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Localization of Wireless Sensors via Nuclear Norm for Rank Minimization

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***Abstract*— The low rank feature of location estimation in Wireless Sensor Networks (WSNs) makes it feasible to use nuclear norm minimization as an accurate and fast solution for low-dimensional embedding problems. In this paper, a novel localization algorithm for WSNs is proposed by using nuclear norm for rank minimization. We formulate the location finding problem from only a small fraction of random entries of Euclidean Distance Matrix (EDM) as a low-rank matrix recovery problem, subject to a set of linear equality constraints. We show that a measurement matrix using orthogonal projection obeys the RIP and thus, supports a sufficient condition for the recovery of the low-rank matrix with overwhelming probability. For simplicity, Singular Value Thresholding (SVT) algorithm, a standard convex optimization approach, is used for the nuclear norm minimization. Simulation results demonstrate that in a 100 m $\times$ 100 m area, for a small scale network with 100 nodes, only 20% of measurements is needed to achieve a 0.5 m localization error, while 3% needed to achieve a 0.05 m error for a comparatively large scale network with 1000 nodes.**

Keywords- Wireless sensor networks, Nuclear norm minimization, Rank minimization, Low-dimensional euclidean embedding

I. Introduction

Technological advances have led to the construction of ad hoc networks using inexpensive sensor nodes to exchange messages with one another [1][2]. Accurate, low cost and real-time sensor localization is an essential and challenging problem that arises in various applications, such as collision warning in Vehicular Ad Hoc Networks (VANets) [3], distance warning for automatic parking [4], monitoring and emergency response in smart buildings, etc.

Manifold Learning (ML)-based algorithms (e.g., multidimensional scaling (MDS), and isometric mapping (Isomap), etc.) have been extensively studied in machine learning, which formulate the localization problem from pair-wise distance measurements as a dimensionality reduction problem on a Riemann manifold. Recently, theoretical guarantees for extracting the underlying geometric structure of distance data, known as *Euclidean Distance Matrix* (EDM) embedding, have been proven for MDS, which can be used to accurately estimate the relative positions of a group of sensor nodes, under the assumption that EDM is mainly known [5]. However, this assumption may not realistically hold when sensor networks have of a large number of sensors (in the order of thousands or higher for example), in which cases it may not be energy efficient to take a large number of pair-wise measurements for each sensor, and communicate with the central node. Therefore, it is more interesting to localize the sensor nodes with high accuracy, under the knowledge of only a small fraction of noisy entries of EDM that are transmitted to the central node due to noisy environment [6], limited communication range and energy constraints. A similar problem is solved in [7], where a connectivity graph and the shortest path are exploited for the EDM completion and the embedding dimension learning based on the incomplete distance measurements. However, to achieve a high localization accuracy, the graph should be connected well. Thus, each sensor node is required to have the distance measuring ability from its vicinity, and a large number of measurements is still a necessity, especially for large scale networks.

In our previous work, we addressed the localization problem in WSNs using the theory of *Compressive Sensing* (CS) [8], which offers accurate recovery of sparse signals from a small number of measurements by solving an ℓ_1-minimization problem [9]. Each column of the incomplete EDM, recording the distance measurements from neighboring nodes, is formulated as a sparse signal and thus, can be recovered for further MDS. A sparse approximation is made and a two stage algorithm consisting of a recovering procedure based on CS and followed by an MDS is conducted. The number of measurements needed for a successful recovery by CS should obey $O(k{\cdot}n\log n)$, where n is the total number of sensor

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nodes, and k is the number of neighbors for each node, with $2 < k \ll n$. It is further realized that a low rank matrix, but not necessarily sparse, exists in the problem due to its intrinsic low-dimensional characteristic. Recent research in signal processing shows that the ℓ_1 heuristic is a special case of nuclear norm heuristic (see Section II) [10], which further suggests that our previous work based on the CS could be extended by using the nuclear norm to provide location estimates, possibly with a smaller number of measurements.

