Linear Processing for the Downlink in Multiuser MIMO Systems with Multiple Data Streams

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Abstract— **In this paper we solve the problem of linear precoding for the downlink in multiuser multiple-input multiple-output (MIMO) systems. The transmitter and the receivers may be equipped with multiple antennas and each user may receive multiple data streams. Our objective is to jointly optimize the power allocation and transmit-receive filters for all users. We develop the optimization for two different criteria; (1) minimizing the total transmitted power while satisfying SINR constraints and (2) minimizing the sum mean squared error given a total power budget. We take advantage of the duality between the uplink and downlink to derive the solution.**

I. Introduction

It is now well accepted that in wireless communications, exploiting the spatial dimension using antenna arrays at the transmitter and/or receiver can increase both the reliability and data rate of a transmission. More recently, researchers have investigated using such a multiple-input multiple-output (MIMO) system to service multiple users. This paper investigates a multiuser MIMO system in the downlink – a single transmitter communicating with multiple users. The system under consideration is general with an arbitrary number of transmit antennas and an arbitrary number of receive antennas at each user, who may possibly receive multiple data streams.

The performance of such a multiuser system is limited by mutual interference. Assuming knowledge of the channel state information (CSI) at the transmitter, our objective is to find a jointly optimal combination of power allocation and transmit-receive filters for given criteria of optimality. We investigate two criteria here: minimizing the total power required to meet a Quality of Service (QoS) constraint for every data stream, measured by the signal-to-interference plus noise ratio (SINR), or minimizing the sum mean squared error (SMSE) between the transmitted and received signals.

The available literature on the above optimization problem can be classified into two different approaches. The first approach uses block diagonalization, also known as block channel inversion. The inter-user interference among users is eliminated, leaving each user to deal with interference among its own data streams. Bourdoux and Khaled [1] solve the problem of inter-stream interference by minimizing the MSE for each user, while Spencer *et al.* [2] find the optimal transmit matrix to maximize the overall system throughput. The drawback of these null-space approaches is the requirement that the number of transmit antennas must be greater than or equal to the total number of receive antennas.

The second approach employs iterative algorithms that repeatedly cycle through power control, optimizing the transmit filter and optimizing the receive filter. This approach removes the previously stated constraint on the number of transmit antennas. The problem of transmitted power minimization with SINR constraints has been comprehensively investigated in literature. Schubert and Boche [3] propose a solution to solve the multiuser downlink problem with individual SINR constraints in a multiple-input single-output (MISO) system, i.e., a system where users have a single antenna. Their algorithm is based on the duality between the uplink and the downlink, and solves the problem iteratively in the uplink before switching to the downlink. This solution is extended in [4] to minimizing the SMSE. Chang *et al.* [5] also solve the downlink problem for single data stream MIMO systems. However, the algorithm diverges when some target SINR scenarios are infeasible. In [6], Doostnejad *et al.* combine the proposed scheme with dirty paper precoding and present a suboptimal solution. In our work, we focus on linear processing methods and generalize the scheme in [3] to MIMO systems with multiple data stream transmission. Our proposed algorithm does not have the drawback of diverging in some scenarios as in [5], which allows us to examine the problem of feasibility of a set of target SINRs.

In [7], the work by Sampath et al. on single user MIMO systems [8] is extended to the multiuser domain using an iterative joint optimization algorithm based on SMSE minimization and a per-user power constraint. A numerical method is also proposed that solves the problem with sum power constraint. Serbetli and Yener [9] study the problem in the uplink with per-user power constraint. They consider the problem of joint transmit-receive optimization while minimizing the SMSE. The proposed scheme allows for each user to transmit multiple data streams. In this paper, we generalize the scheme in [4] to MIMO systems with sum power constraint by making use of the uplink solution in [9] and by exploiting duality.

This paper is organized as follows. Section II states the assumptions made and describes the system model used. Section III studies the joint optimization problem of power minimization given a set of SINR constraints. Section IV investigates the problem of minimizing SMSE with a sum power constraint. Simulation results are presented in Section V, and the paper concludes with Section VI.

II. SYSTEM MODEL

Consider a single base station equipped with M antennas transmitting to K decentralized users. User k is equipped with N_k antennas and $N = \sum_{k=1}^{K} N_k$. In this general setup, user k receives L_k data streams from the base station and $L = \sum_{k=1}^{K} L_k$. Thus we have M transmit antennas transmitting a total of L symbols to K users, who have a total of N receive antennas. The symbols of each user are collected in the data vector $\mathbf{x}_k = [x_{k1}, x_{k2}, \ldots, x_{kL_k}]^T$ and the overall data vector is $\mathbf{x} = \left[\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_K^T\right]^T$.

User k's data streams are processed by the transmit filter $\mathbf{U}_k \in \mathcal{C}^{M \times L_k}$ before being transmitted over the M antennas. These individual precoders together form the global transmitter precoder matrix $\mathbf{U}_{M\times L} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_K]$. Let the downlink transmit power vector for user k be $\mathbf{p}_k = [p_{k1}, p_{k2}, \ldots, p_{kL_k}]^T$, with $\mathbf{p} = \left[\mathbf{p}_1^T, \ldots, \mathbf{p}_K^T\right]^T$, and define $\mathbf{P}_k = diag\{\mathbf{p}_k\}$ and $\mathbf{P} = diag\{\mathbf{p}\}$. The channel between the transmitter and user k is assumed flat and is represented by the $N_k \times M$ matrix $\mathbf{H}_k^H$. The resulting $N \times M$ channel matrix is $\mathbf{H}^H$, with $\mathbf{H} = [\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_K]$.

