Distributed Real-Time Phase Balancing for Power Grids with Energy Storage

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***Abstract*— Phase balancing is of paramount importance for power system operation. We consider a substation connected to multiple buses, each with single phase loads, generation, and energy storage. A representative of the substation operates the system and aims to minimize the cost of all buses as well as balancing loads among phases. We first consider ideal energy storage with perfect charging and discharging efficiency, and propose a distributed real-time algorithm taking into account system uncertainty. The proposed algorithm does not require any system statistics and can ensure a certain performance guarantee. We further extend the algorithm to accommodate non-ideal energy storage. The algorithm is evaluated through numerical examples and compared with a greedy algorithm.**

I. Introduction

In North America, many residential customers are connected to power grids through single phase transmission lines. Phase balancing, i.e., maintaining the balance of loads among phases, is crucial for power grid operation [1]. This is because phase imbalance can increase energy losses and the risk of failures, and can also degrade system power quality. With the spread of single phase renewable generators, such as wind and solar, and large loads, such as electric vehicles, phase imbalance issues could be aggravated and thus deserve a careful study. For example, in [2], the impact of electric vehicles on phase imbalance is investigated considering different charging modes and different penetration levels.

Previous works on phase balancing have considered methods such as phase swapping (e.g., [3]) and feeder reconfiguration (e.g., [4]). However, these approaches can be ineffective or can incur extra costs on human resources, maintenance expenses, and planned outage duration [3]. An alternative method is to employ energy storage to mitigate the imbalance among phases, which is the focus of this paper.

Energy storage has been used widely in power grids for applications such as energy arbitrage, regulation, and load following [5]. The control of energy storage is, however, generally a challenging problem due to storage characteristics and system uncertainty. There are many existing works on storage control in power grids. For example, using stochastic dynamic programming, the authors of [6] propose a stationary optimal policy for power balancing, and the authors of [7] investigate both optimal and suboptimal polices for energy balancing. Nevertheless, the derivation of an optimal policy under dynamic programming generally relies on system statistics and some specific form of the problem structure, and therefore cannot be easily extended. There are some other works employing Lyapunov optimization [8] for storage control. For example, the authors of [9] consider power balancing, the authors of [10] study demand side management, and the authors of [11] investigate the management of networked storage with a DC power flow model. The algorithms there do not require system statistics, and the main technical challenge is to show the analytical performance of the algorithm. Besides the above two approaches, the authors of [12] consider stochastic model predictive control. However, the algorithm performance can only be evaluated through numerical examples.

In this paper, we study the problem of phase balancing with energy storage in the presence of system uncertainty. To the best of our knowledge, this is the first work that employs energy storage for phase balancing in power grids. In particular, we consider a substation connected to multiple buses, each with single phase loads, generation, and energy storage. Aiming at minimizing the cost of all buses as well as phase imbalances, we propose a distributed real-time algorithm, in which each bus can determine its own control variables. The main contribution is summarized as follows.

- We formulate phase balancing as a stochastic optimization problem by incorporating system uncertainty, storage characteristics, and power network constraints.
- For ideal energy storage with perfect charging and discharging efficiency, we provide a real-time algorithm building on the Lyapunov optimization framework and prove its analytical performance guarantee. Moreover, we offer distributed implementation for the algorithm.
- We extend the algorithm to accommodate non-ideal energy storage with inefficient charging and discharging.
- We evaluate the proposed algorithm through numerical examples and reveal several interesting insights.

Our paper is most related to [11], in which a distributed real-time algorithm is also proposed for power grids with energy storage. However, these two papers are different in terms of the application, objective, communication topology, and power network constraints. Therefore, the problem formulation and the design of distributed implementation are largely different. Moreover, imperfect storage charging and discharging are not considered in [11], but they are addressed in this paper.

The remainder of this paper is organized as follows. In Section II, we describe the system model and formulate

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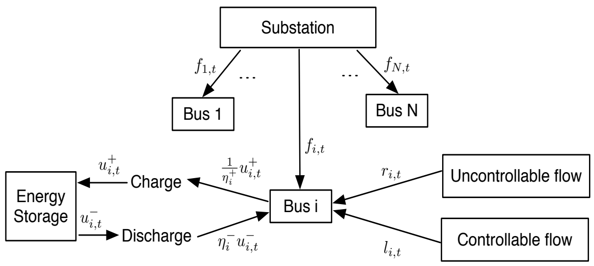

Fig. 1. System model.

the optimization problem. In Section III, we propose a distributed real-time algorithm for ideal energy storage. In Section IV, we extend the algorithm to accommodate non-ideal energy storage with imperfect charging and discharging. Numerical results are presented in Section V, and we conclude in Section VI.

