Quick Introduction to Max-Plus Network Calculus

ECE 466

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Max-Plus Network Calculus

1. Min-plus network calculus: Arrival, departures, service are functions of time.

2. Max-plus network calculus: Arrival, departures, service are functions of space (bits).

3. Functions are related by a reflection at the diagonal!
Representing Arrivals

(a) Time Domain.

(b) Space Domain.
Definitions

$T_A(\nu)$  Arrival time of bit $\nu$

$T_D(\nu)$  Departure time of bit $\nu$, with $T_D(\nu) \geq T_A(\nu)$

$W(\nu)$  Delay of bit $\nu$:  

$$W(\nu) = T_D(\nu) - T_A(\nu)$$

$B^a(\nu)$  Backlog at arrival of $\nu$:  

$$B^a(\nu) = \inf \{ \kappa > 0 \mid T_D(\nu - \kappa) \leq T_A(\nu) \}$$

$B^d(\nu)$  Backlog at departure of $\nu$:  

$$B^d(\nu) = \inf \{ \kappa > 0 \mid T_A(\nu + \kappa) \geq T_D(\nu) \}$$
A busy sequence is a maximal contiguous set of bits that experience non-zero delays.

Begin of a busy sequence with respect to \( \nu \geq 0 \) is

\[
\nu = \sup \{ \kappa \mid 0 \leq \kappa \leq \nu, W(\kappa) = 0 \}.
\]
A function in the space domain, $F : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

$\mathcal{T} = \text{all non-decreasing and right-continuous functions}$

$\mathcal{T}_o = \text{Functions in } \mathcal{T}, \text{ with } F(\nu) = -\infty \text{ for } \nu < 0$

and $F(\nu) \geq 0 \text{ for } \nu \geq 0$
Meet the \((\max, +)\) algebra

<table>
<thead>
<tr>
<th>Conventional Algebra</th>
<th>Max-Plus Algebra</th>
</tr>
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<tbody>
<tr>
<td>Addition ((+))</td>
<td>(\max (\max, \lor))</td>
</tr>
<tr>
<td>Multiplication ((\cdot))</td>
<td>(\text{Addition } (+))</td>
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\[
\int_{\mathbb{R}} F(s)G(t - s) \, ds \quad \rightarrow \quad \sup_{\kappa \in \mathbb{R}} \{ F(\kappa) + G(\nu - \kappa) \}
\]

\[
(= F \ast G(t)) \quad \rightarrow \quad (= F\overline{\otimes} G(\nu))
\]
Burst function

$$\tilde{\delta}(\nu) = \begin{cases} -\infty, & \nu < 0 \\ 0, & \nu \geq 0 \end{cases}$$

Delay function

$$\tilde{\delta}_d(\nu) = \tilde{\delta}(\nu) + d$$
Properties of max-plus convolution

\[ F, G, H \in \mathcal{T}_o: \]

- **Closure.** \( F \boxtimes G \in \mathcal{T}_o. \)
- **Associativity.** \( (F \boxtimes G) \boxtimes H = F \boxtimes (G \boxtimes H). \)
- **Commutativity.** \( (F \boxtimes G) \boxtimes H = F \boxtimes (G \boxtimes H). \)
- **Distributivity.** \( (F \vee G) \boxtimes H = (F \boxtimes H) \vee (G \boxtimes H). \)
- **Neutral element.** \( F \boxtimes \delta = F. \)
- **Time shift.** \( F \boxtimes \delta_T(\nu) = F(\nu) + T. \)
- **Monotonicity.** If \( F \leq G \) then \( F \boxtimes H \leq G \boxtimes H. \)
- **Boundedness.** \( F \boxtimes G \geq F, \) in particular, \( F \boxtimes F \geq F. \)

Except boundedness, properties hold for \( F, G, H \in \mathcal{T} \)

If \( F, G \in \mathcal{T}_o: \)

\[
\sup_{\kappa \in \mathbb{R}} \{ F(\kappa) + G(\nu - \kappa) \} = \sup_{0 \leq \kappa \leq \nu} \{ F(\kappa) + G(\nu - \kappa) \}
\]
Buffered link: \( \gamma(\nu) = \begin{cases} \frac{\nu}{C}, & \nu \geq 0 \\ -\infty, & \nu < 0 \end{cases} \Rightarrow T_D(\nu) = T_A \overline{\otimes} \gamma_S(\nu) \)

Service curve is a generalization:

For arbitrary arrivals \( T_A \) and resulting departures \( T_D \) at a network system, a process \( \gamma_S \in \mathcal{T}_o \) is an exact service curve, if for all \( t \),

\[
T_D(\nu) = T_A \overline{\otimes} \gamma_S(\nu).
\]

Lower service curve \( \Rightarrow T_D(\nu) \geq T_A \overline{\otimes} \gamma_S(\nu) \)

Upper service curve \( \Rightarrow T_D(\nu) \leq T_A \overline{\otimes} \gamma_S(\nu) \)
Service curves express **time-invariant** and **space-invariant** service guarantees.