Based on this model, user k receives a length N_k vector

$$\mathbf{y}_k = \mathbf{H}_k^H \mathbf{U}\sqrt{\mathbf{P}}\mathbf{x} + \mathbf{n}_k, \tag{1}$$

where $\mathbf{n}_k$ represents the additive white Gaussian noise (AWGN) at the user's receive antennas with power σ^2; that is, $\mathbb{E}[\mathbf{n}_k\mathbf{n}_k^H] = \sigma^2\mathbf{I}_{N_k}$, where $\mathbb{E}[\cdot]$ represents the expectation operator. To estimate its L_k symbols $\mathbf{x}_k$, user k processes $\mathbf{y}_k$ with its $L_k \times N_k$ decoder matrix $\mathbf{V}_k^H$ resulting in

$$\hat{\mathbf{x}}_k^{DL} = \mathbf{V}_k^H \mathbf{H}_k^H \mathbf{U}\sqrt{\mathbf{P}}\mathbf{x} + \mathbf{V}_k^H \mathbf{n}_k. \tag{2}$$

The global receive filter $\mathbf{V}^H$ is a block diagonal decoder matrix of dimension $L \times N$, $\mathbf{V} = diag\left[\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_K\right]$.

We now construct a virtual uplink system that will prove very useful when exploiting the duality between the downlink and the uplink. Let the uplink transmit power vector for user k be $\mathbf{q}_k = [q_{k1}, q_{k2}, \ldots, q_{kL_k}]^T$, with $\mathbf{q} = [\mathbf{q}_1^T, \ldots, \mathbf{q}_K^T]^T$, and define $\mathbf{Q}_k = diag\{\mathbf{q}_k\}$ and $\mathbf{Q} = diag\{\mathbf{q}\}$. The transmit and receive filters for user k become $\mathbf{V}_k$ and $\mathbf{U}_k^H$ respectively. Fig. 1 illustrates the linear processing involved in the downlink and the virtual uplink for user k. The received vector at the base station and the estimated symbol vector for user k are

$$\begin{aligned} \mathbf{y} &= \sum_{i=1}^{K} \mathbf{H}_i \mathbf{V}_i \sqrt{\mathbf{Q}_i}\mathbf{x}_i + \mathbf{n}, && (3) \\ \hat{\mathbf{x}}_k^{UL} &= \sum_{i=1}^{K} \mathbf{U}_k^H \mathbf{H}_i \mathbf{V}_i \sqrt{\mathbf{Q}_i}\mathbf{x}_i + \mathbf{U}_k^H \mathbf{n}. && (4) \end{aligned}$$

Assume that the transmitted symbols are independent with unit power, i.e., $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}_L$. The noise, $\mathbf{n}$, is modelled as AWGN with $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \sigma^2\mathbf{I}_M$. To ensure resolvability, in the uplink and downlink, $L \le M$ and $L_k \le N_k$, $\forall k$.

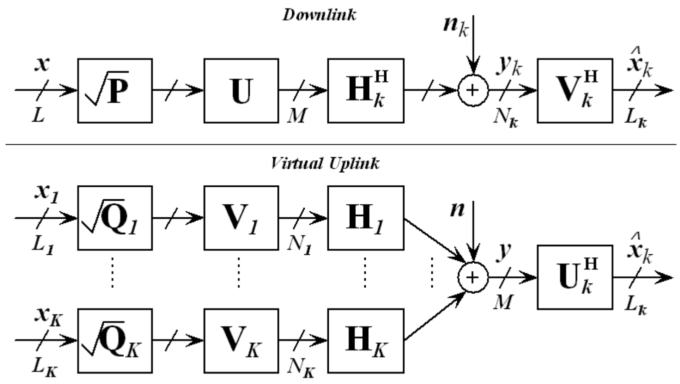

Fig. 1. Processing for user k in downlink and virtual uplink.

III. POWER MINIMIZATION WITH SINR CONSTRAINTS

Let γ_{kj} be the target SINR for the j^{th} substream of user k and the total transmitted power be $P_{max} = \|\mathbf{p}\|_1$. The power minimization problem can then formulated as

$$\begin{aligned} \min_{\mathbf{p},\mathbf{U},\mathbf{V}} P_{max} &= \sum_{k=1}^{K}\sum_{j=1}^{L_k} p_{kj} \\ \text{subject to :} \quad & SINR_{kj} \ge \gamma_{kj}. \end{aligned} \tag{5}$$

Let $\mathbf{u}_{kj}(M \times 1)$ and $\mathbf{v}_{kj}(N_k \times 1)$ be the transmit and receive beamforming vectors for substream j of user k respectively, i.e., $\mathbf{U}_k = [\mathbf{u}_{k1}, \mathbf{u}_{k2}, \ldots, \mathbf{u}_{kL_k}]$ and $\mathbf{V}_k = [\mathbf{v}_{k1}, \mathbf{v}_{k2}, \ldots, \mathbf{v}_{kL_k}]$. Using (2), the SINR of each stream in the downlink is

$$SINR_{kj}^{DL} = p_{kj}\frac{\mathbf{v}_{kj}^H \mathbf{S}_{kj}^{DL}\mathbf{v}_{kj}}{\mathbf{v}_{kj}^H \mathbf{T}_{kj}^{DL}\mathbf{v}_{kj}}, \tag{6}$$