In this paper, we propose a novel localization algorithm in sensor networks based on nuclear norm minimization. We formulate the localization problem from only a small fraction of random entries of EDM as a low-rank matrix recovery problem, subject to a set of linear equality constraints. Since the general rank minimization problem is NP-hard [11], a *nuclear norm*, referred to as the sum of the singular values of a matrix, is used to find the minimum rank solution. A relationship between the EDM and the low rank matrix, known as the *Gram Matrix* (see Section III-C), is exploited to build the linear equality constraints. A measurement matrix by orthogonal projection obeys the *Restricted Isometry Property* (RIP) (see Section III-B) that supports a sufficient condition for the recovery of the low-rank matrix with overwhelming probability, provided that the number of known entries of EDM obeys $O(d \cdot n \log n)$, where d is the intrinsic dimension. A list of numerical methods are developed for solving the nuclear norm minimization problem [12], such as SDPT3 [13], SeDuMi [14], etc. Developing more efficient algorithms is beyond the scope of this paper. We simply use the Singular Value Thresholding (SVT) algorithm, a standard convex optimization approach, for nuclear norm minimization, due to its first-order and easy-to-implement characteristic [15]. It is demonstrated that for small scale networks, only 20% of entries is needed for accurate location estimation, while 3% needed for large scale networks.

The reminder of this paper is organized as follows. System model and notations are given in Section II. In Section III, the localization algorithm is described, including the problem formulation with a brief introduction of the SVT algorithm, an orthogonal projection to induce incoherence needed by SVT, and exploiting the relationship of EDM and the Gram Matrix to build the linear equality constraints for SVT. The performance of the algorithm regarding the localization accuracy versus the number of measurements is demonstrated and compared in Section IV. Finally, Section V concludes the paper.

II. Preliminaries

We start with a typical localization scenario in WSNs, where a large number of wireless sensors are randomly deployed in an area, taking distance measurements (by RSS readings, hop count, etc.) from each other randomly, and passing the measured data to a central node, as illustrated in Fig. 1. We assume that the measurement data is incomplete, with only a small fraction of noisy distance information transmitted to the central node because of the noisy environment, the limit of communication range, and the energy constraints.

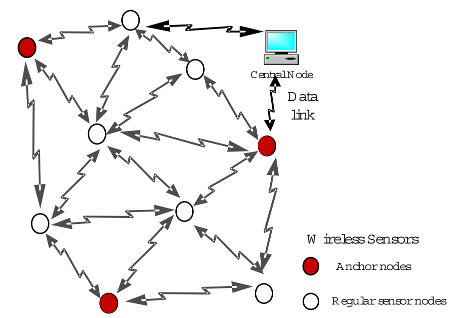

Fig. 1: A typical localization scenario in WSNs.

Among the total number of n sensor nodes, p of them are anchor nodes (e.g., equipped with GPS), whose positions are known, with $p \ll n$. Let $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ in $\mathbb{R}^d$ represent the coordinates of these n sensor nodes in d dimensional Euclidean space. A symmetric matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$, defined as the Euclidean Distance Matrix, has the property $D_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|^2$, and $D_{ii} = 0, i, j \in \{1, 2, ..., n\}$. Let $\mathbf{J} := \mathbf{I}_n - \frac{1}{n}\mathbf{e}\mathbf{e}^T$, where $\mathbf{e} = [1, 1, ..., 1]^T$, with T as the transpose operator. Further define the Gram Matrix $\mathbf{G} = -\frac{1}{2}\mathbf{JDJ}$, where $\mathbf{G}$ is positive semi-definite ($\mathbf{G} \succeq 0$), and rank$(\mathbf{G}) \leq d$, as illustrated in [16]. Since all sensors lie on 2D or 3D Euclidean space, the intrinsic dimension (d) is either 2 or 3, thus $d \ll n$, which implies low rank characteristic of the Gram Matrix. If the matrix $\mathbf{D}$ is fully known, the relative positions of all sensor nodes can be obtained by simply taking a square root of the Gram Matrix [5]. However, in many applications, only a small fraction of random sampling of the noisy distance information can be obtained at the central node. Thus, the objective is to determine the physical positions of all sensor nodes simultaneously in a central node from the incomplete noisy EDM, while achieving a high level of accuracy. For simplicity, our methods are illustrated with 2D sensor networks and they can easily be extended to 3D cases under the same methodology.

