A NEW APPROACH TO SPATIAL POWER SPECTRAL DENSITY ESTIMATION FOR MULTIPLE INCOHERENTLY DISTRIBUTED SOURCES

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ABSTRACT

In this paper, a new technique is proposed for estimating the total spatial power spectral density (PSD) caused by multiple incoherently distributed sources. Our approach is based on the fact that the array covariance matrix can be represented through the moments of the spatial PSD. Based on this representation, we develop a computationally efficient technique to estimate the moments from the array covariance matrix. The so-obtained moments are then used to estimate the total spatial PSD.

Index Terms— Source localization, spread source, covariance fitting, distributed source, spatial power spectral density.

1. INTRODUCTION

Array processing based on point source modeling has been the focus of numerous research efforts during the past three decades. However, in applications such as underwater acoustics, passive sonar, and wireless communications, the signal emitted by a source is scattered by the objects in the vicinity of the source. As a result, point source modeling cannot be a realistic signal representation in such applications. In fact, a distributed source modeling seems to be a promising way to characterize the signal sources [1].

Several methods have been presented in the literature for localization of distributed sources [1]-[8]. A majority of localization methods published in this field assumes that the signal distribution can be parameterized by the source central angle and its angular spread. This assumption may however not be realistic.

In this paper, we study the problem of estimating the total spatial power spectral density caused by multiple incoherently distributed (ID) sources. we show that when ID sources are assumed to be uncorrelated, there is a certain ambiguity that does not allow the individual source PSD be estimated from the array covariance matrix. However, if the array covariance matrix is represented through the total spatial PSD of all ID sources, one can estimate this PSD without any ambiguity. Based on this fact, we develop a covariance fitting technique to estimate the total spatial PSD from the array covariance matrix. Our approach is based on estimating the moments of the total spatial PSD from the covariance matrix. Then the so-obtained estimates are used to estimate the total spatial PSD.

2. DATA MODEL

Assume that the signals of q distributed sources impinge on a uniform linear array (ULA) of p sensors that are spaced half a wavelength apart. The sources are assumed to be stationary and narrowband with the same central frequency. Based on such assumptions, the baseband representation of the array output is given by [1], [2], [3]

$$\mathbf{x}(t) = \sum_{m=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\nu) s_m(\nu, t) d\nu + \mathbf{n}(t), \tag{1}$$

where $\mathbf{x}(t)$ is the $p \times 1$ vector of the complex array outputs, $\mathbf{n}(t)$ is the $p \times 1$ vector of noise, $s_m(\nu, t)$ is the complex random time-varying signal distribution of the mth source at spatial frequency $\nu \triangleq \pi \sin\theta$, and θ is the direction of arrival. Also, $\mathbf{a}(\nu)$ is the $p \times 1$ vector of the array response to a source with spatial frequency ν and it is defined as

$$\mathbf{a}(\nu) = [1 \;\; e^{j\nu} \;\; e^{j2\nu} \;\; \cdots, e^{j(p-1)\nu}]^T \tag{2}$$

where $(\cdot)^T$ denotes the transpose. Note that $\theta = 0$ corresponds to the broadside of the array, and therefore $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Assuming that the sources and noise are uncorrelated, the array covariance matrix can be written as

$$\mathbf{R}_{xx} \triangleq \mathrm{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \sigma^2 \mathbf{I} + \sum_{m,n=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_{mn}(\nu, \nu')\mathbf{a}(\nu)\mathbf{a}^H(\nu')\, d\nu\, d\nu' \tag{3}$$

where σ^2 is the unknown noise power, $\mathbf{I}$ is the identity matrix, $\mathrm{E}\{\cdot\}$ is the statistical expectation operator, $(\cdot)^H$ denotes the Hermitian transpose, the function $p_{mn}(\nu, \nu')$ is defined as

$$p_{mn}(\nu, \nu') \triangleq \mathrm{E}\{s_m(\nu, t) s_n^*(\nu', t)\} \tag{4}$$