where

$$\begin{aligned} \mathbf{S}_{kj}^{DL} &= \mathbf{H}_k^H \mathbf{u}_{kj}\mathbf{u}_{kj}^H \mathbf{H}_k, && (7) \\ \mathbf{T}_{kj}^{DL} &= \underbrace{\sum_{l=1, l\neq j}^{L_k} p_{kl}\mathbf{H}_k^H \mathbf{u}_{kl}\mathbf{u}_{kl}^H \mathbf{H}_k}_{\text{intra-user interference}} \\ &\quad + \underbrace{\sum_{i=1, i\neq k}^{K}\sum_{m=1}^{L_i} p_{im}\mathbf{H}_k^H \mathbf{u}_{im}\mathbf{u}_{im}^H \mathbf{H}_k}_{\text{inter-user interference}} + \sigma^2\mathbf{I}. && (8) \end{aligned}$$

In the virtual uplink, using (4), the SINR expression is

$$SINR_{kj}^{UL} = q_{kj}\frac{\mathbf{u}_{kj}^H \mathbf{S}_{kj}^{UL}\mathbf{u}_{kj}}{\mathbf{u}_{kj}^H \mathbf{T}_{kj}^{UL}\mathbf{u}_{kj}}, \tag{9}$$

where

$$\begin{aligned} \mathbf{S}_{kj}^{UL} &= \mathbf{H}_k \mathbf{v}_{kj}\mathbf{v}_{kj}^H \mathbf{H}_k^H, && (10) \\ \mathbf{T}_{kj}^{UL} &= \sum_{l=1, l\neq j}^{L_k} q_{kl}\mathbf{H}_k \mathbf{v}_{kl}\mathbf{v}_{kl}^H \mathbf{H}_k^H \\ &\quad + \sum_{i=1, i\neq k}^{K}\sum_{m=1}^{L_i} q_{im}\mathbf{H}_i \mathbf{v}_{im}\mathbf{v}_{im}^H \mathbf{H}_i^H + \sigma^2\mathbf{I}. && (11) \end{aligned}$$

A. Power Allocation for Fixed Transmit-Receive Filters

In this section, we extend the results obtained in [3] from single data stream MISO systems to multiple data stream MIMO systems. The authors define the problem

$$C^{DL} = \max \min_{1 \le k \le K, 1 \le j \le L_k} \frac{SINR_{kj}^{DL}}{\gamma_{kj}} \text{ s.t. } \|\mathbf{p}\|_1 \le P_{max}. \quad (12)$$

If $C^{DL} \ge 1$, then the set of SINR targets is feasible. Otherwise, we have infeasible targets and other methods like dropping some users or lowering the SINR targets are necessary. In [3], it is shown that for a fixed transmit filter $\mathbf{U}$, and due to C^{DL} being monotonically increasing in P_{max}, the global optimum $\mathbf{p}$ achieves active SINR constraints, i.e.,

$$C^{DL} = \frac{SINR_{kj}^{DL}}{\gamma_{kj}} \qquad \forall\, k \text{ and } j. \quad (13)$$

It can be easily proven that (13) still holds for the MIMO case, when $\mathbf{U}$ and $\mathbf{V}$ are fixed. Similar to the scheme in [3], by collecting the L equations from (13) we can construct an eigensystem in the downlink and the virtual uplink:

$$\mathbf{\Upsilon} \mathbf{p}_{ext} = \frac{1}{C^{DL}} \mathbf{p}_{ext} \text{ and } \mathbf{\Lambda} \mathbf{q}_{ext} = \frac{1}{C^{UL}} \mathbf{q}_{ext}, \quad (14)$$

where

$$\mathbf{\Upsilon} = \begin{bmatrix} \mathbf{D\Psi} & \mathbf{D}\boldsymbol{\sigma} \\ \frac{1}{P_{max}} \mathbf{1}^T \mathbf{D\Psi} & \frac{1}{P_{max}} \mathbf{1}^T \mathbf{D}\boldsymbol{\sigma} \end{bmatrix} \quad (15)$$

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{D\Psi}^T & \mathbf{D}\boldsymbol{\sigma} \\ \frac{1}{P_{max}} \mathbf{1}^T \mathbf{D\Psi}^T & \frac{1}{P_{max}} \mathbf{1}^T \mathbf{D}\boldsymbol{\sigma} \end{bmatrix} \quad (16)$$

and $\mathbf{D} = diag\left\{ \frac{\gamma_{11}}{|\mathbf{v}_{11}^H \mathbf{H}_1^H \mathbf{u}_{11}|^2}, \ldots, \frac{\gamma_{KL_K}}{|\mathbf{v}_{KL_K}^H \mathbf{H}_K^H \mathbf{u}_{KL_K}|^2} \right\}$, $\mathbf{p}_{ext} = \left(\mathbf{p}^T \ 1\right)^T$, $\mathbf{q}_{ext} = \left(\mathbf{q}^T \ 1\right)^T$, $\boldsymbol{\sigma} = \sigma^2 \mathbf{1}$ (where $\mathbf{1}$ is the all ones vector of appropriate dimension), and $\mathbf{\Psi}$ is the coupling matrix. Refer to Appendix A for the structure of $\mathbf{\Psi}$.

