Symbol Error Rate of Selection Amplify-and-Forward Relay Systems

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Abstract

Cooperative diversity schemes significantly improve the performance of wireless networks by transmitting the same information through several nodes. The amplify-and-forward (AF) relaying method is one of the most attractive cooperative diversity schemes due to its low complexity. Selection AF relaying has recently been proven to achieve the same diversity order as and lower outage probability than All-Participate relays. In this letter, we present an asymptotic analysis of the symbol error rates of a selection AF network, and compare it with the conventional all-participate scheme.

I. INTRODUCTION

Cooperative diversity gain is available in distributed wireless networks with nodes that help each other to relay transmissions [1]–[3]. Amplify-and-Forward (AF) relaying is one of the most popular cooperation protocols where the relay simply amplifies the signal received from the source and transmits the amplified signal to the destination. It requires no decoding at relay nodes and hence is well-suited to systems with simple relay units such as wireless sensor networks.

Conventional AF was studied in [1], where given m potential relays, the available channel resources are split into $m+1$ orthogonal channels through either time or frequency division. All m relays then help the source, achieving order-$(m+1)$ diversity. However, the throughput of such an "all participate" AF (AP-AF) network is limited by the orthogonal partition of the system resources, especially when the number of relay nodes is large. To solve this problem, a new cooperation structure called Selection AF (S-AF) was introduced in our work [4], extending the selection scheme of [5] to the AF protocol. In S-AF, only the node that contributes the most to the received signal-to-noise-ratio (SNR) is chosen as the active relay. The selection algorithm is implemented at the destination, which is assumed to have knowledge of all channel gains, including those between the source and all relays.

The outage performances of the two AF schemes were examined in [4]. We showed that both S-AF and AP-AF achieve the maximum diversity order of $m+1$, but S-AF achieves a higher throughput and lower outage probability than AP-AF except when SNR is very low. In addition to outage probability, the symbol error rate (SER) as a performance metric has recently drawn growing attention. The SER of the AP-AF scheme was derived in [6]. We expect that S-AF has the same advantage in terms of SER over AP-AF as it does in terms of outage probability, due to its more efficient use of channel resources.

In this letter, we first derive the SER for S-AF systems using the outage probability expression derived in [4]. The SER result verifies that S-AF achieves full diversity gain of the cooperative system. Next, using the technique developed in [7], we obtain the coding gain improvement of S-AF over AP-AF in closed-form.

II. SYSTEM MODEL

Consider a wireless system where a source node transmits to a destination with the help of m relay nodes. Transmissions are orthogonal, either through time or frequency division. Two relay schemes are considered and compared in this letter. In AP-AF systems all relay nodes actively forward the signal they receive from the source, and thus the destination receives $m+1$ duplicates of the original signal. After Maximum Ratio Combining (MRC), the received SNR is [4]

$$\gamma_r^{AP} = \frac{|h_{s,d}|^2 E_s}{N_{s,d}} + \sum_{i=1}^{m} \frac{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} \frac{|h_{i,d}|^2 E_i}{N_{i,d}}}{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} + \frac{|h_{i,d}|^2 E_i}{N_{i,d}} + 1}, \tag{1}$$

where $h_{s,d}$, $h_{s,i}$ and $h_{i,d}$ are independent complex Gaussian distributed channel coefficients of the source-destination, source-relay i and relay i-destination channels, respectively. E_s and E_i are the average energy transmitted at the source and the ith relay nodes. They can also be considered as the transmission power assuming each transmission has unit duration. $N_{s,d}$, $N_{s,i}$ and $N_{i,d}$ are single-sided noise power spectral densities of the additive white Gaussian noise (AWGN) in the corresponding channels.

