

Throughput Improvement Through Precoding in OFDMA Systems with Limited Feedback

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Abstract—In this paper, we study the possibility of throughput improvement through precoding in OFDMA based wireless systems with limited channel feedback. Precoding can increase the overall system throughput by selecting the use of higher modulation and coding schemes (if there are any) in each physical resource unit (PRU) under a maximum bit error rate constraint. For a specific form of precoding that we proposed in [1], we analytically derive a formula for the maximum PRU throughput. Sufficient conditions guaranteeing that our proposed technique outperforms the conventional technique are derived. Finally, numerical results show the amount of throughput gain achieved by our proposed technique.

I. INTRODUCTION

Future generation wireless systems, such as mobile WiMAX [2] and Beyond 3G (B3G) [3] networks, are expected to provide high speed wireless connectivity. Orthogonal Frequency Division Multiple Access (OFDMA), with a large number of subcarriers, has been chosen as the multiple access technique in these systems to achieve high rate wireless transmission. The choice of OFDMA arises due to its excellent performance in combating various wireless channel impairments such as multipath fading and inter-symbol interference. Moreover, OFDMA is very efficient in exploiting multiuser diversity by employing adaptive resource allocation [4]–[8] to different mobile receivers according to their instantaneous channel qualities.

Since resource scheduling in OFDMA wireless networks is performed at the base stations (BS), adaptive subcarrier allocation and bit loading techniques cannot be implemented in downlink without the knowledge of downlink channel qualities for all connected receivers. This necessitates the feedback of all this information to the BS, which results in a huge overhead. To reduce the amount of this feedback, current standards [2] [3] partition their spectral and temporal resources into *physical resource units (PRUs)* and allow receivers to report their channel qualities only once for the PRUs sharing the same subcarriers. In general, the subcarriers spanning both the spectral and temporal dimensions of the PRU are referred to as transmission units (TUs). To unify scheduling in both downlink and uplink, current standards assume that a PRU is the smallest allocatable unit. It must be assigned to only one

receiver and only one modulation and coding scheme (MCS) is allowed to be used over all its TUs.

The MCS employed over the TUs of each PRU is determined from the average signal-to-interference-noise ratio (SINR) reported by the receiver that was granted this PRU, so as to satisfy the bit error rate requirement imposed by the network. If distinct symbols are sent over all the TUs (to which we will refer to as the conventional scheme), it is impossible to employ a higher MCS without violating the bit error rate constraint. The only way to use a higher MCS is then to reduce the number of distinct symbols transmitted in the PRU and introduce other alternatives to receive them. This procedure is generally referred to as *precoding*. In [9], [10], it has been shown that precoding can reduce the symbol error probability. Thus, we can employ precoding for any higher MCS to achieve a lower symbol error rate than the one achieved by the conventional scheme for the same MCS and SINR. Obviously, if the symbol error probability is reduced, the bit error rate will also decrease and can attain the desired level. At this point, two questions arise: *Which precoding scheme should be employed?* and *would that lead to a throughput improvement?*

The answer to the first question is obviously the precoding scheme that can achieve both the lowest symbol error reduction and an equal error probability to all transmitted symbols. We refer to the former property as the best improvement property and the latter as the symmetry property. The justification for the latter property is quite intuitive since it is necessary not to favor symbols over others in terms of error probability. The former property is justified by the fact that the stronger the ability of the scheme to reduce the symbol error probability, the lower the number of TUs needed to introduce redundancy and thus the higher the achieved throughput. In [1], we proved that a specific class of symmetric precoding schemes outperforms most other known symmetric precoding classes. In this paper, we will consider this precoding class and study its ability to improve the overall PRU throughput compared to the conventional scheme in such multicarrier environment with limited channel quality feedback. We show that we can indeed achieve throughput improvement for practical WiMAX and B3G systems using this precoding scheme proposed in [1].

The rest of the paper is organized as follows. Section II illustrates the system model. In Section III, we introduce

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the precoding class we explore in this paper. In Section IV, we derive an expression for the maximum PRU throughput achieved by this precoding scheme and sufficient conditions for that throughput to outperform the one achieved by the conventional scheme. Section V shows numerically the amount of throughput gains that can be achieved using our proposed technique. Section VI concludes the paper.