The optimal power allocation (downlink and uplink) that maximizes the SINRs satisfies (14). In other words, the best strategy to find $\mathbf{p}$ (or $\mathbf{q}$) is to obtain the dominant eigenvector of $\mathbf{\Upsilon}$ (or $\mathbf{\Lambda}$). As shown in [3] and its references, this solution to (14) yields the only positive pair $(C^{DL}, \mathbf{p}_{ext})$ or $(C^{UL}, \mathbf{q}_{ext})$. If $C^{DL} \ge 1$, then the set of SINR targets is feasible. To minimize the total transmit power while meeting the required SINR targets, the power allocation policy sets $SINR_{kj} = \gamma_{kj}$ for all L streams. The resulting power vectors are

$$\mathbf{p} = \sigma^2 (\mathbf{D}^{-1} - \mathbf{\Psi})^{-1} \mathbf{1} \text{ and } \mathbf{q} = \sigma^2 (\mathbf{D}^{-1} - \mathbf{\Psi}^T)^{-1} \mathbf{1}. \quad (17)$$

Several studies have identified an interesting duality between the uplink and the downlink. This duality states that given a fixed transmit filter and a sum power constraint, the same balanced SINR level is achieved in both directions, i.e., $C^{DL} = C^{UL}$. In [6], the authors extend the duality proof from MISO to MIMO systems with dirty paper precoding. We use a similar approach to state the following theorem:

Theorem 1: With linear processing matrices $\mathbf{U}$ at the base station and $\mathbf{V}$ over all users, and $\|\mathbf{p}\|_1 = \|\mathbf{q}\|_1$, $C^{DL} = C^{UL}$.

Proof: See Appendix A.

Consequently, for a fixed $\mathbf{U}$, $\mathbf{V}$ and P_{max}, there exist downlink/uplink power allocations $\mathbf{p}$ and $\mathbf{q}$ such that $\|\mathbf{p}\|_1 = \|\mathbf{q}\|_1 = P_{max}$, and $SINR_{kj}^{DL} = SINR_{kj}^{UL}$ for all L streams. This remarkable result is very useful to our problem since it allows us to solve the optimization problem in either the downlink or the virtual uplink.

B. Transmit-Receive Filters for Fixed Power Allocation

We now investigate the reverse problem; assuming a fixed power allocation $\mathbf{p}$ and $\mathbf{q}$, we are interested in finding the optimal transmit-receive filters $\mathbf{U}$ and $\mathbf{V}$. Examining (6), we observe that in the downlink, and for a fixed $\mathbf{U}$, $SINR_{kj}^{DL}$ is a function of $\mathbf{v}_{kj}$ and is independent of all other receive beamformers. Hence, the receive beamformers can be optimized independently such that $\mathbf{v}_{kj}^{opt} = \arg\max_{\mathbf{v}_{kj}} SINR_{kj}^{DL} = \hat{\mathbf{e}}_{max}(\mathbf{S}_{kj}^{DL}, \mathbf{T}_{kj}^{DL})$, where $\hat{\mathbf{e}}_{max}(\mathbf{A}, \mathbf{B})$ is the dominant generalized eigenvector of the matrix pair $(\mathbf{A}, \mathbf{B})$. Equivalently in the uplink, for a fixed $\mathbf{V}$, $\mathbf{u}_{kj}^{opt} = \hat{\mathbf{e}}_{max}(\mathbf{S}_{kj}^{UL}, \mathbf{T}_{kj}^{UL})$. Note that this solution is equivalent to the MMSE filter due to the fact that $\mathbf{S}_{kj}^{UL}$ and $\mathbf{S}_{kj}^{DL}$ are rank-1 matrices.

The proposed algorithm, summarized in Table I, optimizes each variable by fixing the other variables, and iterating between the uplink and the downlink. Unlike the algorithm in [5], this algorithm does not diverge even if the set of target SINRs is infeasible, such as in the case when $M < K$. Convergence is ensured by using an initial power control policy in (14) that accounts for infeasibility, i.e., when $C^{DL} = C^{UL} < 1$.

TABLE I

POWER MINIMIZATION ALGORITHM

Initialization: $C = 0, \mathbf{U} = [\mathbf{1}, \ldots, \mathbf{1}]$ and $\mathbf{p} = (P_{max}/L)\,[1, \ldots, 1]^T$
Iteration:
1- *Downlink Receive Beamforming* (for $k = 1:K, j = 1:L_k$)
 $\mathbf{v}_{kj} = \hat{\mathbf{e}}_{max}(\mathbf{S}_{kj}^{DL}, \mathbf{T}_{kj}^{DL})$
 $\mathbf{v}_{kj} = \mathbf{v}_{kj}/\|\mathbf{v}_{kj}\|$
2- *Virtual Uplink Power Allocation*
 If $C < 1$
  solve $\mathbf{\Lambda}\mathbf{q}_{ext} = \frac{1}{\lambda_{max}}\mathbf{q}_{ext}$, let $C = 1/\lambda_{max}$
 else
  $\mathbf{q} = \sigma^2(\mathbf{D}^{-1} - \mathbf{\Psi}^T)^{-1}\mathbf{1}$
3- *Virtual Uplink Receive Beamforming* (for $k = 1:K, j = 1:L_k$)
 $\mathbf{u}_{kj} = \hat{\mathbf{e}}_{max}(\mathbf{S}_{kj}^{UL}, \mathbf{T}_{kj}^{UL})$
 $\mathbf{u}_{kj} = \mathbf{u}_{kj}/\|\mathbf{u}_{kj}\|$
4- *Downlink Power Allocation*
 If $C < 1$
  solve $\mathbf{\Upsilon}\mathbf{p}_{ext} = \frac{1}{\lambda_{max}}\mathbf{p}_{ext}$, let $C = 1/\lambda_{max}$
 else
  $\mathbf{p} = \sigma^2(\mathbf{D}^{-1} - \mathbf{\Psi})^{-1}\mathbf{1}$
  $P_{min} = \|\mathbf{p}\|_1$
5- *Repeat steps 1-4 until convergence*