In S-AF systems only the "best" relay which contributes most to received SNR is chosen for re-transmission. The received SNR is

$$\gamma_r^{S} = \frac{|h_{s,d}|^2 E_s}{N_{s,d}} + \max_{i} \frac{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} \frac{|h_{i,d}|^2 E_i}{N_{i,d}}}{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} + \frac{|h_{i,d}|^2 E_i}{N_{i,d}} + 1}. \tag{2}$$

III. Symbol Error Rate Analysis

A. SER of S-AF Systems

Under the assumption of linear modulation and AWGN, the SER conditioned on the instantaneous received SNR is approximately $P_e = Q\left(\sqrt{c\gamma_r}\right)$, where c is a constant determined by the modulation format (e.g. $c=2$ for phase shift keying (PSK)), and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-t^2/2)dt$. The average SER over the distribution of the received SNR is therefore

$$P_e = \mathcal{E}_{\gamma_r}\left\{Q\left(\sqrt{c\gamma_r}\right)\right\}. \tag{3}$$

Denote the probability density function (PDF) and cumulative distribution function (CDF) of the received SNR γ_r as $f_{\Gamma_R}()$ and $F_{\Gamma_R}()$, respectively. After introducing a new random variable (RV) with standard Normal distribution or $X \sim N(0,1)$, the average SER can be rewritten as

$$P_e = P\left\{X > \sqrt{c\gamma_r}\right\} = P\left\{\gamma_r < X^2/c\right\} = \mathcal{E}_X\left\{F_{\Gamma_R}(\frac{X^2}{c})\right\}. \tag{4}$$

Assuming all the noise variances are equal, i.e., $N_{s,d} = N_{s,i} = N_{i,d} = N_0 = 1/\gamma$, γ is proportional to all the transmit SNRs (at the source and relay nodes) and the receive SNR. Therefore, γ can be used to represent the system SNR. Defining $\alpha_0 = |h_{s,d}|^2 E_s$, $\alpha_i = |h_{s,i}|^2 E_s$ and $\beta_i = |h_{i,d}|^2 E_i$, the received SNR of the S-AF scheme in (2) can be written as

$$\gamma_r^S = \alpha_0 \gamma + \max_i \frac{\alpha_i \beta_i \gamma^2}{\alpha_i \gamma + \beta_i \gamma + 1}. \tag{5}$$

In [4], we derived the high SNR approximation of the CDF of the received SNR for S-AF system under Rayleigh fading as

$$F^S_{\Gamma_R}(\gamma_r) \approx \frac{\lambda_0 \prod_{i=1}^m (\lambda_i + \xi_i)}{m+1} \left(\frac{\gamma_r}{\gamma}\right)^{m+1}, \tag{6}$$

where λ_0 is the exponential distribution parameter of α_0, or $f_{\alpha_0}(x) = \lambda_0 \exp(-\lambda_0 x)$. Similarly, λ_i and ξ_i are the exponential distribution parameters of α_i and β_i, respectively.

Substituting (6) into (4) and using the fact that $\int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}}$ [8], we obtain the high SNR approximation of the SER for S-AF scheme as

$$\begin{aligned} P_e^S &\approx \int_0^\infty \frac{\lambda_0 \prod_{i=1}^m (\lambda_i + \xi_i)}{m+1} \left(\frac{x^2}{c\gamma}\right)^{m+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{(2m+1)!\lambda_0 \prod_{i=1}^m (\lambda_i + \xi_i)}{(m+1)!(2c\gamma)^{m+1}}. \end{aligned} \tag{7}$$

From (7) it is obvious that full diversity order of $(m+1)$ is achieved by S-AF.

Figure 1 shows the SER of an S-AF system with two relay nodes and unit transmission power per node. All channel coefficients are assumed to be complex Gaussian distributed with zero mean and unit variance. The theoretical curve is from averaging the Q-function with the distribution of received SNR derived in [9]. From the figure we can see that the approximation is valid starting from practical values of SNR.

[Figure 1 about here.]

B. SER Improvement: S-AF vs. AP-AF

In this section we derive the SER improvement of S-AF over AP-AF. To ensure a fair comparison, unit transmit energy and equal power division among cooperating nodes are assumed for both systems. As a result, the transmitted power at each node in the AP-AF system is $E_s = E_i = 1/(m+1)$. On the other hand, the source and chosen relay use $1/2$ unit of power for transmission in S-AF.