and $(\cdot)^*$ stands for the complex conjugate. In fact, $p_{mn}(\nu, \nu')$ represents the cross-correlation between the signals of the mth and nth sources that arrive at the array with the spatial frequencies ν and ν', respectively. Let us assume that all distributed sources are mutually uncorrelated. This means that

$$p_{mn}(\nu, \nu') = p_{mm}(\nu, \nu')\delta_{mn} \tag{5}$$

where δ_{mn} is the Kronecker delta. Also, throughout the paper, we consider the ID source model. A distributed source is said to be ID if its components arriving from different directions (or with different spatial frequencies) are uncorrelated. That is, for the mth source, we assume that

$$p_{mm}(\nu, \nu') = \sigma_m^2\, \rho_m(\nu)\, \delta(\nu - \nu') \tag{6}$$

where $\delta(\nu - \nu')$ is the Dirac delta-function, σ_m^2 is the power of the mth source, and $\rho_m(\nu)$ is mth source *normalized spatial power spectral density (SPSD)* satisfying

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho_m(\nu)\, d\nu = 1, \quad m = 1, 2, \ldots, q\,. \tag{7}$$

The index m in $\rho_m(\nu)$ is used to emphasize that the sources can have different SPSDs. Using (5) and (6), we can rewrite (3) as

$$\mathbf{R}_{xx} = \sum_{m=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sigma_m^2\, \rho_m(\nu)\, \mathbf{a}(\nu)\, \mathbf{a}^H(\nu)\, d\nu + \sigma^2 \mathbf{I}\,. \tag{8}$$

Using (8), we have proposed a covariance fitting based method, for parametric localization of multiple ID sources [3]. This method is based on two assumptions. First, it is assumed that q, the number of ID sources, is known, and second, the source SPSDs $\{\rho_m(\nu)\}_{m=1}^q$ are assumed to be parameterized functions that are known up to two parameters: the source central angles and angular spreads[1]. However, these two assumptions may not be realistic. The number of sources may not be known, and to the best of our knowledge, no method has been presented in the literature for detecting the number of multiple ID sources. This leaves the problem of ID source enumeration open for future research efforts. Also, in practice, the shape of source SPSDs may not be known, and as a result, the second assumption of [3] may not be realistic.

It is also noteworthy that using the covariance matrix $\mathbf{R}_{xx}$ in (8) for parametric localization of ID sources (i.e., estimating the source parametric SPSDs) causes a certain ambiguity. More specifically, there exist infinite number of sets $\{\rho_m(\nu)\}_{m=1}^q$ that all can produce the same covariance matrix. To show this, we need to introduce the notion of the total SPSD that is caused by several ID sources.

3. TOTAL SPATIAL POWER SPECTRAL DENSITY

Let us rewrite (8) as

$$\mathbf{R}_{xx} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sigma_s^2 \rho_T(\nu)\, \mathbf{a}(\nu)\, \mathbf{a}^H(\nu)\, d\nu + \sigma^2 \mathbf{I} \tag{9}$$

where the following definitions are used:

$$\sigma_s^2 \triangleq \sum_{m=1}^{q} \sigma_m^2 \tag{10}$$

$$\rho_T(\nu) \triangleq \sum_{m=1}^{q} \frac{\sigma_m^2}{\sigma_s^2} \rho_m(\nu)\,. \tag{11}$$

We call $\rho(\nu)$ the *total spatial power spectral density*. It follows from (7), (10), and (11) that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho_T(\nu) d\nu = 1. \tag{12}$$

As can be seen from (11), there exists infinite number of sets $\{\rho_m(\nu)\}_{m=1}^q$ that all can result in the same total SPSD, or equivalently, in the same covariance matrix $\mathbf{R}_{xx}$. Therefore, using the covariance matrix $\mathbf{R}_{xx}$ to estimate the source SPSDs can result in ambiguous estimates of these SPSDs. However, one might still be able to use the covariance matrix to estimate the total SPSD $\rho_T(\nu)$ without any ambiguity. We will show how this is possible in the next section.