We first give some notations about matrix and vector norms that will be used throughout the paper. For a symmetric matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$, $\sigma_i(\mathbf{X})$ denotes the i-th largest singular value of $\mathbf{X}$. The rank of $\mathbf{X}$ is denoted by d, and is equal to the number of non-zero singular values.

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1) The *Frobenius norm* of a matrix $\mathbf{X}$ is defined as:

$$\|\mathbf{X}\|_F := \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{X}_{ij}^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{i=1}^{d} \sigma_i^2 \right\}^{\frac{1}{2}}. \tag{1}$$

where $\langle \mathbf{X}, \mathbf{X} \rangle$ is the scalar product.
2) The ℓ_2 norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is defined as:

$$\|\mathbf{x}\| = \left\{ \sum_{i=1}^{n} \mathbf{x}_i^2 \right\}^{\frac{1}{2}}. \tag{2}$$

III. Localization via Nuclear Norm Minimization

A. Nuclear norm for rank minimization

The problem of finding the smallest embedding dimension of a valid Gram Matrix $\mathbf{G}$ with the known noisy and incomplete EDM, defined in Section II, can be expressed as the following rank minimization problem, subject to a set of linear equality constraints.

$$\begin{aligned} &\text{minimize} \;\; rank(\mathbf{G}) \\ &\text{subject to} \;\; \mathbf{G} \succeq 0 \\ &\mathbf{b} = \mathbf{\Phi} \cdot vec(\mathbf{D}) + \varepsilon, \end{aligned} \tag{3}$$

where $\mathbf{b}$ is the actual distance measurement data, $\mathbf{b} \in \mathbb{R}^m$, with m representing the total number of measurements, and $\mathbf{\Phi}$ is a $m \times n^2$ random sampling matrix that maps the EDM matrix from the high-dimensional space $\mathbb{R}^{n \times n}$ to the low dimensional measurement space $\mathbb{R}^m$. $vec(\mathbf{D})$ is the columnwise vector form of matrix $\mathbf{D}$, and ε is the measurement noise.

Since the rank minimization problem is NP-hard in general [11], a number of heuristic algorithms have been proposed in the literatures [13][14][15]. Of particular importance is the nuclear norm, which replaces (3) with

$$\begin{aligned} &\text{minimize} \;\; \|\mathbf{G}\|_* \\ &\text{subject to} \;\; \mathbf{G} \succeq 0 \\ &\mathbf{b} = \mathbf{\Phi} \cdot vec(\mathbf{D}) + \varepsilon, \end{aligned} \tag{4}$$

Similar to the result of using ℓ_1 norm for the CS, *B. Recht* [11] provided the necessary and sufficient conditions for the success of the nuclear norm for rank minimization – that is matrix $\mathbf{\Phi}$ should obey the following RIP property.

Definition 1: Restricted Isometry Property (RIP)

For every integer $1 \leq d \leq n$, define the d-restricted isometry constant to be the smallest number δ_d, $(0 < \delta_d < 1)$, such that

$$(1 - \delta_d)\|\mathbf{X}\|_F \leq \|\mathbf{\Phi} \cdot vec(\mathbf{X})\| \leq (1 + \delta_d)\|\mathbf{X}\|_F \tag{5}$$

holds for all matrices $\mathbf{X}$ of rank at most d.

A list of numerical methods are developed for solving the problem in (4), such as SDPT3 [13], SeDuMi [14], etc. Developing more efficient algorithms is beyond the scope of this paper. We simply use the SVT algorithm [15], a standard convex optimization approach, for nuclear norm minimization, due to its first-order and easy-to-implement characteristic. It is also indicated in [10] that if $\mathbf{\Phi}$ obeys the RIP, with very high probability, (4) gives a sufficient solution for (3), provided that the number of known entries obeys $O(d \cdot n \log n)$.