The technique developed in [6], [7] can be used to compare the SER's of the two AF schemes. If the instantaneous SER can be written as $P_e|\gamma_r = Q\left(\sqrt{c\gamma_r}\right)$ and the pdf of γ_r can be approximated as

$$f_{\gamma_r} = a\gamma_r^t + o(\gamma_r^t), \tag{8}$$

where t is a positive ingeger, then the asymptotic average SER is given by [6]

$$P_e \to \frac{\prod_{i=1}^{t+1}(2i-1)}{2(t+1)c^{t+1}t!}\frac{\partial^t f_{\gamma_r}}{\partial \gamma_r^t}(0), \tag{9}$$

where

$$\frac{\partial^t f_{\gamma_r}}{\partial \gamma_r^t}(0) = \lim_{\gamma_r \to 0^+} \frac{\partial^t f_{\gamma_r}}{\partial \gamma_r^t}.$$

It is easy to verify that the received SNR for both AF schemes satisfy (8) with $t = m$. To obtain the mth order derivative of the PDF of γ_r, consider two RVs:

$$\begin{aligned} Y_1 &= \frac{X_0 + \sum_{i=1}^m X_i}{m+1}, \\ Y_2 &= \frac{X_0 + \max_i X_i}{2} = \frac{X_0 + V}{2}, \end{aligned} \tag{10}$$

where $\{X_i : i = 0 \cdots m\}$ are independent RV's whose CDF and PDF satisfy $F_{X_i}(0) = 0$ and $f_{X_i}(0) \neq 0$. The denominators guarantee unit energy for each transmitted signal. From first principles, the Moment Generating Function (MGF) of Y_1 can be shown to be

$$M_{Y_1}(s) = \prod_{i=0}^m M_{X_i}\left(\frac{s}{m+1}\right), \tag{11}$$

where $M_{X_i}(s)$ is the MGF of X_i.

Using the initial value theorem, it is proven in [6] that

$$\begin{aligned} \frac{\partial^m f_{Y_1}}{\partial y_1^m}(0) &= \lim_{s\to\infty} s^{m+1} M_{Y_1}(s) = \lim_{s\to\infty} s^{m+1} \prod_{i=0}^m M_{X_i}\left(\frac{s}{m+1}\right) \\ &= (m+1)^{m+1} \lim_{s\to\infty} \prod_{i=0}^m \frac{s}{m+1} M_{X_i}\left(\frac{s}{m+1}\right) \\ &= (m+1)^{m+1} \prod_{i=0}^m f_{X_i}(0). \end{aligned} \tag{12}$$

On the other hand, it is straightforward that the CDF of $V = \max_i X_i$ can be written as

$$F_V(v) = \prod_{i=1}^m F_{X_i}(v). \tag{13}$$

Thus we have

$$\frac{\partial^{m-1} f_V}{\partial v^{m-1}}(0) = \frac{\partial^m F_V}{\partial v^m}(0) = m! \prod_{i=1}^{m} f_{X_i}(0) + o(0),$$

where the last equation is directly from (13), and $o(v)$ denotes all the other terms. Note that each term in $o(v)$ is in product form and contains at least one CDF of X_i (otherwise it belongs to the term $m! \prod_{i=1}^m f_{X_i}(0)$). Since $F_{X_i}(0) = 0$ we have $o(0) = 0$, which yields

$$\frac{\partial^{m-1} f_V}{\partial v^{m-1}}(0) = m! \prod_{i=1}^{m} f_{X_i}(0). \tag{14}$$