It is also noteworthy that the method of [3] cannot be used to estimate the total SPSD $\rho_T(\nu)$, (for example, by assuming $q = 1$). Indeed the method of [3] is applicable to the case where the source SPSD is a parametric function of the source central angle and angular spread parameters. However, $\rho_T(\nu)$ may not be parameterized by only these two parameters.

[1]Note that the notation used here is slightly different from that used in [3]. In fact, we herein use the source SPSD, $\rho_m(\nu)$ to characterize the mth source while in [3], the angular power density, $\rho_m(\theta)$ has been used for the same purpose.

4. TOTAL SPATIAL PSD ESTIMATION

In this section, we develop a method to estimate the total SPSD. Let us define the matrix $\mathbf{A}(\nu)$ as $\mathbf{A}(\nu) \triangleq \mathbf{a}(\nu)\mathbf{a}^H(\nu)$. Noting that the element (k, l) of $\mathbf{A}(\nu)$ is given by $[\mathbf{A}(\nu)]_{kl} = e^{j(k-l)\nu}$, the Taylor series representation of $\mathbf{A}(\nu)$ is given by

$$\mathbf{A}(\nu) = \sum_{n=0}^{\infty} \frac{(\nu - \tilde{\nu}_0)^n}{n!} \mathbf{A}^{(n)}(\tilde{\nu}_0) \tag{13}$$

where ν_0 is an arbitrary spatial frequency, $\mathbf{A}^{(n)}(\nu)$ is the nth derivative of $\mathbf{A}(\nu)$ and its (k, l) element is given by $[\mathbf{A}^{(n)}(\nu)]_{kl} = (j(k - l))^n e^{j(k-l)\nu}$. Using (13), we can rewrite (9) as

$$\mathbf{R}_{xx} = \sum_{n=0}^{\infty} \frac{\sigma_s^2 M_n(\tilde{\nu}_0)}{n!} \mathbf{A}^{(n)}(\tilde{\nu}_0) \tag{14}$$

where $M_n(\tilde{\nu}_0)$ is the nth moment, computed at $\tilde{\nu}_0$, of the total SPSD $\rho_T(\nu)$ and it is defined as

$$M_n(\tilde{\nu}_0) \triangleq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\nu - \tilde{\nu}_0)^n \rho_T(\nu) d\nu\,. \tag{15}$$

Let us define the center mass of $\rho_T(\nu)$ as

$$\nu_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu \rho_T(\nu) d\nu\,. \tag{16}$$

If we choose $\tilde{\nu}_0 = \nu_0$, then the moments in (15) (i.e. $\{M_n(\nu_0)\}_{n=0}^{\infty}$) represent the central moments of the total SPSD. We now show that the total SPSD can be represented through its central moments. To do this, we define the function $\phi(z)$ as

$$\phi(z) = e^{-jz\nu_0} \mathcal{F}_z\{\rho_T(\nu)\} \tag{17}$$

where $\mathcal{F}_z\{\cdot\}$ denotes the Fourier transform of the function in the argument. Noting that

$$\phi(z) = \int_{-\pi}^{+\pi} \rho_T(\nu) e^{-jz(\nu-\nu_0)} d\nu \tag{18}$$

and using the fact that the nth derivative of $\phi(z)$ at $z = 0$ is given by

$$\phi^{(n)}(0) = (-j)^n \int_{-\infty}^{+\infty} (\nu - \nu_0)^n \rho_T(\nu) d\nu = (-j)^n M_n(\nu_0) \tag{19}$$

the Maclaurant series expansion for $\phi(z)$ can be written as

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{M_n(\nu_0)}{n!} (-jz)^n. \tag{20}$$

Note that $\rho_T(\nu)$ can be obtained from $\phi(z)$ as

$$\rho_T(\nu) = \mathcal{F}_z^{-1}\{e^{jz\nu_0}\phi(z)\} \tag{21}$$

where $\mathcal{F}_z^{-1}\{\cdot\}$ represents the inverse Fourier transform of the function in the argument. Using (20) and (21), we obtain that

$$\rho_T(\nu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{n=0}^{\infty}\frac{M_n(\nu_0)}{n!}(-jz)^n e^{-jz(\nu-\nu_0)}dz\,. \tag{22}$$

Equation (22) represents the relationship between the total SPSD and its central moments. Therefore, if negligible terms in (22) are ignored, $\rho_T(\nu)$ can be obtained through its prominent central moments. In the next section, we show how the central moment of $\rho_T(\nu)$ can be obtained from the covariance matrix $\mathbf{R}_{xx}$.