II. System Model and Problem Statement

Consider a discrete-time model with time $t \in \{0, 1, 2, \ldots\}$. For simplicity of notation, throughout the paper we work with energy units instead of power units. The system model is depicted in Fig. 1. A substation is connected with $N \geq 2$ buses, each with single phase loads and generation. For the deployment of energy storage, we consider a general case where it is optional for each bus to deploy storage. Denote the set of buses that deploy storage by $\mathcal{E} \subseteq \{1, 2, \ldots, N\}$. Below we first describe the components of each phase.

A. System Model of Each Phase

At the i-th bus, denote the amount of uncontrollable energy flow during time slot t by $r_{i,t}$. The uncontrollable flow can represent renewable generation such as wind and solar, base loads, or the difference between renewable generation and base loads. Since the uncontrollable flow is generally governed by nature or uncertain human behavior we assume that $r_{i,t}$ is random, but it is confined within an interval $[r_{i,\min}, r_{i,\max}]$. Throughout the paper we use a bold letter to denote a vector that contains the elements of N buses. Here, we define $\mathbf{r}_t \triangleq [r_{1,t}, \ldots, r_{N,t}]$ to represent the uncontrollable energy flow vector during time slot t. The other vectors in the following are defined similarly. Denote the amount of the controllable energy flow at the i-th bus during time slot t by $l_{i,t}$. The controllable flow can represent the output of conventional generators, or the consumption of flexible loads. We associate a cost function with the controllable flow and denote the function by $C_i(\cdot)$.

The energy flow between the substation and the i-th bus during time slot t is represented by $f_{i,t}$. Due to the capacity constraints of power lines, we assume that the energy flow vector $\mathbf{f}_t \in \mathcal{F}$, where the set $\mathcal{F}$ is non-empty, compact, and convex. For example, $\mathcal{F} \triangleq \{\mathbf{f}_t | f_{i,t} \in [f_{i,\min}, f_{i,\max}], \forall i\}$.

Remark: The values of $r_{i,t}, l_{i,t}$, and $f_{i,t}$ can be positive or negative. We use the positive sign to indicate energy injection to the i-th bus, and the negative sign to indicate energy extraction from the i-th bus.

Assume that the i-th bus is equipped with an energy storage unit. Denote its charging and discharging amounts during time slot t by $u^+_{i,t} \in [0, u_{i,\max}]$ and $u^-_{i,t} \in [0, u_{i,\max}]$, respectively, where $u_{i,\max}$ is the maximum charging and discharging amount. Denote the energy state of the i-th storage at the beginning of time slot t by $s_{i,t}$, which evolves as $s_{i,t+1} = s_{i,t} + u^+_{i,t} - u^-_{i,t}$. To protect the storage device, the energy state $s_{i,t}$ is required to be within a preferred interval $[s_{i,\min}, s_{i,\max}]$. The initial energy state $s_{i,0} \in [s_{i,\min}, s_{i,\max}]$. For the i-th storage, we denote the charging efficiency by $\eta_i^+ \in (0, 1]$ and the discharging efficiency by $\eta_i^- \in (0, 1]$. Due to the round-trip efficiency or other operating constraints, simultaneous charging and discharging is forbidden in our model. If the i-th bus does not deploy storage, we simply set the values of $s_{i,t}, u^+_{i,t}$, and $u^-_{i,t}$ to zero for all t.

The energy storage can additionally be used for arbitrage. Denote the electricity price at time slot t by $p_t \in [p_{\min}, p_{\max}]$, which is random over time. Then the cost associated with energy arbitrage during time slot t is $p_t(\frac{1}{\eta_i^+} u^+_{i,t} - \eta_i^- u^-_{i,t})$. Moreover, frequent charging and discharging can shorten the lifetime of storage [13]. To model this effect, we introduce a degradation cost function $D_i(\cdot)$, with the negative input indicating discharging and positive input indicating charging.

B. Problem Statement

Since phase imbalance is harmful for power system operation, it is critical to balance the energy flows $f_{i,t}$ among buses. To this end, we introduce a loss function $F(\cdot)$ to characterize the deviation of $f_{i,t}$ from the average energy flow during each time slot. In particular, at the i-th bus, $F(\cdot)$ is a function of $f_{i,t} - \overline{f}_t$, where $\overline{f}_t \triangleq \frac{1}{N} \sum_{j=1}^N f_{j,t}$.