IV. SMSE MINIMIZATION WITH POWER CONSTRAINT

We now address the popular problem of minimizing the sum mean squared error. This problem is solved in a computationally intensive manner using sequential quadratic programming (SQP) in [7]. Let $\mathbf{E}_k^{DL}$ be the $L_k \times L_k$ error covariance matrix of user k in the downlink, where

$$\mathbf{E}_k^{DL} = \mathbb{E}\left[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^H\right]. \quad (18)$$

The diagonal entries of $\mathbf{E}_k^{DL}$ are the MSEs of the L_k substreams of user k and thus $SMSE_k^{DL}$=tr[$\mathbf{E}_k^{DL}$], where tr[$\cdot$] is the trace operator. The SMSE minimization problem is

$$\min_{\mathbf{p},\mathbf{U},\mathbf{V}} \sum_{k=1}^{K} \text{tr}[\mathbf{E}_k^{DL}] \tag{19}$$
$$\text{subject to} : \|\mathbf{p}\|_1 \leq P_{max}.$$

The authors of [4] solve the same problem for MISO systems. They prove that under a total power constraint, for a given $\mathbf{U}$, the uplink and the downlink have the same normalized MSE achievable region. We extend these results to MIMO systems with possibly multiple data streams per user:

Theorem 2: For a given $\mathbf{U}$, $\mathbf{V}$ and P_{max}, there exist power allocations $\mathbf{p}$ and $\mathbf{q}$ such that $MSE_{kj}^{UL} = MSE_{kj}^{DL}$, where MSE_{kj} is the MSE of the j^{th} substream of user k.

Proof: See Appendix B.

We can therefore solve problem (19) in the virtual uplink and transform the resulting transmit-receive filters to the downlink. Using (4) and expanding (18) in the virtual uplink, the resulting error covariance matrix of user k is

$$\begin{aligned}\mathbf{E}_k^{UL} \;=\; & \mathbf{U}_k^H\mathbf{H}\mathbf{V}\mathbf{Q}\mathbf{V}^H\mathbf{H}^H\mathbf{U}_k + \sigma^2\mathbf{U}_k^H\mathbf{U}_k + \mathbf{I}_{L_k} \\ & -\mathbf{U}_k^H\mathbf{H}_k\mathbf{V}_k\sqrt{\mathbf{Q}_k} - \sqrt{\mathbf{Q}_k}\mathbf{V}_k^H\mathbf{H}_k^H\mathbf{U}_k. \end{aligned} \tag{20}$$

The optimum uplink MMSE receiver is

$$\mathbf{U}_k^{MMSE} = \mathbf{J}^{-1}\mathbf{H}_k\mathbf{V}_k\sqrt{\mathbf{Q}_k}, \tag{21}$$

$$\text{where} \quad \mathbf{J} = \mathbf{H}\mathbf{V}\mathbf{Q}\mathbf{V}^H\mathbf{H}^H + \sigma^2\mathbf{I}_M. \tag{22}$$

$$\Rightarrow \mathbf{E}_k^{UL,MMSE} = \mathbf{I}_{L_k} - \sqrt{\mathbf{Q}_k}\mathbf{V}_k^H\mathbf{H}_k^H\mathbf{J}^{-1}\mathbf{H}_k\mathbf{V}_k\sqrt{\mathbf{Q}_k}. \tag{23}$$

The sum MSE of the whole system is therefore

$$SMSE = \sum_{k=1}^{K} \text{tr}[\mathbf{E}_k^{UL,MMSE}] = L - M + \sigma^2\text{tr}[\mathbf{J}^{-1}]. \tag{24}$$

The SMSE expression in (24) is a function of two variables; uplink power allocation $\mathbf{Q}$ and uplink global transmit filter $\mathbf{V}$. We first assume that $\mathbf{V}$ is fixed. Therefore, minimizing SMSE is equivalent to minimizing the trace of $\mathbf{J}^{-1}$. The resulting optimization problem is convex in $\mathbf{Q}$ [4], making it relatively easy to solve:

$$\mathbf{Q}^{opt} = \arg\min_{\mathbf{Q}} \text{tr}[\mathbf{J}^{-1}], \text{ subject to tr}[\mathbf{Q}] = P_{max}. \tag{25}$$

The next step is to optimize $\mathbf{V}$ for a fixed power allocation $\mathbf{Q}$. The authors of [9] propose a scheme for calculating $\mathbf{V}$ given per-user power constraints. Similarly, we are interested in generalizing the scheme to allow for a sum power constraint. Using the matrix inversion lemma,

$$\mathbf{J}^{-1} = \mathbf{J}_{kj}^{-1} - q_{kj}\frac{\mathbf{J}_{kj}^{-1}\mathbf{H}_k\mathbf{v}_{kj}\mathbf{v}_{kj}^H\mathbf{H}_k^H\mathbf{J}_{kj}^{-1}}{1 + q_{kj}\mathbf{v}_{kj}^H\mathbf{H}_k^H\mathbf{J}_{kj}^{-1}\mathbf{H}_k\mathbf{v}_{kj}}, \tag{26}$$

$$\text{where} \quad \mathbf{J}_{kj} = \mathbf{J} - q_{kj}\mathbf{H}_k\mathbf{v}_{kj}\mathbf{v}_{kj}^H\mathbf{H}_k^H. \tag{27}$$