II. SYSTEM MODEL

Our model consists of a WiMAX or a B3G base station that employs OFDMA as the multiple access technology. Our focus is on downlink transmission but all our results apply to uplink transmission as well. The number of subcarriers in the cell can be as large as 2048 [2] [3] and even higher in newer standards. Each N_f of these subcarriers are grouped together into subchannels. The subcarriers in each subchannel can be adjacent or distributed according to the subchannelization mode. Each subchannel is generally allocated to a receiver for N_s symbols, which we generally term as PRU. Thus, each PRU includes $N = N_f \times N_s$ TUs. Current standards and pre-standards set the value of N_f between 10 and 25 subcarriers and the value of N_s between 8 and 16 OFDMA symbols [2] [3], which makes practical PRU sizes (N) in the range of 80 to 400 TUs.

The PRU allocation among different receivers is out of the scope of this paper. We only focus on the MCS determination for a PRU allocated to a given receiver. In current standards, the MCS is determined according to the SINR reported by the receiver, to which this PRU was granted, so as to transmit N distinct symbols over the PRU TUs with a maximum threshold bit error rate P_b^{th} . The number of bits in each symbol is determined by the spectral efficiency of the MCS employed in this PRU.

In the rest of the paper, we will use the following terminology and notation. The Spectral Efficiency (SE) of a MCS is defined as the effective number of information bits this MCS transmits on one TU. We define b_c as the SE of the MCS allowed in the conventional scheme, B as the set of MCSs whose SE are higher than b_c . Define i as an index to the elements of B , b_i as the SE of the i^{th} MCS in B , R_i as the PRU throughput achieved by the i^{th} MCS and R_{max} as the maximum PRU throughput achieved over all $i \in B$.

III. PRECODING SCHEME

Precoding has been introduced in the literature [9], [10] as a symbol protection scheme against fading conditions. In the context described in Section II, the idea of precoding can be introduced as reducing the number of distinct transmitted symbols in a PRU below N , and exploiting the remaining TUs in transmitting redundancy of these symbols. A symbol can thus be recovered through different alternatives which reduces the symbol error probability. One simple form of precoding is simple diversity in which the original symbols are simply repeated. In this case, the number of distinct symbols per PRU should be a divisor of the number of TUs. This diversity precoding is employed in several current standards. However,

it is intuitively not suitable for the purpose of this study since a huge decrease in the transmitted distinct symbols will result as the employed diversity order is increased which will not lead to the desired throughput improvement.

Other forms of precoding can be obtained by performing finite field operations on the distinct data symbols. To explain this idea, assume that we send the $1 \times N$ vector \underline{s} of coded symbols that is derived from the $1 \times U$ vector \underline{u} of original distinct symbols ($U \leq N$) as $\underline{s} = \underline{u} \mathbf{A}$. In this case, \mathbf{A} is a $U \times N$ matrix that is generally referred to as the *precoding matrix*. The previous multiplication is performed over a finite field of suitable dimension. A non-zero entry $a_{i,j}$ in the precoding matrix determines the involvement of the original symbol u_i in the coded symbol s_j . Note that the conventional scheme can be regarded as a special case of precoding where $U = N$, \mathbf{A} is the $N \times N$ identity matrix and the SE is set to b_c .

In [1], we defined different classes of precoding matrices that satisfy the symmetry property, defined in Section I, and proved that a specific class of precoding matrices \mathbf{A}^* achieves the lowest symbol error probability among these classes. The matrices of this class must include all the columns of the $U \times U$ identity matrix and the rest of the columns must be linearly independent vectors having no zero entries. In other words, any matrix of this class \mathbf{A}^* must be a column permutation of the following matrix:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,U+1} & a_{1,U+2} & \dots & a_{1,N} \\ 0 & 1 & \dots & 0 & a_{2,U+1} & a_{2,U+2} & \dots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{U,U+1} & a_{U,U+2} & \dots & a_{U,N} \end{pmatrix}$$

where $a_{i,j} \neq 0 \forall i \in \{1, \dots, U\}, j \in \{U+1, \dots, N\}$ and are chosen so that any U columns of \mathbf{A}^* are linearly independent. In [11], it has been proved that the aforementioned precoding matrix is also the optimum precoding matrix among all matrices having the symmetry property. Consequently, this is the best matrix that serves our purpose since the stronger the ability of the precoding matrix to reduce the symbol error probability, the lower the amount of redundancy needed for a MCS in B to satisfy the bit error rate constraint. In the next section, we will provide an analysis of the maximum PRU throughput achievable by applying this precoding scheme on the MCSs in B and derive the sufficient conditions needed for our scheme to outperform the conventional one in terms of PRU throughput.