We assume that the system is operated by a representative of the substation, who aims to minimize the long-term system cost, which includes the costs of all buses as well as phase imbalances. Specifically, based on the model described above, the system cost during time slot t is given by

$$
\begin{aligned}
w_t =& \textstyle\sum_{i \in \mathcal{E}} \Big[p_t(\frac{1}{\eta_i^+} u^+_{i,t} - \eta_i^- u^-_{i,t}) + D_i(u^+_{i,t}) + D_i(-u^-_{i,t}) \Big] \\
&+ \textstyle\sum_{i=1}^N \big[C_i(l_{i,t}) + F(f_{i,t} - \overline{f}_t) \big].
\end{aligned}
$$

Denote the random system state at time slot t by $\mathbf{q}_t \triangleq [\mathbf{r}_t, p_t]$, which includes the uncontrollable flow of N buses and the electricity price. Denote the control action at time slot t by $\mathbf{a}_t \triangleq [\mathbf{l}_t, \mathbf{u}^+_t, \mathbf{u}^-_t, \mathbf{f}_t]$, which contains the controllable flow, the charging and discharging amounts, and the energy flow between each bus and the substation. We formulate the problem as the following stochastic program.

$$
\begin{aligned}
\textbf{P1:} \quad \min_{\{\mathbf{a}_t\}} \quad & \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[w_t] \\
\text{s.t.} \quad & 0 \leq u^+_{i,t}, u^-_{i,t} \leq u_{i,\max}, \forall i \in \mathcal{E}, \forall t, && (1) \\
& u^+_{i,t} \cdot u^-_{i,t} = 0, \forall i \in \mathcal{E}, \forall t, && (2) \\
& s_{i,t+1} = s_{i,t} + u^+_{i,t} - u^-_{i,t}, \forall i \in \mathcal{E}, \forall t, && (3) \\
& s_{i,\min} \leq s_{i,t} \leq s_{i,\max}, \forall i \in \mathcal{E}, \forall t, && (4) \\
& u^-_{i,t} = u^+_{i,t} = 0, \forall i \notin \mathcal{E}, \forall t, && (5) \\
& \mathbf{f}_t \in \mathcal{F}, \forall t, && (6)
\end{aligned}
$$

$$f_{i,t} + r_{i,t} + l_{i,t} + \eta_i^- u_{i,t}^- - \frac{1}{\eta_i^+} u_{i,t}^+ = 0, \forall i, \forall t. \quad (7)$$

The expectation on the objective is taken over the randomness of $\mathbf{q}_t$ and the possibly random control action. Constraint (7) requires energy balance at each bus during each time slot.

To keep mathematical exposition simple, we assume that the cost functions $C_i(\cdot)$ and $D_i(\cdot)$ are continuously differentiable and convex. This assumption is not too restrictive since many practical cost functions can be well approximated by such functions. Denote the derivatives of $C_i(\cdot)$ and $D_i(\cdot)$ by $C_i'(\cdot)$ and $D_i'(\cdot)$, respectively. Since the variables $u_{i,t}^+, u_{i,t}^-$, and $l_{i,t}$ are bounded based on the constraints of **P1**, the cost functions and their derivatives are bounded in the feasible set. For the cost function $C_i(\cdot)$, we denote its range by $[C_{i,\min}, C_{i,\max}]$ and its range of the derivative by $[C'_{i,\min}, C'_{i,\max}]$ in the feasible set. The range of the cost function $D_i(\cdot)$ and that of its derivative are defined similarly. In addition, we assume that the loss function $F(\cdot)$ is convex and continuously differentiable.

We are interested in designing a distributed real-time algorithm for solving **P1** that does not require any system statistics. This is motivated by privacy concerns of each bus, the potential requirement of real-time operation, and the fact that accurate statistical information is generally difficult to obtain in reality. However, this is a challenging task due to system uncertainty, the coupling of all buses through the objective, the non-simultaneous charging and discharging constraint (2), and the energy state constraint (4).

III. Distributed Real-Time Algorithm for Ideal Energy Storage

In this section, we propose a distributed real-time algorithm for ideal energy storage with $\eta_i^+ = \eta_i^- = 1, \forall i \in \mathcal{E}$. We first propose a real-time algorithm and show its analytical performance and then provide distributed implementation for the algorithm. The case of non-ideal energy storage is studied in Section IV.

A. Description and Analysis of Real-Time Algorithm

For ideal energy storage, without loss of optimality, we can combine the variables $u_{i,t}^+$ and $u_{i,t}^-$ into one by introducing a new variable $u_{i,t} \triangleq u_{i,t}^+ - u_{i,t}^-$, which can represent the net charging and discharging amount. If $u_{i,t} > 0$ it indicates charging, and if $u_{i,t} < 0$ it indicates discharging. Therefore, the non-simultaneous charging and discharging constraint (2) can be eliminated. The control action at time slot t is now $\mathbf{a}_t \triangleq [\mathbf{l}_t, \mathbf{u}_t, \mathbf{f}_t]$, and the system cost can be rewritten as $w_t = \sum_{i\in\mathcal{E}} \left[p_t u_{i,t} + D_i(u_{i,t})\right] + \sum_{i=1}^{N} \left[C_i(l_{i,t}) + F(f_{i,t} - \overline{f}_t)\right]$.