It is noticed that in the problem of localization of wireless sensors, there are two problems that should be addressed before applying the nuclear norm minimization to find the locations of wireless sensors. First, how to find the measurement operator $\mathbf{\Phi}$ such that it obeys the RIP. Second, since a small collection of random entries of EDM is known, a relationship between the EDM and the Gram Matrix is needed to be exploited, such that the linear equality constraints in (4) can be re-formulated based on the low rank Gram Matrix. We will solve the two problems in the following subsections.

B. Measurement matrix by orthogonal projection

In the localization problem, the measurement matrix $\mathbf{\Phi}$ represents a random selection matrix. Each row of $\mathbf{\Phi}$ is a $1 \times n^2$ vector with all elements equal to zero except $\phi(\ell) = 1$, where ℓ is the index of element in $vec(\mathbf{D})$ that is selected. Further define an operator $\mathbf{\Phi}_\Omega$ as an orthogonal projection onto the span of $n \times n$ matrices, such that the (i,j)-th component of $\mathbf{\Phi}_\Omega(\mathbf{D})$ is equal to D_{ij} if $(i,j) \in \Omega$ and zero otherwise, where Ω indicates the positions of known entries of $\mathbf{D}$. Thus, we can write a matrix representation of $\mathbf{\Phi} \cdot vec(\mathbf{D})$ as $\mathbf{\Phi}_\Omega(\mathbf{D})$. In other words, $\mathbf{\Phi} \cdot vec(\mathbf{D})$ is the vector consisting of the columnwise non-zero entries of $\mathbf{\Phi}_\Omega(\mathbf{D})$.

Proposition 1: A random orthogonal projection $\mathbf{\Phi}_\Omega$ defined above holds the RIP in (5).

Proof: According to the definitions in (1) and (2), we have:

$$\|\mathbf{\Phi} \cdot vec(\mathbf{D})\| = \|\mathbf{\Phi}_\Omega(\mathbf{D})\|_F. \tag{6}$$

where the known positions of entries of $\mathbf{D}$ is randomly selected, with $|\Omega| = m$, and $m = O(d \cdot n \log n)$. Since $\mathbf{\Phi}_\Omega(\mathbf{D})$ only keeps all the elements of $\mathbf{D}$ inside of Ω, there always exists a $\delta_d \in (0,1)$ such that the following inequality holds.

$$(1-\delta_d)\|\mathbf{D}\|_F \leq \|\mathbf{\Phi}_\Omega(\mathbf{D})\|_F \leq \|\mathbf{D}\|_F < (1+\delta_d)\|\mathbf{D}\|_F. \tag{7}$$

■

C. EDM and Gram Matrix

The Gram Matrix $\mathbf{G}$ can also be represented as the matrix of inner products [5], i.e., $\mathbf{G} = \mathbf{X}^T\mathbf{X}$, where $\mathbf{X} = [\mathbf{x}_1, ...\mathbf{x}_i, ..., \mathbf{x}_n], \mathbf{x} \in \mathbb{R}^d$. Since for each element in

$\mathbf{D}$, it is derived that $D_{ij} = \|x_i\|^2 + \|x_j\|^2 - 2x_i^T x_j = G_{ii} + G_{jj} - 2G_{ij}, \forall i, j \in \{0, 1, ..., n\}$. The following relationship between the EDM and the Gram Matrix holds.

$$\mathbf{D} = diag(\mathbf{G}) \cdot \mathbf{e}^T + \mathbf{e} \cdot diag(\mathbf{G})^T - 2\mathbf{G}, \tag{8}$$

with $\mathbf{G} = -\frac{1}{2}\mathbf{JDJ}$.