5. COVARIANCE FITTING

Let us define the N-term approximation of the covariance matrix $\mathbf{R}_{xx}$ as

$$\tilde{\mathbf{R}}_{xx}^N(\tilde{\nu}_0) = \sum_{n=0}^{N-1}\frac{\sigma_s^2 M_n(\tilde{\nu}_0)}{n!}\mathbf{A}^{(n)}(\tilde{\nu}_0)\,. \tag{23}$$

Denoting the corresponding approximation error as

$$\xi_N(\tilde{\nu}_0) = \|\mathbf{R}_{xx} - \tilde{\mathbf{R}}_{xx}^N(\tilde{\nu}_0) - \sigma^2\mathbf{I}\|^2 \tag{24}$$

where $\|\cdot\|$ represents the Frobenius norm, the non-central moments can then be estimated by solving the following optimization problem:

$$\min_{\mathbf{m}(\tilde{\nu}_0)} \xi_N(\tilde{\nu}_0) \tag{25}$$

where $\mathbf{m}(\tilde{\nu}_0)$ is the $(N+1)\times 1$ vector whose nth entry is given by

$$[\mathbf{m}(\tilde{\nu}_0)]_{n+1} = \begin{cases} \sigma_s^2 M_n(\tilde{\nu}_0) & \text{for} \quad n = 0,1,2,\cdots,N-1 \\ \sigma^2 & \text{for} \quad n = N \end{cases} \tag{26}$$

It can be readily shown that the non-central moments can be obtained by solving the following set of linear equations:

$$\mathbf{Q}(\tilde{\nu}_0)\mathbf{m}(\tilde{\nu}_0) = \mathbf{b}(\tilde{\nu}_0)\,. \tag{27}$$

In (27), $\mathbf{Q}(\tilde{\nu}_0)$ is an $(N+1)\times(N+1)$ matrix whose (r,s) element is defined as

$$[\mathbf{Q}(\tilde{\nu}_0)]_{rs} = \boldsymbol{\alpha}_r^H\boldsymbol{\alpha}_s \tag{28}$$

and $\mathbf{b}(\tilde{\nu}_0)$ is an $(N+1)\times 1$ vector whose rth element is given by

$$[\mathbf{b}(\tilde{\nu}_0)]_r = \boldsymbol{\alpha}_r^H\mathrm{vec}\{\mathbf{R}_{xx}\} \tag{29}$$

where vec$\{\cdot\}$ is the vectorization operator, and $\boldsymbol{\alpha}_r$ is defined as

$$\boldsymbol{\alpha}_{r+1} \triangleq \begin{cases} \frac{1}{r!}\mathrm{vec}\{\mathbf{A}^r(\tilde{\nu}_0)\} & \text{for} \quad 0 \le r \le N-1 \\ \mathrm{vec}\{\mathbf{I}\} & \text{for} \quad r = N\,. \end{cases} \tag{30}$$

The solution to (27) is given by

$$\mathbf{m}(\tilde{\nu}_0) = \mathbf{Q}^{-1}(\tilde{\nu}_0)\mathbf{b}(\tilde{\nu}_0)\,. \tag{31}$$

Therefore, using the fact that $[\mathbf{m}(\tilde{\nu}_0)]_1 = \sigma^2 M_0(\tilde{\nu}_0) = \sigma^2$ holds true for any $\tilde{\nu}_0$, the remaining moments $\{M_n(\tilde{\nu}_0)\}_{n=1}^{N-1}$ can be obtained as

$$M_n(\tilde{\nu}_0) = \frac{[\mathbf{m}(\tilde{\nu}_0)]_{n+1}}{[\mathbf{m}(\tilde{\nu}_0)]_1} \quad \text{for}\ n = 1,2,\cdots,N-1\,. \tag{32}$$