To deal with the energy state constraint, we replace constraints (3) and (4) with a new time-averaged constraint, which only requires the net charging and discharging amount to be zero on average, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[u_{i,t}] = 0, \ \forall i \in \mathcal{E}. \quad (8)$$

It is not difficult to show that (3) and (4) imply (8). In other words, any $u_{i,t}$ that satisfies (3) and (4) also satisfies (8). The reason for this relaxation is to employ Lyapunov optimization techniques [8] for the real-time algorithm design. Later we will prove that our proposed algorithm can ensure that (3) and (4) hold.

With constraint (8), we can form a relaxed stochastic optimization problem as follows.

$$\textbf{P2:} \quad \min_{\{\mathbf{a}_t\}} \quad \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[w_t]$$
$$\text{s.t.} \quad (5),\ (6),\ (7),\ (8)$$
$$-u_{i,\max} \le u_{i,t} \le u_{i,\max}, \forall i \in \mathcal{E}, \forall t. \quad (9)$$

At time slot t, for $i \in \mathcal{E}$, define a Lyapunov function $L(s_{i,t}) \triangleq \frac{1}{2}(s_{i,t} - \beta_i)^2$, where β_i is a perturbation parameter introduced to ensure the boundedness of the energy state and is defined as

$$\beta_i \triangleq s_{i,\min} + u_{i,\max} + V_i(p_{\max} + D'_{i,\max} + C'_{i,\max}). \quad (10)$$

The parameter V_i belongs to an interval $(0, V_{i,\max}]$ where $V_{i,\max} \triangleq \frac{s_{i,\max} - s_{i,\min} - 2u_{i,\max}}{p_{\max} - p_{\min} + D'_{i,\max} - D'_{i,\min} + C'_{i,\max} - C'_{i,\min}}$. To ensure the positivity of $V_{i,\max}$, we need $s_{i,\max} - s_{i,\min} - 2u_{i,\max} > 0$. This is generally true in real-time applications, in which the length of each time interval is small.

We define the one-slot conditional Lyapunov drift as $\Delta(\mathbf{s}_t) \triangleq \mathbb{E}\big[\sum_{i\in\mathcal{E}} \frac{L(s_{i,t+1}) - L(s_{i,t})}{V_i} | \mathbf{s}_t\big]$. In the following lemma, we show that this drift function is upper bounded.

Lemma 1: For all possible decisions and all possible values of $s_{i,t}, i \in \mathcal{E}$, in each time slot t, the drift function is upper bounded as follows

$$\Delta(\mathbf{s}_t) \le \sum_{i\in\mathcal{E}} \frac{u_{i,\max}^2}{2V_i} + \frac{s_{i,t} - \beta_i}{V_i} \mathbb{E}\left[u_{i,t} | \mathbf{s}_t\right].$$

Proof: See Appendix VI-A. ∎

Define a drift-plus-cost function at time slot t by $\Delta(\mathbf{s}_t) + \mathbb{E}[w_t | \mathbf{s}_t]$. We propose a real-time algorithm that intends to minimize this function. The idea is to minimize the current system cost and also to keep the energy state bounded in an intelligent way. Instead of using $\Delta(\mathbf{s}_t)$ directly in the objective, we employ its upper bound in Lemma 1 and formulate the optimization problem at time slot t as follows.

$$\textbf{P3:} \quad \min_{\mathbf{a}_t} \quad w_t + \sum_{i\in\mathcal{E}} \frac{(s_{i,t} - \beta_i) u_{i,t}}{V_i}$$
$$\text{s.t.} \quad (5),\ (6),\ (7),\ (9).$$

The overall algorithm is summarized in Algorithm 1, in which we use the superscript notation $*$ to indicate the solution to **P3**.

Algorithm 1 Real-time algorithm for ideal storage.

At time slot t, the substation executes the following steps sequentially:

1: observe the system state $\mathbf{q}_t$ and the energy state $s_{i,t}$;
2: solve **P3** and obtain a solution $\mathbf{a}_t^*$; and
3: update $s_{i,t}$ by (3) using $u_{i,t}^*$.

Denote the optimal objective value of **P1** by w^{opt}. Under Algorithm 1, denote the objective value of **P1** by w^* and the system cost at time slot t by w_t^*. The performance of Algorithm 1 is shown in the following theorem.

Theorem 1: Assume that the system state $\mathbf{q}_t$ is i.i.d. over time, and $\eta_i^+ = \eta_i^- = 1, \forall i \in \mathcal{E}$. Under Algorithm 1 the following statements hold:

1) $\{\mathbf{a}_t^*\}$ is feasible for **P1**.
2) $w^* - w^{\text{opt}} \leq \sum_{i\in\mathcal{E}} \frac{u_{i,\max}^2}{2V_i}$.
3) $\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[w_t^*] - w^{\text{opt}} \leq \sum_{i\in\mathcal{E}} \frac{u_{i,\max}^2}{2V_i} + \frac{\mathbb{E}[L(s_{i,0})]}{TV_i}$.

