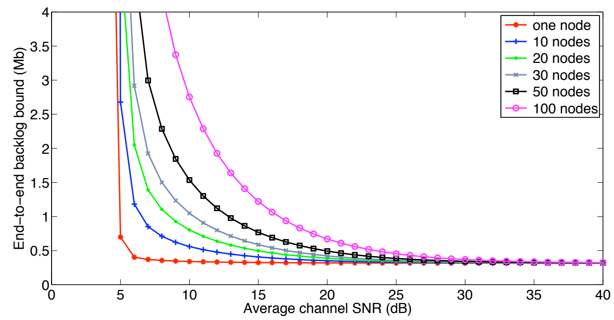

Fig. 4. End-to-end backlog bound ($b_{\text{net}}^{\varepsilon}$) vs. average channel SNR ($\bar{\gamma}$) for multihop Rayleigh fading channels with $\varepsilon = 10^{-4}$, $(\sigma(s), \rho(s))$-bounded traffic with $\sigma(s) = 50$ kb and $\rho(s) = 30$ kbps, and $W = 20$ kHz.

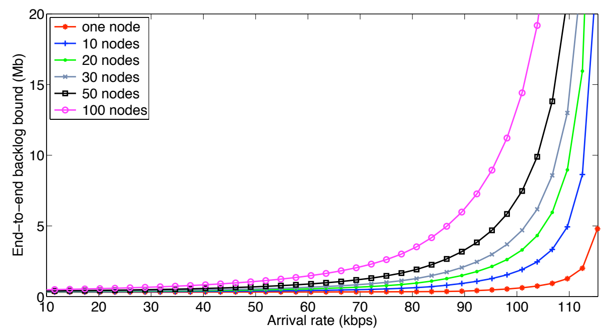

Fig. 5. End-to-end backlog bound ($b_{\text{net}}^{\varepsilon}$) vs. arrival rate ($\rho(s)$) for multi-hop Rayleigh fading channels with $\varepsilon = 10^{-4}$, $(\sigma(s), \rho(s))$-bounded traffic with $\sigma(s) = 50$ kb, $\bar{\gamma} = 20$ dB, and $W = 20$ kHz.

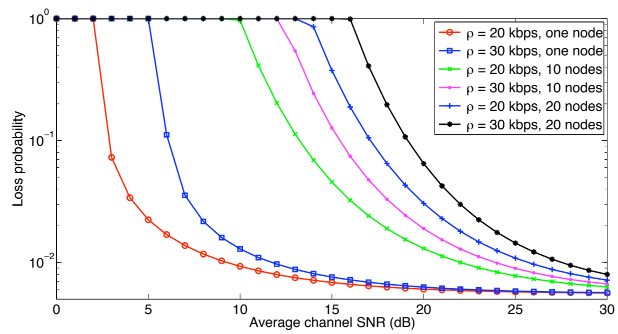

Fig. 6. Loss probability ($\varepsilon(b)$) vs. average channel SNR ($\bar{\gamma}$) for multi-hop Rayleigh fading channels for $N = 1, 10$ and 20, with buffer size 200 kb,

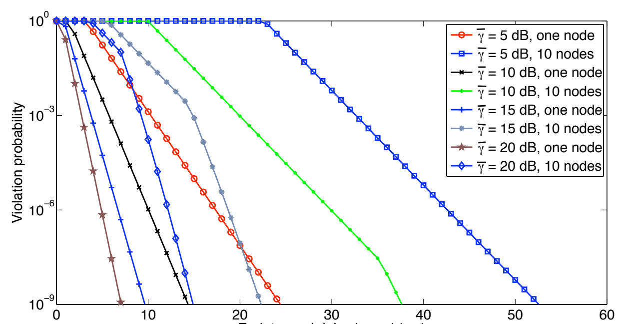

Fig. 7. Delay bound violation probability ($\varepsilon(w)$) vs. end-to-end delays for multi-hop Rayleigh fading channels for $N = 1, 10$, $\bar{\gamma} = 5, 10, 15, 20$ dB, $(\sigma(s), \rho(s))$ bounded traffic with $\sigma(s) = 50$ kb and $\rho(s) = 20$ kbps and $W = 20$ kHz.

Information as given in Figs. 4 and 5 could assist the planning of a multi-hop wireless network where predefined QoS bounds are desired for a given transmission rate. Since the average SNR depends largely on the path loss, which, in turn, is a function of the transmission radius, the graphs could help with determining the maximum distance between nodes to support a desired transmission rate and QoS.

Suppose that buffer sizes are set to satisfy the end-to-end backlog. For a fixed buffer size $b_{\max}$, we can then use the probability $Pr(B_{\text{net}}(t) > b_{\max})$ as an estimate of the probability of dropped traffic, and refer to it as the *loss probability*. In Fig. 6, we show the loss probability as a function of the average channel SNR for a fixed value of $b_{\max} = 200$ kb, for traffic with a rate of $\rho(s) = 20$ and 30 kbps, and for $N = 1, 10$, and 20 nodes. The figure shows that the minimum SNR needed to support a given loss probability is very sensitive to the number of network nodes.

In Fig. 7, we show the violation probability for given end-to-end delay bounds, where we compare the delays at a single node ($N = 1$) with a multi-hop network ($N = 10$) for different average channel SNR values, using Eqs. (22) and (28). The traffic parameters are $\sigma(s) = 50$ kb and $\rho(s) = 20$ kbps. The figure illustrates that for sufficiently large SNR values, low delays are achieved even when traffic traverses 10 links. When the SNR is decreased, we can observe how the delay performance deteriorates in the multi-hop scenario.

V. Conclusion

We have developed a novel network calculus that can incorporate fading channel distributions, without the need for secondary models, such as FSMC. We use the calculus to compute statistical bounds on delay and backlog of multi-hop wireless networks with fading channels. We took a fresh point of view, where the descriptions of the arrivals and the fading channels reside in different domains, referred to as bit domain and SNR domain. We found that by mapping arrival processes to the SNR domain, an end-to-end analysis with fading channels becomes tractable. We discovered that arrivals and service in the SNR domain obey the laws of a $(\min, \times)$ dioid algebra. The analytical framework developed in this paper appears suitable to study a broad class of fading channels and their impact on the network-layer performance in wireless networks. Even though we computed numerical examples for simple networks and strong independence assumptions for the fading channels, our $(\min, \times)$ network calculus is applicable to networks where these assumptions are relaxed. Generalizing our framework and obtaining a more profound understanding of the dioid algebra and computational methods in the SNR domain is the subject of future research.

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