Note however that in order to approximate $\rho_T(\nu)$, we need to obtain the cental moments, i.e., we have to obtain $\mathbf{m}(\tilde{\nu}_0)$ for $\tilde{\nu}_0 = \nu_0$, but ν_0 is yet to be estimated. To cope with this problem, one can use the iterative method proposed in [3] to estimate ν_0. The essence of this iterative method is based on an interesting property of $M_1(\tilde{\nu}_0)$ given by

$$M_1(\tilde{\nu}_0) = \nu_0 - \tilde{\nu}_0\,. \tag{33}$$

It follows from (33) that given $M_1(\tilde{\nu}_0)$, the mass center ν_0 can be obtained as

$$\nu_0 = \tilde{\nu}_0 + M_1(\tilde{\nu}_0). \tag{34}$$

Note that in practice, only an estimate of $M_1(\tilde{\nu}_0)$ is available not its true value. Hence, in order to improve the quality of the estimate of ν_0, the iterative method of [3] uses ν_0 obtained from (34) as a new value for $\tilde{\nu}_0$ in (31) and computes a new value for $\mathbf{m}(\tilde{\nu}_0)$, and consequently, a new value for $M_1(\tilde{\nu}_0)$. Then (34) is used to update ν_0. This technique is repeated a few times to improve the accuracy of the so-obtained ν_0.

We end up this section by mentioning that choosing N, plays an important role in obtaining the moments precisely. To choose N, one can use the normalized covariance fitting error (NCFE) error defined as

$$e_N \triangleq \frac{\xi_N(\nu_0)}{\|\mathbf{R}_{xx}\|^2}\,. \tag{35}$$

More specifically, one can start with a reasonable N, compute e_N, and increase N by one. if $e_N - e_{N+1}$ is negligible, the iteration is stopped. Otherwise N is increased by one and the algorithm is repeated.

The proposed algorithm is summarized as it follows.

1. Estimate the covariance matrix with its sample covariance matrix $\hat{\mathbf{R}}_{xx} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}(t)\mathbf{x}^H(t)$ where T is the number of available snapshots.
2. Choose $\tilde{\nu}_0$.
3. Select an initial value for N and set e_{N-1} to 1.
4. Construct the set of matrices $\mathbf{A}^{(n)}(\tilde{\nu}_0)$, $n = 0,1,2,\ldots,N-1$.
5. Compute $\mathbf{Q}(\tilde{\nu}_0)$ and $\mathbf{b}(\tilde{\nu}_0)$ as in (28) and (29), respectively.
6. Obtain $\mathbf{m}(\tilde{\nu}_0) = \mathbf{Q}^{-1}(\tilde{\nu}_0)\mathbf{b}(\tilde{\nu}_0)$.
7. Calculate ν_0 as in (34).
8. Set $\tilde{\nu}_0 = \nu_0$ and repeat steps 4-7, a few times, to improve the accuracy of ν_0 estimate.
9. Compute the NCFE as in (35).
10. If $e_N - e_{N-1}$ is negligible go to 11,
    else $N = N+1$ and go to 4.
11. Compute $\rho_T(\nu)$ as in (22).

6. SIMULATION RESULTS

We consider a scenario where the signals of two ID sources with Gaussian SPSD impinge on an array of $p = 21$ sensors. The sources are assumed to have the same signal-to-noise ratios (SNRs) defined as σ_m^2/σ^2. The sources' central angles are 5^o and 35^o and their angular spreads amount to a spatial frequency standard deviation of 0.2 in radians. We use $T = 500$ independent snapshots to calculate the sample covariance matrix. It is easy to show that in this scenario, the center spatial frequency is given by $\nu_0 = \frac{1}{2\pi}(\sin^{-1}5^o + \sin^{-1}35^o)$. Also, we choose $\tilde{\nu}_0 = 1.2$rad.