Proof: See Appendix VI-B. ■

Remarks:

- Theorem 1.1 shows that although the problem **P1** is relaxed to obtain the real-time problem **P3**, the boundedness of the energy state is ensured by Algorithm 1.
- For Theorem 1.2, first, if $\mathcal{E}$ is empty, i.e., no bus deploys storage, then Algorithm 1 achieves the optimal objective value. In fact, in this case, Algorithm 1 reduces to a greedy algorithm that only minimizes the current system cost at each time. Second, if $\mathcal{E}$ is non-empty, to minimize the gap to the optimal objective value, we should set $V_i = V_{i,\max}$. Asymptotically, if the energy capacity $s_{i,\max}$ is large and thus $V_{i,\max}$ is large, Algorithm 1 achieves the optimal objective value.
- In Theorem 1.3, we characterize the performance of Algorithm 1 under a finite time horizon. An extra gap is incurred due to the initialization. However, if the time horizon T is large, this gap is negligible.
- The i.i.d. assumption of the system state $\mathbf{q}_t$ can be relaxed to accommodate $\mathbf{q}_t$ that follows a finite state irreducible and aperiodic Markov chain. Using a multi-slot drift technique [8], we can show similar conclusions which are omitted here.

B. Distributed Implementation of Real-Time Algorithm

Due to the privacy concerns or the difficulty of collecting information from all buses, in Algorithm 1, the substation may not be able to solve **P3** in a centralized way. In this subsection, we provide a distributed algorithm for solving **P3**. For ease of notation, we suppress the time index t.

The provided distributed algorithm is based on the two-block ADMM [14]. To facilitate the algorithm development, we rewrite **P3** as follows.

$$\begin{aligned} \min_{\mathbf{a}} \quad & \iota(\mathbf{f}\in\mathcal{F}) + \sum_{i=1}^{N}\Big[H_i(l_i,u_i) + F(f_i - \overline{f})\Big] \\ \text{s.t.} \quad & f_i + r_i + l_i - u_i = 0, \forall i \end{aligned} \tag{11}$$

where $\iota(\cdot)$ is the indicator function that equals 0 (resp. $+\infty$) when the enclosed event is true (resp. false), and

$$H_i(l_i,u_i) \triangleq \begin{cases} \frac{(s_i-\beta_i)u_i}{V_i} + pu_i + D_i(u_i) + C_i(l_i) & \\ \quad + \iota(-u_{i,\max} \leq u_i \leq u_{i,\max}), & \text{if } i \in \mathcal{E} \\ C_i(l_i) + \iota(u_i = 0), & \text{if } i \notin \mathcal{E}. \end{cases}$$

We associate a Lagrange multiplier λ_i with equality (11).

By treating the variables $(\mathbf{l},\mathbf{u})$ as one block and the variable $\mathbf{f}$ as the other, we express the updates at the $(k+1)$-th iteration as follows based on the ADMM algorithm.

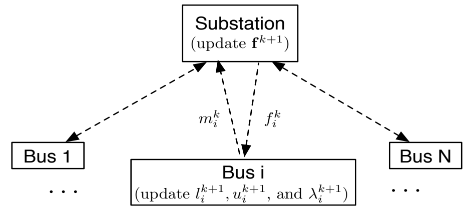

Fig. 2. Distributed implementation for solving **P3**.

$$(l_i,u_i)^{k+1} \leftarrow \operatorname*{argmin}_{l_i,u_i}\Big[H_i(l_i,u_i) + \frac{\rho}{2}(f_i^k + r_i + l_i - u_i + \frac{\lambda_i^k}{\rho})^2\Big]$$

$$\mathbf{f}^{k+1} \leftarrow \operatorname*{argmin}_{\mathbf{f}\in\mathcal{F}} \sum_{i=1}^{N}\Big[F(f_i - \overline{f}) + \frac{\rho}{2}(f_i + r_i + l_i^k - u_i^k + \frac{\lambda_i^k}{\rho})^2\Big]$$

$$\lambda_i^{k+1} \leftarrow \lambda_i^k + \rho(f_i^{k+1} + r_i + l_i^{k+1} - u_i^{k+1})$$

where $\rho > 0$ is a parameter. From the above updates, each bus updates l_i, u_i, and λ_i, and the substation updates $\mathbf{f}$. To accomplish the updates, at the $(k+1)$-th iteration, the substation sends f_i^k to each bus, and each bus provides $m_i^k \triangleq r_i + l_i^k - u_i^k + \frac{\lambda_i^k}{\rho}$ to the substation. The schematic representation of the distributed implementation is given in Fig. 2.

The convergence behavior of the distributed algorithm is summarized in the following theorem. The proof follows Theorem 2 in [15] and thus is omitted.