We find from (8) that,

$$2\mathbf{\Phi}_\Omega(\mathbf{G}) = \mathbf{\Phi}_\Omega(diag(\mathbf{G})\mathbf{e}^T) + \mathbf{\Phi}_\Omega(\mathbf{e}diag(\mathbf{G})^T) - \mathbf{\Phi}_\Omega(\mathbf{D}) \tag{9}$$

where

$$diag(\mathbf{G}) = -\frac{1}{2}diag(\mathbf{JDJ}) = \frac{1}{n}[\sum_{i=1}^{n} D_{i1}, ..., \sum_{i=1}^{n} D_{in}]^T - \frac{1}{2n^2}[\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}, ..., \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}]^T \tag{10}$$

In large sensor networks, since $\mathbf{\Phi}$ represents a random orthogonal sampling, we can assume that

$$\frac{1}{n}\sum_{i=1}^{n} D_{ij} \approx \frac{1}{|n_j|}\sum_{i \in n_j} D_{ij}, j = 1, 2, ..., n \tag{11}$$

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} \approx \frac{1}{m}\sum_{i \in n_j}\sum_{j=1}^{n} D_{ij} \tag{12}$$

where n_j represents the set of sensors with which that sensor j takes measurement, and $|n_j|$ represents the total number of measurements in the set, with $\sum_{j=1}^{n} |n_j| = m$.

Based on (9)-(12), an approximate set of linear equality constraints on the Gram Matrix can be derived, and the localization problem is fully formulated from only a small fraction of random entries of EDM as a low-rank matrix recovery problem subject to a set of linear equality constraints, as (3) shows. For simplicity, we use SVT algorithm in the simulation to find the minimum rank solution, provided that the number of known entries of EDM obeys $O(d \cdot n \log n)$. Note that the above solution only provides a 2D relative position map, thus a position alignment for obtaining the global positions is still needed by scaling, rotating and shifting, *i.e.*,

$$\mathbf{x}'_i = s \cdot r \cdot \hat{\mathbf{x}}_i + t, \quad \forall i \in \{1, 2, ..., n\} \tag{13}$$

where $\hat{\mathbf{x}}_i$ is the estimated relative position for sensor i, and $\mathbf{x}'_i$ is its global position. s, r, t are the variables that can be found based on the prior location knowledge of the p anchor nodes.

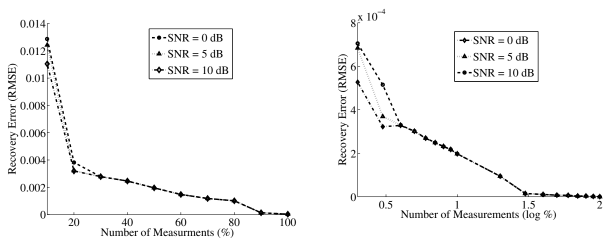
IV. Performance Evaluation

The effectiveness and properties of the proposed localization scheme are studied through simulations. Sensors are randomly placed in a 100 m $\times 100$ m area. Two different scale of networks are considered, using 100 nodes and 1000 nodes to represent small and large scale networks, respectively. 4 anchor nodes are placed on the four corners for position alignment. The localization accuracy is evaluated by:

$$E = \frac{1}{n}\|\mathbf{X} - \mathbf{X}'\|_F \tag{14}$$

where, $\mathbf{X}' = [\mathbf{x}'_1, ..., \mathbf{x}'_i, ..., \mathbf{x}'_n]$, $\mathbf{x}'_i$ is the estimated global position of sensor i, and $\mathbf{x}$ is its actual position, $\mathbf{x}_i, \mathbf{x}'_i \in \mathbb{R}^d, d = 2$.