**Constant-rate server:**  \( \gamma_{S_1}(\nu) = \frac{\nu}{R} \)

**Delay server:**  \( \gamma_{S_2}(\nu) = T \)

**Latency-rate server:**

\[
\gamma_{S_3}(\nu) = \gamma_{S_1} \otimes \gamma_{S_2}(\nu) = \frac{\nu}{R} + T
\]
For a sequence of $N$ service elements where the $n$-th element offers a lower/exact/upper service curve $\gamma_{S_n}$ ($n = 1, \ldots, N$), the sequence as a whole offers an lower/exact/upper service curve

$$\gamma_{S_1} \boxtimes \gamma_{S_2} \boxtimes \cdots \boxtimes \gamma_{S_N}$$

The service curve of the sequence is called *network service curve*. 
Meet the \((\max, +)\) deconvolution

For two processes \(F, G \in \mathcal{T}\), the max-plus deconvolution \(F \oslash G\) is

\[
F \oslash G(\nu) = \inf_{\kappa \in \mathbb{R}} \{F(\nu + \kappa) - G(\kappa)\}
\]

- If \(F, G \in \mathcal{T}_o\), then \(F \oslash G(\nu) = \inf_{\kappa \geq 0} \{F(\nu + \kappa) - G(\kappa)\}\).
- Weak properties: not closed, not associative, not commutative.

\(F, G, H \in \mathcal{T}_o\)

- **Composition of \(\oslash\) and \(\oslash\)**. \((F \oslash G) \oslash H = F \oslash (G \oslash H)\).
- **Duality**. \(F \leq G \oslash H\) if and only if \(F \oslash H \leq G\).
A function $F$ is superadditive, if for all $\kappa, \nu \in \mathbb{R}$

$$F(\nu + \kappa) \geq F(\nu) + F(\kappa).$$

- $F$ convex $\Rightarrow$ $F$ superadditive
- $F$ superadditive:
  - $F = F \otimes F$
  - $F = F \otimes F$
- $F \in \mathcal{T}_o$:
  - $F \otimes F \in \mathcal{T}_o$
  - $F \otimes F$ superadditive
Traffic envelopes put a lower bound on the amount of time needed for a given amount of traffic.

A function $\lambda_E$ is a max-plus traffic envelope for $T_A$ ($T_A \sim \lambda_E$) if

$$\lambda_E(\mu) \leq T_A(\nu + \mu) - T_A(\nu), \quad \forall \nu \geq 0, \forall \mu \geq 0.$$ 

- $T_A \sim \lambda_E \implies T_A = T_A \overline{\otimes} \lambda_E$
- Good envelopes are superadditive
Empirical envelopes

What is the best traffic envelope?

The empirical envelope $\lambda^E_A$ of an arrival function $A$ satisfies

$$\lambda^E_A(\nu) = T_A \ominus T_A(\nu) \ , \ \nu \geq 0$$

- $\lambda^E_A$ is superadditive
- $\lambda^E_A$ is largest superadditive function with $\lambda_E \leq T_A$
Traffic shaping (Greedy shaper)

**Greedy shaper**: A network element that

- limits arrivals to a network to a given specification (traffic envelope),
- buffers non-compliant traffic, and
- releases buffered traffic when it becomes compliant.

If $\lambda_E$ is subadditive, greedy shaper offers an exact service curve, i.e.,

$$T_D = T_A \boxtimes \lambda_E$$
Example of greedy shaper: Token Bucket

\[ \lambda_E(\nu) = \left[ \frac{\nu - b}{r} \right]^+ \]
Given a work-conserving buffered link with rate $C$ with a through flow and cross traffic. If $\lambda_c$ is a max-plus traffic envelope for the cross traffic then

$$\gamma_S(\nu) = \frac{1}{C} \left( \inf \{ \mu \geq 0 \mid \lambda_c(\mu) \geq \frac{\nu + \mu}{C} \} + \nu \right)$$

is a lower service curve for the through.

This residual service curve is a (pessimistic) benchmark for the service experienced at a link with multiplexing.
Three Performance Bounds

Given arrival function $T_A$ with traffic envelope $\lambda_E$, and a network system with lower service curve $\gamma_S$:

1. Envelope for departures: $T_D \sim \left[\lambda_E \bar{\gamma}_S\right]^+$ (for $\nu \geq 0$)
2. Delay bound: $W(\nu) \leq -\lambda_E \bar{\gamma}_S(0)$
3. Backlog bound: $B^a(\nu) \leq \inf \{b \geq 0 \mid \lambda_E \bar{\gamma}_S(b) \geq 0\}$
Lower service curves are not good for rate guarantees \((γ_S(ν) = \frac{ν}{R})\)

A strict max-plus service curve satisfies for all \(ν\) and \(μ\) (\(μ < ν\)) in the same busy sequence

\[
TD(ν) - TD(μ) \leq γ_S(ν - μ), \text{ if } ν < μ,
\]

\[
TD(ν) - TA(μ) \leq γ_S(ν - μ), \text{ if } ν = μ.
\]
Adaptive Max-Plus Service Curves

Strict service curves are not good for delay guarantees.

Define:

\[ F \otimes \mu G(\nu) = \sup_{\mu \leq \kappa \leq \nu} \{ F(\kappa) + G(\nu - \kappa) \} \]

An adaptive max-plus service curve \( \gamma_S \) satisfies for all \( \nu \geq 0 \) and \( \mu \leq \nu \):

\[ T_D(\nu) \leq T_D(\mu) + \gamma_S(\nu - \mu) \quad \text{or} \quad T_D(\nu) \leq T_A \otimes \mu \gamma_S(\nu) \]

This can be re-written as a single condition:

\[ T_D(\nu) \leq \inf_{\mu \leq \nu} \left\{ \max \left[ T_D(\mu) + \gamma_S(\nu - \mu) , T_A \otimes \mu \gamma_S(\nu) \right] \right\} , \quad \forall \nu \geq 0 \]