Theorem 2: Assume that the functions $D_i(\cdot), C_i(\cdot)$, and $F(\cdot)$ are closed, proper, and convex. Then the sequence $\{\mathbf{l}^k, \mathbf{u}^k, \mathbf{f}^k, \lambda^k\}$ converges to an optimal primal-dual solution of **P3** with the worst case convergence rate $O(1/k)$.

Remark: With the proposed distributed algorithm, each bus only needs to provide m_i^k to the substation without revealing the cost functions or the other parameters. Therefore, the communication requirement is limited and the privacy of each bus is well protected.

IV. Extension to Non-ideal Energy Storage

In this section, we discuss the algorithm design for non-ideal storage with imperfect charging and discharging.

The framework of the algorithm design follows that of ideal storage. However, due to the charging and discharging inefficiency, the variables $u_{i,t}^+$ and $u_{i,t}^-$ cannot be combined into one as we did in Section III, and therefore, the non-simultaneous charging and discharging constraint (2) cannot be eliminated. The strategy we take is to first ignore constraint (2) and then adjust the solution to fit (2). Specifically, the real-time problem at time slot t is designed as follows.

$$\begin{aligned} \textbf{P3'}: \quad \min_{\mathbf{a}_t} \quad & w_t + \sum_{i\in\mathcal{E}} \frac{(s_{i,t}-\beta_i)}{V_i}(u_{i,t}^+ - u_{i,t}^-) \\ \text{s.t.} \quad & (1),\ (5),\ (6),\ (7) \end{aligned}$$

where we have defined the perturbation parameter $\beta_i \triangleq s_{i,\min} + u_{i,\max} + V_i(\frac{p_{\max}}{\eta_i^+} + \frac{1}{\eta_i^+}C'_{i,\max} + D'_{i,\max})$. The parameter V_i lies in the interval $(0, V_{i,\max}]$, where $V_{i,\max} \triangleq \frac{s_{i,\max} - s_{i,\min} - 2u_{i,\max}}{\frac{p_{\max}}{\eta_i^+} - p_{\min}\eta_i^- + D'_{i,\max} - D'_{i,\min} + \frac{1}{\eta_i^+}C'_{i,\max} - \eta_i^- C'_{i,\min}}$.

Algorithm 2 Real-time algorithm for non-ideal storage.

At time slot t, the substation executes the following steps sequentially:

1: observe the system state $\mathbf{q}_t$ and the energy state $s_{i,t}$;
2: solve **P3'** and obtain a solution $\hat{\mathbf{a}}_t \triangleq [\hat{\mathbf{l}}_t, \hat{\mathbf{u}}_t^+, \hat{\mathbf{u}}_t^-, \hat{\mathbf{f}}_t]$;
3: generate a new solution $\mathbf{a}_t^*$ where $u_{i,t}^{+*} = \max\{\hat{u}_{i,t}^+ - \hat{u}_{i,t}^-, 0\}$, $u_{i,t}^{-*} = \max\{\hat{u}_{i,t}^- - \hat{u}_{i,t}^+, 0\}$, $l_{i,t}^* = \hat{l}_{i,t} + \eta_i^- \hat{u}_{i,t}^- - \frac{1}{\eta_i^+}\hat{u}_{i,t}^+ - \eta_i^- u_{i,t}^{-*} + \frac{1}{\eta_i^+} u_{i,t}^{+*}$, and $\mathbf{f}_t^* = \hat{\mathbf{f}}_t$; and
4: update $s_{i,t}$ by (3) using $u_{i,t}^{+*}$ and $u_{i,t}^{-*}$.

TABLE I
SETUP OF PARAMETERS AND FUNCTIONS IN EXPERIMENT

Par.	Setup	Par. (Fun.)	Setup
$[r_{i,\min}, r_{i,\max}]$	$[-8, 8]$	η_i^+, η_i^-	1
$[f_{i,\min}, f_{i,\max}]$	$[-5, 5]$	$C_i(x)$	$1.5x^2$
$[s_{i,\min}, s_{i,\max}]$	$[2, 10]$	$D_i(x)$	$0.2x^2$
$[p_{\min}, p_{\max}]$	$[7, 12]$	$F(x)$	$10x^2$
$u_{i,\max}$	1		

The overall algorithm is summarized in Algorithm 2, where we use the superscript notations $\hat{}$ and $*$ to indicate the intermediate solution derived from **P3'** and the final solution, respectively. To ensure that the final solution satisfies constraint (2), in Step 3, we adjust the intermediate solutions $\hat{u}_{i,t}^+$, $\hat{u}_{i,t}^-$, and $\hat{l}_{i,t}$ derived from **P3'**, so that simultaneous charging and discharging cannot happen. Note that since the non-convex constraint (2) is ignored in **P3'**, **P3'** is a convex problem and can be solved by a similar distributed algorithm as that in Section III-B.