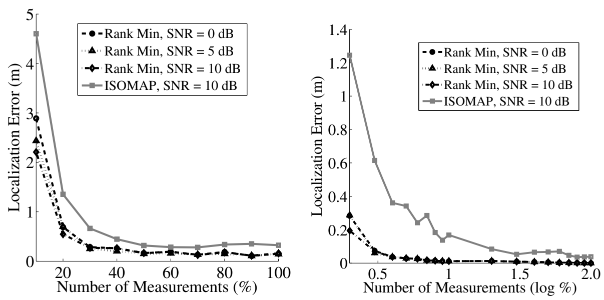
Fig. 2 shows the recovery of the Gram Matrix based on (4) with respect to the number of entries in $\mathbf{D}$ that are known under different noisy environment, in small and large networks, respectively. Signal-to-noise ratio (SNR) varies from 0 dB to 10 dB, which is defined as the ratio of the transmit signal power to the noise power at the receiver. The recovery error is evaluated by the relative Root Mean Squared Error (RMSE). It is demonstrated that only 20% of entries of EDM are necessary for a good recovery when $n = 100$, and 3% ($\log 3 \approx 0.48$) needed when $n = 1000$.



(a) n=100 (b) n=1000

Fig. 2: The recovery of Gram Matrix (G) with respect to the number of entries known in EDM, under different SNR, in small and large scale networks.

Fig. 3 shows the whole performance of the proposed scheme in terms of the localization error, with respect to the number of measurements needed. SNR varies from 0 dB to 10 dB, and two different scenarios are still considered, with 100 and 1000 sensor nodes, respectively. We compare the proposed scheme with traditional Isomap, in which graph connectivity and the shortest path are exploited for the EDM completion and the embedding dimension learning, under the same number of measurements, when SNR $= 10$ dB. In both cases, the proposed scheme achieves a better performance in terms of both the localization accuracy and the number of measurements reduction, especially when the scale of the network is large. In the proposed scheme, an error of 0.5 m is obtained when $n = 100$, and 0.05 m when $n = 1000$, as long as the number of measurements (m) conforms with proposition (1). Specifically, in small scale network with $n = 100$, when the number of measurements reaches 20%, which approximately obeys $O(d{\cdot}n\log n)/n^2 \approx 13\%$, the localization error decreases sharply. The turning point exists at 3% ($\log 3 \approx 0.48$) when $n = 1000$, which also conforms with $O(d \cdot n\log n)/n^2 \approx 2\%$.



(a) n=100 (b) n=1000

Fig. 3: The comparison of localization error with respect to the number of entries known in EDM, under different SNR, in small and large scale networks, respectively.

Fig. 4 shows an intuitive localization result of 100 randomly deployed sensor nodes, provided by the nuclear norm minimization solution. 30% compressive measurements are transmitted to the central node under SNR = 5 dB. Circles are the actual positions of these 100 nodes, while squares represent the estimated locations.

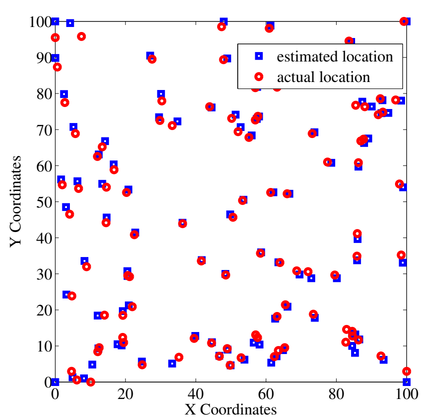


Fig. 4: An intuitive localization result, n = 100, SNR = 5 dB, with 30% compressive measurements.

V. Conclusion

In this paper, we have proposed a novel localization algorithm for WSNs based on nuclear norm for rank minimization. The intuition behind this technique is that the intrinsic dimension of the measured data for localization is low and thus, location estimation can be formulated as a low rank problem. The location can be well recovered from only a small number of noisy measurements through a nuclear norm minimization program. We have addressed that a random measurement matrix by orthogonal projection obeys the RIP that supports a sufficient recovery of the low-rank Gram Matrix with overwhelming probability, provided that the number of known entries of EDM obeys $O(d{\cdot}n\log n)$. A position alignment is conducted by scaling, rotating and shifting of the relative positions that are obtained from the SVT algorithm. The simulation results demonstrate that the proposed algorithm leads to a better performance in terms of the localization accuracy, and the number of measurements reduction, that can be considered as an extension of the traditional manifold learning approaches.

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