IV. ANALYSIS OF THE PRU THROUGHPUT

A. Derivation of the Maximum Throughput

In this section, we aim to find an expression for the maximum throughput achieved by our proposed precoding scheme using \mathbf{A}^* . To make the problem general, precoding can be performed by either coding on all the N TUs of the PRU or partitioning these TUs into K subsets, each of size F_k ($\sum_{k=1}^K F_k = N$), and employing precoding on each of them using U_k symbols ($U_k < F_k$). One can infer that the former option is a special case of the latter with $K = 1$. In

the following analysis, we will consider the general case and derive an expression for R_{max} . For notation simplicity, we will replace F_k by F and U_k by U when we derive formulae for one precoding block.

Defining P_{e_i} as the symbol error probability for a single transmission on a TU using the i^{th} MCS, the overall probability of receiving an original symbol in error, denoted by P_e , can be expressed as:

$$P_e = P_{e_i} \cdot \sum_{j=F-U}^{F-1} \binom{F-1}{j} P_{e_i}^j (1 - P_{e_i})^{F-j-1} \quad (1)$$

In words, an original symbol is received in error if symbol errors occur on both the TU carrying the original symbol and more than $F - U - 1$ TUs from the remaining $F - 1$ ones. Since error occurrences on individual TUs are independent of each other, we can use the central limit theorem (CLT) to approximate this probability as:

$$P_e \approx P_{e_i} \cdot Q \left(\frac{(1 - P_{e_i})(F - 1) - U}{\sqrt{(F - 1)(1 - P_{e_i})P_{e_i}}} \right) \quad (2)$$

Since the Q-function is a decreasing function with the increase of its argument, the lower U , the lower P_e . Thus, for any $i \in B$, we can reduce U until the symbol error probability is reduced such that the corresponding bit error probability drops below the threshold bit error rate. We denote U satisfying this condition on F TUs using the i^{th} MCS by $U_{i,F}$. To ensure that the bit error rate achieved by our scheme is always below the threshold bit error rate, we assume that if a symbol is received in error, all its bits are lost. Thus, defining P_{e_t} as the maximum allowed symbol error probability for the i^{th} MCS, then $P_{e_t} = P_b^{\text{th}}$.

Using the CLT approximation defined in (2), we can calculate an approximate value of $U_{i,F}$ as follows:

$$P_e \approx P_{e_i} \cdot Q \left(\frac{(1 - P_{e_i})(1 - F) - U}{\sqrt{(F - 1)(1 - P_{e_i})P_{e_i}}} \right) \leq P_{e_t} \quad (3)$$

$$U'_{i,F} = \left\lfloor (1 - P_{e_i})(F - 1) - \theta_i \sqrt{P_{e_i}(1 - P_{e_i})(F - 1)^{\frac{1}{2}}} \right\rfloor \quad \forall F > 1$$

where $\theta_i = Q^{-1}(P_{e_t}/P_{e_i})$. Note that $U_{i,F}$ should be upper bounded by $F - 1$. Since in (3), θ_i can be negative for $P_{e_i} \in (P_{e_t}, 2P_{e_t})$ which may lead to an increase in $U'_{i,F}$ above its upper bound $F - 1$, we can express $U_{i,F}$ as:

$$U_{i,F} = \min \left\{ U'_{i,F}, F - 1 \right\} \quad (4)$$

We can thus express the PRU throughput for the i^{th} MCS as:

$$R_i = b_i \sum_{k=1}^K U_{i,F_k} \quad (5)$$

Proposition 1. For practical values of N and P_{e_t} such that $N \in \{1, \dots, N_u\}$ where $N_u = 1 + \frac{1}{2P_{e_t}}$, the maximum value for R_i is achieved when $K = 1$. In other words, $R_i^{\text{max}} = b_i U_{i,N}$.

To prove this proposition, we first need to introduce the following two lemmas. The proofs of these lemmas are found in the appendices.

Lemma 1. If $P_{e_i} \geq 2P_{e_t}$, then $U_{i,F}$ is a superadditive function in F on its domain ($F \geq 1$).

Lemma 2. For $P_{e_i} \in (P_{e_t}, 2P_{e_t})$, if $F \leq F_u$, where $F_u = 1 + \frac{1}{2P_{e_t}}$, then $U_{i,F}$ is superadditive in $F \in \{1, \dots, F_u\}$.