Using the MGF and initial value theorem, we finally get

$$\begin{aligned} \frac{\partial^m f_{Y_2}}{\partial y_2^m}(0) &= \lim_{s\to\infty} s^{m+1} M_{Y_2}(s) \\ &= \lim_{s\to\infty} s^{m+1} M_{X_0}\left(\frac{s}{2}\right) M_V\left(\frac{s}{2}\right) \\ &= 2^{m+1} f_{X_0}(0) \frac{\partial^{m-1} f_V}{\partial v^{m-1}}(0) \\ &= m! 2^{m+1} \prod_{i=0}^{m} f_{X_i}(0). \end{aligned} \tag{15}$$

The received SNR of the two AF schemes are in the form of Y_1 and Y_2. Besides, they share the same parameters as $t = m$ and c is determined by the modulation scheme only. From (9), (12) and (15) we see that

$$\frac{P_e^S}{P_e^{AP}} = m! \left(\frac{2}{m+1}\right)^{m+1}. \tag{16}$$

This ratio is always smaller than 1 for all $m \geq 2$. When $m = 2$, there is a performance gain of 1.74dB for using S-AF over AP-AF. This gain increases to 3.32dB when the number of relays increases to $m = 10$.

The above results are verified in Fig. 2, where the same network as in Fig. 1 is considered. The modulation scheme is 4-QAM. This Figure shows clearly that the S-AF scheme has a better SER of a factor of $m!(\frac{2}{m+1})^{m+1} = 0.593$ over AP-AF, as predicted in (16).

[Figure 2 about here.]

Note that we made no assumptions on the distributions of the channel coefficients except that $F_{X_i}(0) = 0$ and $f_{X_i}(0) \neq 0$. As a result, this performance gain is valid not only for Rayleigh fading, but for almost all other fading models.

References

[1] J. N. Laneman and G. W. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Transactions on Information Theory*, vol. 49, pp. 2415–2425, November 2003.

[2] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Transactions on Information Theory*, vol. 50, pp. 3062–3080, December 2004.

[3] H. El Gamal and D. Aktas, "Distributed space-time filtering for cooperative wireless networks," *Proceedings of IEEE Globecom*, pp. 1826–1830, 2003.

[4] Y. Zhao, R. Adve, and T. J. Lim, "Improving amplify-and-forward relay networks: Optimal power allocation versus selection," *To appear IEEE International Symp. on Information Theory*, 2006.

[5] E. Beres and R. Adve, "On selection cooperation in distributed networks," *Proceedings of IEEE 40th Conference on Information Science and Systems*, March 2006.

[6] A. Ribeiro, X. Cai, and G. B. Giannakis, "Symbol error probability for general cooperative links," *IEEE Transactions on Wireless Communications*, vol. 4, pp. 1264–1273, May 2005.

[7] Z. Wang and G. B. Giannakis, "A simple and general parameterization quantifying performance in fading channels," *IEEE Transactions on Communications*, vol. 51, pp. 1389–1398, August 2003.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integals, Series, and Products: 5th Edition*. Academic Press, 1994.

[9] P. A. Anghel and M. Kaveh, "Exact symbol error probability of a cooperative network in a rayleigh-fading environment," *IEEE Transactions on Wireless Communications*, vol. 3, pp. 1416–1421, September 2004.