Under some conditions, constraint (2) may automatically hold by solving **P3'**, and thus the adjustment in Step 3 is unnecessary, e.g., when the electricity price p_t is positive and the cost function of the controllable flow $C_i(\cdot)$ is increasing. In such a case, Theorem 1 can be easily extended to characterize the performance of Algorithm 2. However, in general, the solution of **P3'** may not meet constraint (2) and thus Step 3 in Algorithm 2 may be necessary. The performance analysis of Algorithm 2 in the general case is more complicated and is left for future work.

V. Numerical Results

In this section, we evaluate the performance of the proposed algorithm using an idealized numerical setup.

In simulation, all buses are equipped with ideal energy storage. The setup of the system parameters and functions are shown in Table I. The random system state $[\mathbf{r}_t, p_t]$ is assumed to be i.i.d. over time. The uncontrollable energy flow $r_{i,t}$ follows the Gaussian distribution $\mathcal{N}(0, 4^2)$ and is truncated within $[r_{i,\min}, r_{i,\max}]$. The electricity price p_t follows the uniform distribution. At each time the control action $\mathbf{a}_t$ is generated by Algorithm 1, and the algorithm is run for $T = 500$ time slots. For comparison, we consider a greedy algorithm, which minimizes the current system cost subject to all constraints of **P1** at each time slot.

In Figs. 3 and 4, the uncontrollable energy flows are modeled as independent among phases. Although the three-phase transmission is dominant in practice, we are interested

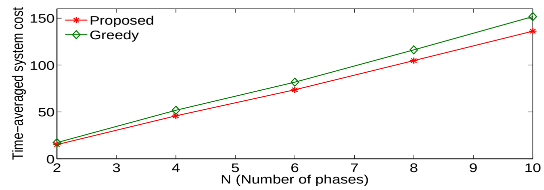

Fig. 3. System cost vs. number of phases.

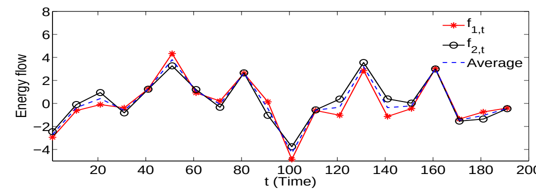

Fig. 4. Trace of energy flow between each phase and substation.

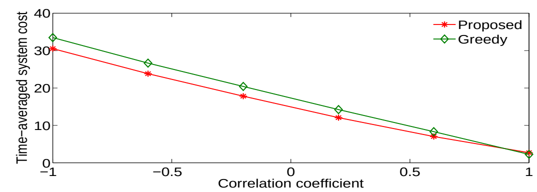

Fig. 5. System cost vs. correlation coefficient of uncontrollable flows.

in finding how the number of phases affects the algorithm performance. In Fig. 3, we increase the number of phases N from 2 to 10. We see that the system cost of both algorithms increases linearly with N. In addition, the proposed algorithm leads to a smaller slope, which indicates that it is more preferable than the greedy algorithm especially with more phases. For $N = 2$, we sample the energy flow between each bus and the substation every 10 time slots. The sampled energy flows and the corresponding average flow are shown in Fig. 4 for the first 200 time slots. For the energy flow of each bus, the average percentage of the deviation from the average flow is around 20%. This value can be decreased by increasing the coefficient of the loss function at the expense of a higher system cost.

In Fig. 5, for $N = 2$, we examine the effect of the correlation of the uncontrollable flows on the system performance. Interestingly, for both algorithms, the system cost decreases with the correlation coefficient. In other words, a lower system cost is incurred if the uncontrollable flows of two buses are more positively correlated, and vice versa, which confirms our intuition. Moreover, the benefit of applying the proposed algorithm increases as the uncontrollable flows are more negatively correlated.

VI. Conclusion and Future Work

We have investigated the problem of phase balancing with energy storage. We have proposed a distributed real-time algorithm for ideal energy storage and also extended the algorithm to accommodate non-ideal energy storage. For future work, we are interested in incorporating system statistics into the algorithm design to further improve performance, and also combing energy storage with traditional methods such as feeder reconfiguration for phase balancing.

APPENDIX

A. Proof of Lemma 1

Based on the definition of $L(s_{i,t})$ and the update of $s_{i,t}$,

$$L(s_{i,t+1}) - L(s_{i,t}) = \frac{1}{2}\left[(s_{i,t+1}-\beta_i)^2 - (s_{i,t}-\beta_i)^2\right] \le (s_{i,t}-\beta_i)u_{i,t} + \frac{1}{2}u_{i,\max}^2.$$

Using the above upper bound for all $i \in \mathcal{E}$ and then taking the conditional expectation give the final upper bound.

B. Proof of Theorem 1

1) Since the real-time problem **P3** includes all constraints of **P1** but the energy state constraint, the key of the feasibility proof is to show that the energy state $s_{i,t}$ is bounded within $[s_{i,\min}, s_{i,\max}]$. We first prove the following lemma which gives a sufficient condition for charging or discharging.