If we substitute (26) in (24), we get

$$SMSE = \mathbf{W}_{kj} - \sigma^2\frac{\mathbf{v}_{kj}^H(\mathbf{H}_k^H\mathbf{J}_{kj}^{-2}\mathbf{H}_k)\mathbf{v}_{kj}}{\mathbf{v}_{kj}^H(\mathbf{I}/q_{kj} + \mathbf{H}_k^H\mathbf{J}_{kj}^{-1}\mathbf{H}_k)\mathbf{v}_{kj}}, \tag{28}$$

where $\mathbf{W}_{kj}$ contains all the terms independent of the beamforming vector $\mathbf{v}_{kj}$ [9]. Thus the optimal $\mathbf{v}_{kj}$, which minimizes SMSE for a given power allocation when the beamformring vectors of all other streams are fixed, is the dominant generalized eigenvector of the matrix pair $(\mathbf{H}_k^H\mathbf{J}_{kj}^{-2}\mathbf{H}_k,\ \mathbf{I}/q_{kj} + \mathbf{H}_k^H\mathbf{J}_{kj}^{-1}\mathbf{H}_k)$. Note that while each step of the iteration is optimal, it is not guaranteed that the algorithm will converge to the globally optimal solution.

TABLE II

SMSE MINIMIZATION ALGORITHM

Initialization: $\mathbf{V_k}$ = SVD($\mathbf{H}_k$) and $\mathbf{q} = (P_{max}/L)\,[1,\ldots,.1]^T$
Iteration:
1- *Virtual Uplink Transmit Beamforming* (for $k = 1:K, j = 1:L_k$)
$\mathbf{v}_{kj} = \hat{\mathbf{e}}_{max}(\mathbf{H}_k^H\mathbf{J}_{kj}^{-2}\mathbf{H}_k, \mathbf{I}/q_{kj} + \mathbf{H}_k^H\mathbf{J}_{kj}^{-1}\mathbf{H}_k)$
$\mathbf{v}_{kj} = \mathbf{v}_{kj}/\|\mathbf{v}_{kj}\|$
2- *Virtual Uplink Power Allocation*
$\mathbf{q} = \arg\min_{\mathbf{q}}$ tr[$\mathbf{J}^{-1}$], subject to $q_{kj} \geq 0$, $\|\mathbf{q}\|_1 = P_{max}$
3- *Repeat 1-2 until* $oldSMSE - newSMSE < \epsilon$
Update:
4- *Downlink Transmit Beamforming* (for $k = 1:K$)
$\mathbf{U}_k = \mathbf{J}^{-1}\mathbf{H}_k\mathbf{V}_k\sqrt{\mathbf{Q}_k}$
5- *Set target SINR to actual SINR* (for $k = 1:K, j = 1:L_k$)
$\gamma_{kj} = SINR_{kj}^{UL}$
6- *Downlink Power Allocation*
$\mathbf{p} = \sigma^2(\mathbf{D}^{-1} - \mathbf{\Psi})^{-1}\mathbf{1}$

In the initialization step above, SVD refers to singular value decomposition.

The proposed algorithm in Table II first solves the problem iteratively in the virtual uplink and then transfers the filters to the downlink. The initialization step starts with the uplink power allocation $\mathbf{Q}$ by distributing P_{max} evenly among all data streams. $\mathbf{V}_k$ = SVD($\mathbf{H}_k$) indicates that the L_k dominant right singular vectors of $\mathbf{H}_k$ (corresponding to the L_k largest singular values) are used to initialize $\mathbf{V}_k$. The next step involves optimizing $\mathbf{V}$ and $\mathbf{Q}$ iteratively. Each iteration begins with optimizing $\mathbf{V}$ for the previous power allocation $\mathbf{Q}$, where the optimum $\mathbf{v}_{kj}$ of every stream is found using the dominant eigenvector method. After the beamforming vectors of all streams are found, power allocation is made by solving the convex optimization problem (25). The iterative step is executed repeatedly until SMSE converges to its final value. The downlink transmit filter $\mathbf{U}$ is then found using the MMSE receiver (21). Finally, the algorithm exploits duality by setting $\gamma_{kj} = SINR_{kj}^{UL}$, which leads to $C^{DL} = C^{UL} = 1$. The last step finds $\mathbf{p}$, the downlink power allocation, using (17).

V. Numerical Examples

In this section we illustrate the performance of the proposed algorithms. The channel is modelled as Rayleigh with the channel matrix $\mathbf{H}$ comprising independent and identically distributed (iid) samples of a complex Gaussian process with zero mean and unit variance. The examples use a noise variance of $\sigma^2 = 1$. The transmitter is assumed to know the channel perfectly.