Proof of Proposition 1:

Since $F_k \leq N \leq 1 + \frac{1}{2P_{e_t}} \Rightarrow U_{i,F_k}$ is superadditive (from Lemmas 1 and 2). Thus

$$\begin{aligned} R_i\{K = 1\} &= b_i U_{i,N} \\ &= b_i U_{i, \sum_{k=1}^K F_k} \quad \forall K > 1 \\ &\geq b_i \sum_{k=1}^K U_{i,F_k} \quad \forall K > 1 \\ &= R_i\{K > 1\} \end{aligned}$$

Thus, we are always sure that “no partitioning” results in the maximum value of R_i and that $R_i^{\text{max}} = b_i U_{i,N}$. ■

From Proposition 1, the maximum PRU throughput is:

$$R_{max} = \max_{i \in B} \{b_i U_{i,N}\} \quad (6)$$

B. Improvement Conditions

Defining R_c as the throughput achieved by the conventional technique, an improvement in the PRU throughput is obtained by our precoding scheme over the conventional one iff $R_{max} > R_c$. That occurs iff:

$$\exists i \in B : U_{i,N} b_i > N b_c \Rightarrow U_{i,N} > \frac{N b_c}{b_i} \quad (7)$$

Defining $\beta_i = b_c/b_i$, $\alpha_i = \theta_i \sqrt{P_{e_i}(1 - P_{e_i})}$ and $\nu_i = 1 - P_{e_i} - \beta_i$, the following proposition provides sufficient conditions on the range of N to guarantee the satisfaction of the improvement condition in (7)

Proposition 2. For practical values of N and P_{e_t} such that $N \leq N_u$, a throughput improvement is achieved by our proposed technique (i.e. $R_{max} > R_c$) if $\nu_i > 0$ and if:

$$N > \begin{cases} \frac{2}{1 - \beta_i} & P_{e_i} \in (P_{e_t}, 2P_{e_t}) \\ 1 + \left(\frac{\alpha_i + \sqrt{\alpha_i^2 + 4\nu_i(1 + \beta_i)}}{2\nu_i} \right)^2 & P_{e_i} \geq 2P_{e_t} \end{cases} \quad (8)$$

Proof:

Case 1: $P_{e_i} \in (P_{e_t}, 2P_{e_t})$

If $N > 2(1 - \beta_i)^{-1} \Rightarrow N - 2 > N\beta_i$. But since $N \leq N_u \Rightarrow U_{i,N} \geq N - 2$ (from proof of Lemma 2), then $U_{i,N} > N\beta_i$.

TABLE I
NUMERICAL VALUES OF θ_i , ν_i AND $N_{i_{min}}$ FOR $P_b^{th} = 10^{-3}$, 10^{-4} AND DIFFERENT VALUES OF SINR

P_b^{th}	SINR	b_c	b_i	θ_i	ν_i	$N_{i_{min}}$
10^{-3}	7 dB	2	4	1.8849	0.4664	6
			6	2.4245	0.5362	8
			8	2.6046	0.5326	10
10^{-3}	15 dB	4	6	-0.3786	0.3318	7
			8	1.9515	0.4608	7
10^{-3}	20 dB	6	8	-2.3170	0.2490	9
10^{-4}	10 dB	2	4	1.9029	0.4965	5
			6	2.8927	0.6143	6
			8	3.2079	0.6004	9
10^{-4}	15 dB	4	6	1.5161	0.3318	7
			8	2.8006	0.4608	8
10^{-4}	20 dB	6	8	1.2874	0.2490	9

Case 2: $P_{e_i} \geq 2P_{e_t}$

If the condition holds, then:

$$(N-1)^{\frac{1}{2}} > \frac{\alpha_i + \sqrt{\alpha_i^2 + 4\nu_i(1+\beta_i)}}{2\nu_i}$$

Since $\nu_i > 0$

$$\text{Thus } \nu_i(N-1) - \alpha_i(N-1)^{\frac{1}{2}} - (1+\beta_i) > 0$$

$$\Rightarrow \hat{U}_{i,N} > N\beta_i + 1 \Rightarrow \left\lceil \hat{U}_{i,N} \right\rceil > N\beta_i$$

$$\Rightarrow U_{i,N} > N\beta_i$$

In the following section, we will show that the practical PRU sizes (ranging from 80 to 400) satisfy these sufficient conditions for different system settings.

V. NUMERICAL EVALUATION

In this numerical evaluation, uncoded modulations were employed. The allowed constellations are (BPSK, QPSK, 16-QAM, 64-QAM, 256-QAM) corresponding to spectral efficiencies of (1, 2, 4, 6, 8), respectively. The value for P_b^{th} is set to 10^{-3} or 10^{-4} for which $N_u > 501, 1001$ TUs respectively. Since practical PRU sizes range from 80 to 400 TUs, the upper bound set by Propositions 1 and 2 is already satisfied.