Lemma 2: Under Algorithm 1, for $i \in \mathcal{E}$,

1) if $s_{i,t} < \beta_i - V_i(p_{\max} + D'_{i,\max} + C'_{i,\max})$, then $u^*_{i,t} = u_{i,\max}$;
2) if $s_{i,t} > \beta_i - V_i(p_{\min} + D'_{i,\min} + C'_{i,\min})$, then $u^*_{i,t} = -u_{i,\max}$.

Proof: For simplicity of notation, we drop the time index t in **P3**. Using constraint (7) we replace l_j with $u_j - f_j - r_j$ in the objective of **P3**. Next we solve **P3** through the partitioning method by first fixing the optimization variables $\mathbf{f}$ and $u_j, j \neq i$, and then minimizing over u_i. The optimization problem with respect to u_i is as follows.

$$\min_{u_i}\ pu_i + D_i(u_i) + C_i(u_i - f_i - r_i) + \frac{(s_i - \beta_i)u_i}{V_i}, \text{s.t.}(9).$$

The derivative of the objective above with respect to u_i is $\frac{\partial(\cdot)}{\partial u_i} = p + D'_i(u_i) + C'_i(u_i - f_i - r_i) + \frac{(s_i-\beta_i)}{V_i}$. Therefore, if s_i is upper bounded as shown in Lemma 2.1), we have $\frac{\partial(\cdot)}{\partial u_i} < 0$ and thus $u^*_{i,t} = u_{i,\max}$. Or, if s_i is lower bounded as shown in Lemma 2.2), we have $\frac{\partial(\cdot)}{\partial u_i} > 0$ and thus $u^*_{i,t} = -u_{i,\max}$. ■

The boundedness of the energy state can be shown by mathematical induction using Lemma 2 and the definition of β_i. We omit the proof here due to limited space.

2) We prove Theorem 1.2 and 1.3 together. Denote $\tilde{w}$ as the optimal value of **P2**. In the following lemma, we show the existence of a special algorithm for **P2**. The proof follows Theorem 4.5 in [8] and is omitted for brevity.

Lemma 3: For **P2**, there exists a stationary and randomized solution $\mathbf{a}^s_t$ that only depends on the system state $\mathbf{q}_t$, and at the same time satisfies the following conditions:

$$\mathbb{E}[w^s_t] \le \tilde{w},\ \forall t, \quad \mathbb{E}[u^s_{i,t}] = 0,\ \forall i \in \mathcal{E}, t,$$

where the expectations are taken over the randomness of the system state and the possible randomness of the actions.

Using Lemmas 1 and 3, the drift-plus-cost function under Algorithm 1 can be upper bounded as follows.

$$\begin{aligned}
&\Delta(\mathbf{s}_t) + \mathbb{E}[w^*_t|\mathbf{s}_t] \\
&\le \mathbb{E}[w^s_t|\mathbf{s}_t] + \sum_{i\in\mathcal{E}}\left[\frac{u^2_{i,\max}}{2V_i} + \frac{s_{i,t}-\beta_i}{V_i}\mathbb{E}\big[u^s_{i,t}|\mathbf{s}_t\big]\right] \qquad (12)\\
&\le \tilde{w} + \sum_{i\in\mathcal{E}}\frac{u^2_{i,\max}}{2V_i} \qquad (13)\\
&\le w^{\text{opt}} + \sum_{i\in\mathcal{E}}\frac{u^2_{i,\max}}{2V_i} \qquad (14)
\end{aligned}$$

where (12) is derived based on Lemma 1 and the fact that **P3** minimizes the upper bound of the drift-plus-cost function, (13) is derived based on Lemma 3 and the fact that the action $\mathbf{a}^s_t$ is independent of $\mathbf{s}_t$, and the inequality in (14) holds since **P2** is a relaxed problem of **P1**.

Taking expectations over $\mathbf{s}_t$ on both sides of (14) and summing over $t \in \{0, \cdots, T-1\}$ yields

$$\sum_{i\in\mathcal{E}}\mathbb{E}\Big[\frac{L(s_{i,T}) - L(s_{i,0})}{V_i}\Big] + \sum_{t=0}^{T-1}\mathbb{E}[w^*_t] \le (w^{\text{opt}} + \sum_{i\in\mathcal{E}}\frac{u^2_{i,\max}}{2V_i})T.$$

Note that $L(s_{i,T})$ is non-negative. Divide both sides of the above inequality by T. After some arrangement, there is

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[w^*_t] - w^{\text{opt}} \le \sum_{i\in\mathcal{E}}\Big[\frac{u^2_{i,\max}}{2V_i} + \frac{\mathbb{E}[L(s_{i,0})]}{TV_i}\Big], \quad (15)$$

which is the conclusion in Theorem 1.3. Taking $\limsup$ on both sides of (15) gives Theorem 1.2.

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