LIST OF FIGURES

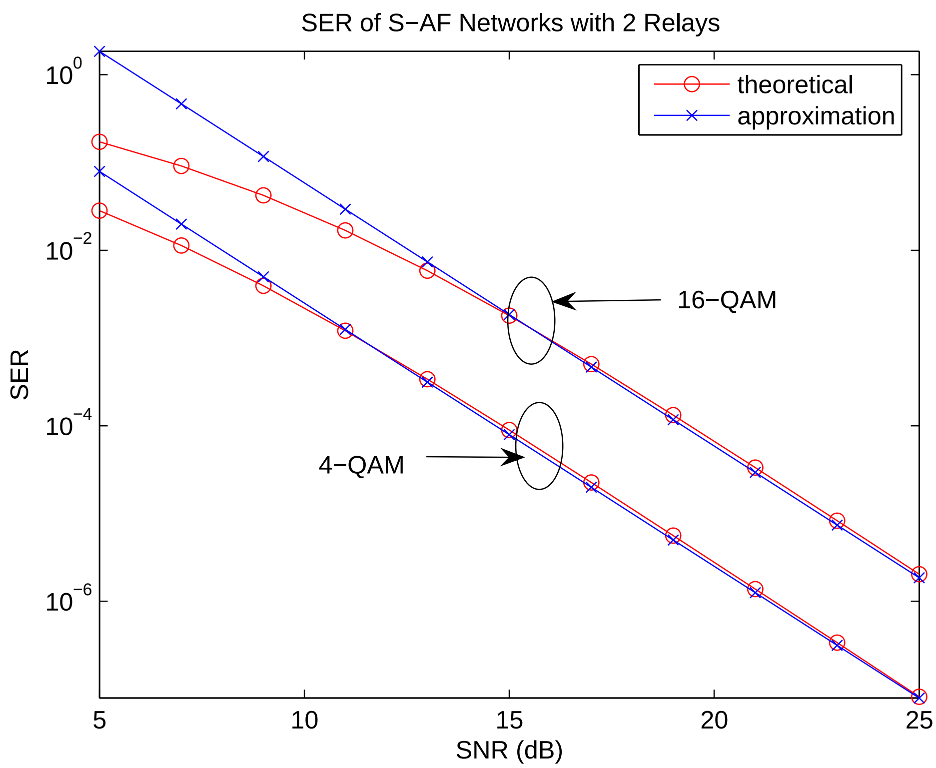

Fig. 1. SER of S-AF network with two relays.

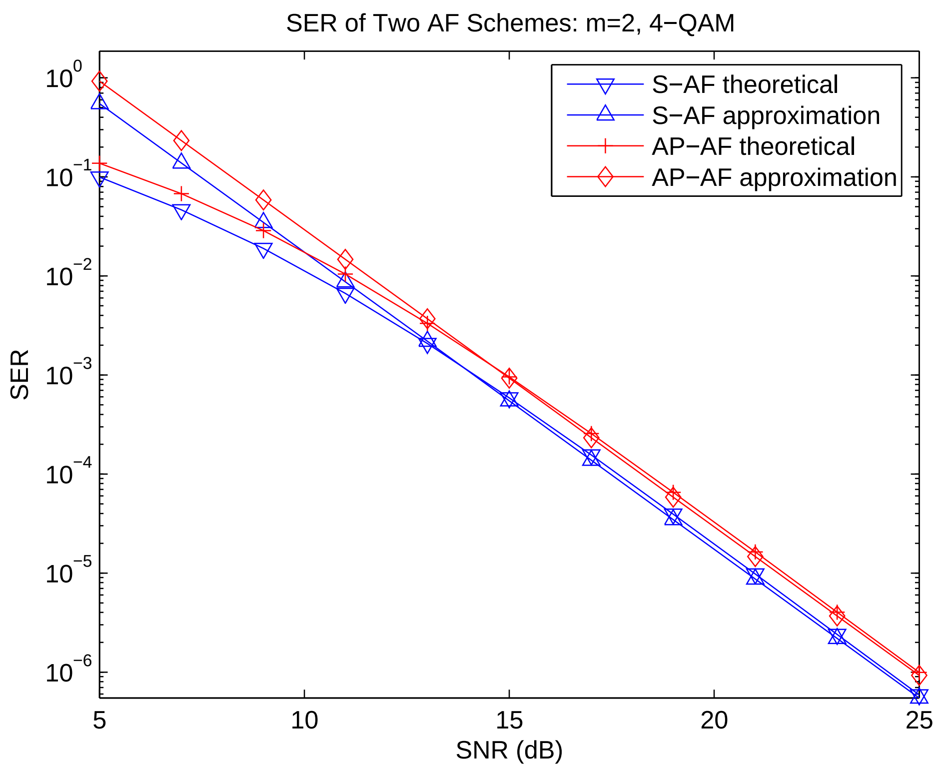

Fig. 2. SER performance: AP-AF vs. S-AF