Defining $N_{i_{min}}$ as the smallest PRU size satisfying the sufficient conditions provided in Proposition 2, Table I depicts the values of θ_i , ν_i and $N_{i_{min}}$ for $P_b^{th} = 10^{-3}$ and 10^{-4} and for different SINR values.

We can clearly observe that the values of ν_i are always positive and that the values of $N_{i_{min}}$ are very low compared to practical PRU sizes that range from 80 to 400 TUs. This shows that our proposed technique guarantees throughput improvement in practical systems.

For $P_b^{th} = 10^{-3}$, Figure 1 depicts a comparison between the throughput performance of the conventional and precoding schemes against SINR for $N = 200$ TUs. Figure 2 depicts the same comparison against the PRU size (N) for SINR = 17 dB. In both figures, the percentage increase of the precoding scheme over the conventional one is also illustrated. The

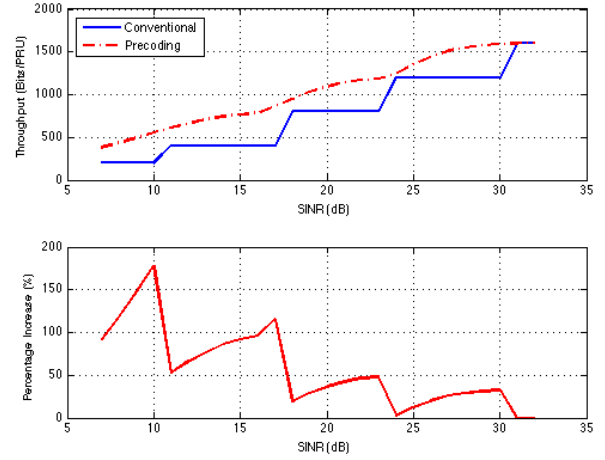


Fig. 1. Comparison between R_c and R_{max} against SINR for $N = 200$ TUs and $P_b^{th} = 10^{-3}$.

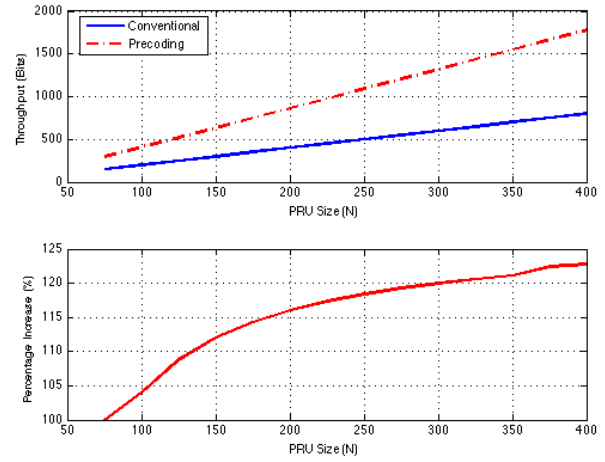


Fig. 2. Comparison between R_c and R_{max} against N for SINR=17 dB and $P_b^{th} = 10^{-3}$

percentage increase is defined as:

$$\text{Percentage Increase} = \frac{R_{max} - R_c}{R_c} \times 100 \quad (9)$$

From both figures, we can observe that a throughput improvement is obtained by our proposed technique for different SINR values and different PRU sizes for the given settings. Similar improvements are also achievable for other combinations of N , SINR and P_b^{th} . We omitted these results due to space limitations.

VI. CONCLUSION

In this paper, we proposed the use of the precoding scheme, we suggested in [1], to increase the MCS level employed in the PRUs of OFDMA based wireless systems with limited feedback. We derived a formula for the number of original distinct data symbols that should be coded in each PRU as

well as the sufficient conditions for our proposed scheme to outperform the conventional scheme. Our analysis and numerical evaluations showed that this technique can indeed achieve a considerable increase in the PRU throughput when the conventional technique fails in employing the highest MCS in this PRU.