Fig. 2 examines the power minimization algorithm for different scenarios. The figure plots the minimal transmitted

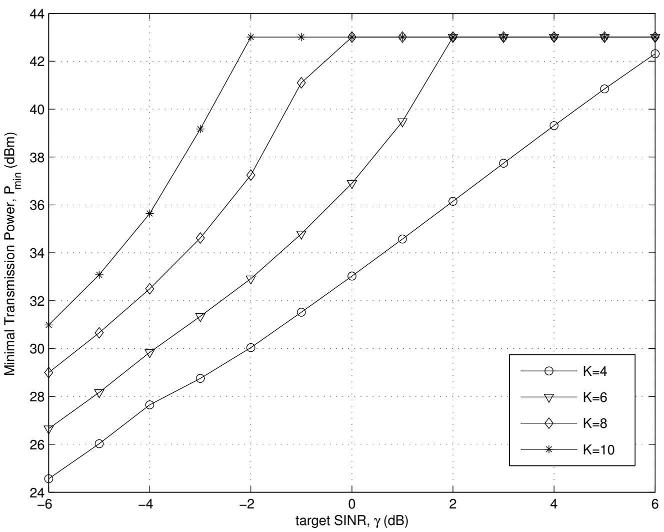

Fig. 2. $P_{\min}$ vs. γ for $M = 8$, $L_k = 2$ and $N_k = 2$.

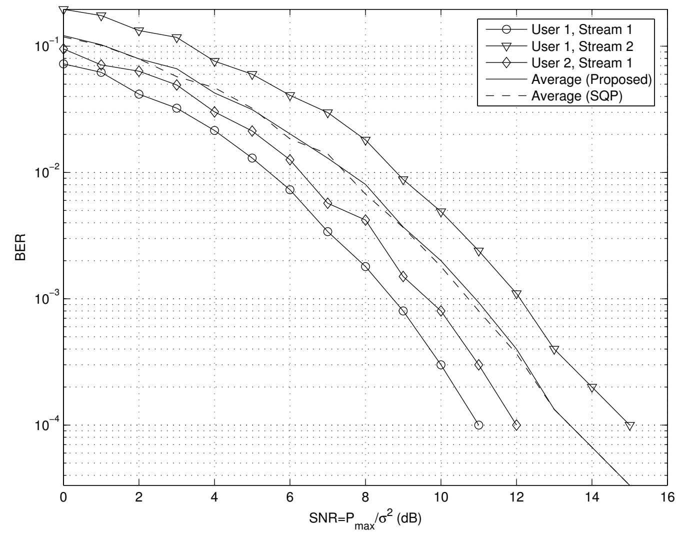

Fig. 4. BER vs. SNR for SMSE minimization.

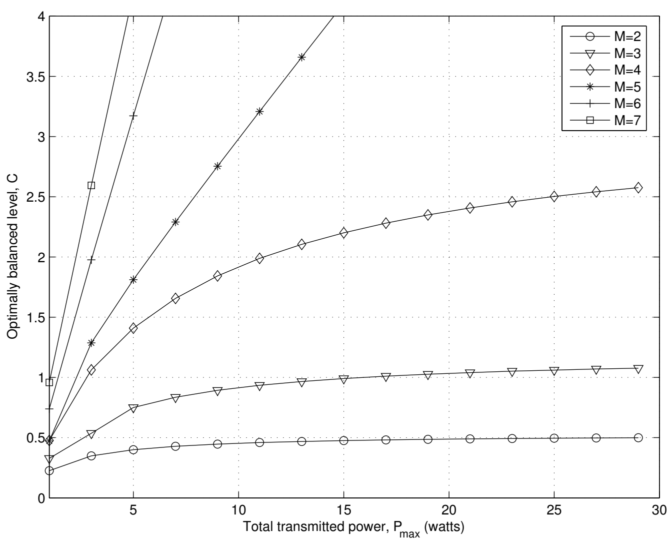

Fig. 3. C vs. P_{max} for γ=1dB, K=3, **L**=[2 1 2], and **N**=[2 2 3].

power required to satisfy the same SINR constraint γ for all K users. The example uses $M = 8$ transmit antennas. Every user is equipped with $N_k = 2$ antennas and receives $L_k = 2$ data streams. The maximum available power is set fixed at $P_{\max} = 43$dBm (setting the level for infeasibility). As expected, increasing either the SINR targets or the number of users requires an increase in the minimum total power needed to successfully complete the transmission. We also notice that the scenario becomes infeasible when $K \geq M$ and $\gamma > 0$dB, but the proposed algorithm doesn't diverge as the case in [5].

In Fig. 3, we plot C versus the total transmission power P_{max}, where $C = C^{UL} = C^{DL}$ is the optimally balanced level of actual to target SINRs. The figure shows C for varying numbers of transmit antennas M. As stated previously, the target SINRs ($\gamma =$1dB) can only be achieved for a given P_{max} if $C \geq 1$. From the figure, we notice that for $M \geq K$, the SINR targets are feasible; i.e., there exists a total power P_{max} that satisfies γ. We notice that as the number of users in the network increases, it becomes more difficult to achieve desired targets for all the users. When the scenario is infeasible, the network has either to drop some users and try to optimize the link again, or the target SINRs have to be relaxed.

Finally, Fig. 4 plots the average bit error rate (BER) versus signal-to-noise ratio (SNR) for the SMSE minimization algorithm, where SNR=P_{max}/σ^2. The figure also plots the BER when using the SQP approach of [7]. The simulated system has $K = 2$ users, receiving $L_1 = 2$ and $L_2 = 1$ data streams respectively. Both users are equipped with $N_k = 2$ antennas each. The figure shows that the proposed algorithm achieves an almost identical performance in BER to the SQP algorithm which suffers from being computationally intensive. However, our algorithm's implementation suggests that the iterative approach has a computational complexity an order-of-magnitude lower than the SQP algorithm.