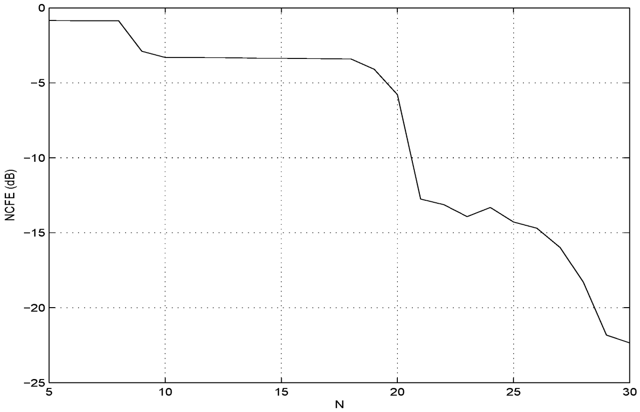

Fig. 1. The normalized covariance fitting error versus N.

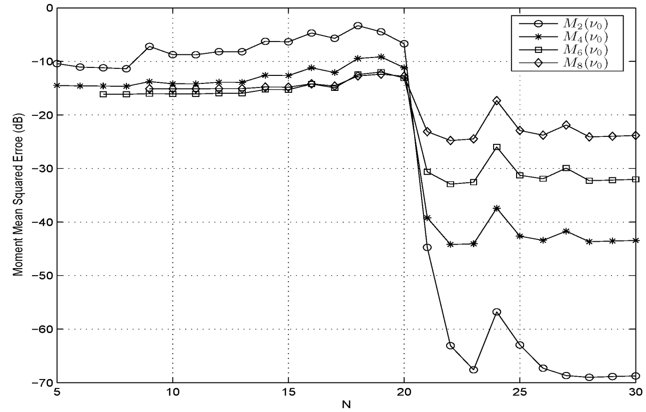

Fig. 2. The normalized mean square error of even-indexed moment estimates versus N.

Fig. 1 shows the NCFE, e_N for different values of N. Figs. 2 and 3 show the mean square error (MSE) of estimates for even and odd indexed moments, respectively. Note that $M_0(\nu_0) = 1$ always holds true and therefore the NMSE for the $M_0(\nu_0)$ is not shown in Fig. 2. Also, as the SPSD is symmetric with respect to ν_0, the odd-indexed moments are zero and therefore their estimates have very small values. It is worth mentioning that as true odd-index moments are all equal to zero, the MSEs shown in Fig. 3 are the square of the corresponding odd-indexed moment estimates.

As can be seen from Fig. 1, the NCFE drops sharply for $N > 20$. Also, Fig. 2 shows that the quality of even-indexed moment estimates is significantly improved for $N > 20$ and at the same time, the MSE for odd-indexed moment estimates remains below -50 (dB).

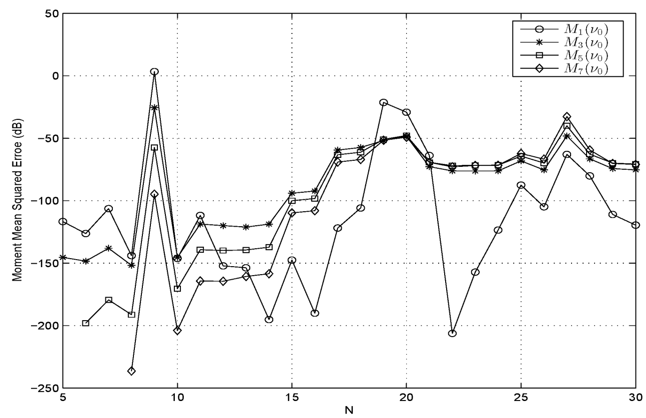

Fig. 3. The normalized mean square error of odd-indexed moment estimates versus N.

7. CONCLUSION

A new method has been proposed for estimating the total spatial power spectral density (SPSD) caused by several incoherently distributed sources. We have shown that this density can be obtained from its prominent central moments. Also, we have proven that the array covariance matrix can be approximated by means of the central moments of the SPSD. Based on such a covariance approximation, we have herein proposed a covariance fitting approach to estimate the prominent central moments of the SPSD. The so-obtained central moments can then be used to estimate the SPSD itself.

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