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APPENDIX A PROOF OF LEMMA 1

If $P_{e_i} \geq 2P_{e_t} \Rightarrow \theta_i \geq 0 \Rightarrow U_{i,F} = U'_{i,F}$. Define $\hat{U}_{i,F}$ as:

$$\hat{U}_{i,F} = (1 - P_{e_i})(F - 1) - \theta_i \sqrt{P_{e_i}(1 - P_{e_i})}(F - 1)^{\frac{1}{2}} \quad (10)$$

Clearly, $U_{i,F} = \lfloor \hat{U}_{i,F} \rfloor$. The superadditivity of $\hat{U}_{i,F}$ follows from the linearity and increasing monotonicity of its first term and the subadditivity of its second term since $\theta_i \geq 0$. Thus, we have:

$$\begin{aligned} \lfloor \hat{U}_{i,F_1+F_2} \rfloor &\geq \lfloor (\hat{U}_{i,F_1} + \hat{U}_{i,F_2}) \rfloor \geq \lfloor \hat{U}_{i,F_1} \rfloor + \lfloor \hat{U}_{i,F_2} \rfloor \\ \Rightarrow U_{i,F_1+F_2} &\geq U_{i,F_1} + U_{i,F_2}. \end{aligned}$$

APPENDIX B PROOF OF LEMMA 2

For $P_{e_i} \in (P_{e_t}, 2P_{e_t})$, $\theta_i < 0$. So we can re-write the expression for $U'_{i,F}$ in (3) as follows:

$$U'_{i,F} = \lfloor (1 - P_{e_i})(F - 1) + |\theta_i| \cdot \sqrt{P_{e_i}(1 - P_{e_i})}(F - 1)^{\frac{1}{2}} \rfloor \quad (11)$$

Thus, for any value of $F > 1$,

$$\begin{aligned} \inf_{P_{e_i} \in (P_{e_t}, 2P_{e_t})} U'_{i,F} &= \inf_{P_{e_i} \in (P_{e_t}, 2P_{e_t})} \left\{ \lfloor (1 - P_{e_i})(F - 1) + |\theta_i| \sqrt{P_{e_i}(1 - P_{e_i})}(F - 1)^{\frac{1}{2}} \rfloor \right\} \\ &= \lfloor \inf_{P_{e_i} \in (P_{e_t}, 2P_{e_t})} \left\{ (1 - P_{e_i})(F - 1) + |\theta_i| \cdot \sqrt{P_{e_i}(1 - P_{e_i})}(F - 1)^{\frac{1}{2}} \right\} \rfloor \\ &= \lfloor (1 - 2P_{e_t})(F - 1) \rfloor \end{aligned}$$

Since $F \leq 1 + \frac{1}{2P_{e_t}} \Rightarrow P_{e_t} \leq \frac{1}{2(F-1)}$, then

$$\inf_{P_{e_i} \in (P_{e_t}, 2P_{e_t})} U'_{i,F} = \lfloor F - 2 \rfloor = F - 2 \quad (12)$$

Thus, for any value of $P_{e_i} \in (P_{e_t}, 2P_{e_t})$, and $F \in \{2, \dots, F_u\}$, $U'_{i,F} \geq F - 2$. Substituting in (4), we can clearly see that $U_{i,F}$ can only be $F - 2$ or $F - 1$. The change from the first to the second formula occurs when:

$$U'_{i,F} = F - 1 \Rightarrow F \leq 1 + \frac{\theta_i^2(1 - P_{e_i})}{P_{e_i}}$$

For any value of the right-hand-side of the above inequality and $F \in \{2, \dots, F_u\}$, we can only have four possible cases:

- 1) If $U_{i,F_1} = U_{i,F_2} = U_{i,F_1+F_2} = F - 1$:

$$U_{i,F_1+F_2} - (U_{i,F_1} + U_{i,F_2}) = 1 \geq 0;$$

- 2) If $U_{i,F_1} = U_{i,F_2} = F - 1$ and $U_{i,F_1+F_2} = F - 2$:

$$U_{i,F_1+F_2} - (U_{i,F_1} + U_{i,F_2}) = 0 \geq 0;$$

- 3) If $U_{i,F_1} = F - 1$ and $U_{i,F_2} = U_{i,F_1+F_2} = F - 2$:

$$U_{i,F_1+F_2} - (U_{i,F_1} + U_{i,F_2}) = 1 \geq 0;$$

- 4) If $U_{i,F_1} = U_{i,F_2} = U_{i,F_1+F_2} = F - 2$:

$$U_{i,F_1+F_2} - (U_{i,F_1} + U_{i,F_2}) = 2 \geq 0.$$

Thus, $U_{i,F}$ is superadditive for the four possibilities \Rightarrow super-additive in $F \in \{2, \dots, F_u\}$. Finally, for $F = 1$, $U_{i,F} = 0$ which maintains the superadditivity of the function in the range $F \in \{1, \dots, F_u\}$. This concludes the proof.