VI. Conclusions

We have proposed a solution to the problem of joint beamforming and power allocation in the downlink of multiuser MIMO systems, in the most general setting of multiple transmit antennas and multiple users with possibly multiple receive antennas receiving possibly multiple data streams. We proposed an algorithm to find the optimal combination of transmit-receive filters and transmission powers to minimize the total transmitted power while satisfying individual SINR targets for each data stream. The second goal was to find the same combination that minimizes the sum mean squared error under a total power constraint. Both solutions are based on uplink-downlink duality.

VII. Appendix

Without loss of generality, we simplify the proof by transforming the system from one having K users with L_k data streams each into a system having L virtual users with single data stream each. The transmit and receive filters become $\mathbf{U}_{M\times L} = [\mathbf{u}_1\ \mathbf{u}_2\ ..\ \mathbf{u}_L]$ and $\mathbf{V}_{N\times L} = diag\{\mathbf{v}_1, \mathbf{v}_2,.., \mathbf{v}_L\}$.

A. Proof of Uplink-Downlink Duality for SINR

Define the coupling matrix $\mathbf{\Psi}$ such that

$$[\mathbf{\Psi}]_{ik} = \begin{cases} |\mathbf{v}_i^H \mathbf{H}_i^H \mathbf{u}_k|^2 = |\mathbf{u}_k^H \mathbf{H}_i \mathbf{v}_i|^2 & k \neq i \\ 0 & k = i. \end{cases} \tag{29}$$

It has been shown (see [10]) that for the set of SINR constraints to be achievable in the downlink, $\lambda_{max}(\mathbf{D\Psi}) < 1$, where $\lambda_{max}(.)$ is the maximum eigenvalue operator. Similary, $\lambda_{max}(\mathbf{D\Psi}^T) < 1$ guarantees achievability in the uplink. The authors in [11] prove that $\lambda_{max}(\mathbf{D\Psi}) = \lambda_{max}(\mathbf{D\Psi}^T)$. Consequently, the SINR achievable region is the same for the uplink and the downlink, with power allocations $\mathbf{q} > 0$ and $\mathbf{p} > 0$ respectively. Using (17) we write

$$\begin{aligned} \|\mathbf{q}\|_1 = \mathbf{1}^T\mathbf{q} &= \sigma^2 \mathbf{1}^T(\mathbf{D}^{-1} - \mathbf{\Psi}^T)^{-1}\mathbf{1} \\ &= \sigma^2 \mathbf{1}^T(\mathbf{D}^{-1} - \mathbf{\Psi})^{-1}\mathbf{1} = \mathbf{1}^T\mathbf{p} = \|\mathbf{p}\|_1. \end{aligned}$$

With the total power in the uplink and the downlink being identical, the SINR targets are achieved and $C^{DL} = C^{UL}$.

B. Proof of Uplink-Downlink Duality for MSE

Along the same lines of the proof presented in [4] for MISO systems, we generalize the MSE duality to systems with multiple antennas at the receiver. In what follows, we adopt the notations $\tilde{\mathbf{v}}_i$ and $\tilde{\mathbf{u}}_i$ for the MMSE receive beamforming vectors in the downlink and the uplink respectively, where

$$\tilde{\mathbf{v}}_i = (\mathbf{H}_i^H \mathbf{UPU}^H \mathbf{H}_i + \sigma^2 \mathbf{I})^{-1} \mathbf{H}_i^H \mathbf{u}_i \sqrt{p_i}, \tag{30}$$

$$\tilde{\mathbf{u}}_i = (\mathbf{HVQV}^H \mathbf{H}^H + \sigma^2 \mathbf{I})^{-1} \mathbf{H}_i \mathbf{v}_i \sqrt{q_i}. \tag{31}$$

Also let $\mathbf{v}_i = \tilde{\mathbf{v}}_i / \|\tilde{\mathbf{v}}_i\|$ and $\mathbf{u}_i = \tilde{\mathbf{u}}_i / \|\tilde{\mathbf{u}}_i\|$. We write the MSE expressions for user i in the downlink and the uplink respectively:

$$\begin{aligned} \varepsilon_i^{DL} &= \tilde{\mathbf{v}}_i^H \mathbf{H}_i^H \mathbf{UPU}^H \mathbf{H}_i \tilde{\mathbf{v}}_i + \sigma^2 \|\tilde{\mathbf{v}}_i\|^2 + 1 \\ &\quad - \sqrt{p_i} \tilde{\mathbf{v}}_i^H \mathbf{H}_i^H \mathbf{u}_i - \sqrt{p_i} \mathbf{u}_i^H \mathbf{H}_i \tilde{\mathbf{v}}_i, \end{aligned} \tag{32}$$

$$\begin{aligned} \varepsilon_i^{UL} &= \tilde{\mathbf{u}}_i^H \mathbf{HVQV}^H \mathbf{H}^H \tilde{\mathbf{u}}_i + \sigma^2 \|\tilde{\mathbf{u}}_i\|^2 + 1 \\ &\quad - \sqrt{q_i} \mathbf{v}_i^H \mathbf{H}_i^H \tilde{\mathbf{u}}_i - \sqrt{q_i} \tilde{\mathbf{u}}_i^H \mathbf{H}_i \mathbf{v}_i. \end{aligned} \tag{33}$$

Setting $SINR_i^{DL} = SINR_i^{UL}$, we can prove by simple mathematical manipulation that when $\|\tilde{\mathbf{v}}_i\| = \beta/\sqrt{p_i}$ and $\|\tilde{\mathbf{u}}_i\| = \beta/\sqrt{q_i}$, where β is a scalar, we have $\varepsilon_i^{DL} = \varepsilon_i^{UL}$.

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