Distributed Source Localization Using ESPRIT Algorithm

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Abstract—A new algorithm based on ESPRIT is proposed for the estimation of central angle and angular extension of distributed sources. The central angles are estimated using TLS-ESPRIT for both incoherently distributed (ID) and coherently distributed (CD) sources. For CD sources, the extension width is estimated by constructing a one-dimensional (1-D) distributed source parameter estimator (DSPE) spectrum for each source. For ID sources, the extension widths are estimated using the central moments of the distribution. The algorithm can be used for sources with different angular distributions.

Index Terms—Array signal processing, coherently distributed source, incoherently distributed source, parametric localization.

I. INTRODUCTION

S EVERAL applications of array processing—such as operating antenna arrays at base stations for mobile communications, passive sonar, and underwater acoustics—require a spatially distributed source modeling: a modeling to which much attention has been recently paid in the literature of array processing [1]–[3]. Depending on the nature of reflection and scattering in the above examples, signal components arriving from different directions exhibit varying degrees of correlation, ranging from totally incorrelated (incoherent) to fully correlated (coherent) cases. Distributed source modeling suffers from a deficiency, namely, seizing the whole observation space by signal components and nullifying the noise subspace. This begets a breakdown of the techniques that exploit the orthogonality of signal and noise subspaces, such as MUSIC [4] and its variants.

Several distributed-source localization techniques have been proposed in the recent literature. The first attempt for generalization of the signal and noise subspace concepts to distributed sources has been done in [1]. Based on these concepts, an algorithm called the distributed source parameter estimator (DSPE) has been proposed, which is the generalization of MUSIC to distributed sources and can be applied to both coherently distributed (CD) and incoherently distributed (ID) sources. Since

DSPE is essentially a MUSIC-type algorithm, it suffers from intrinsic disadvantages of MUSIC such as array manifold measurements and calibration.

In [5], a maximum likelihood (ML) algorithm has been proposed for localization of Gaussian distributed sources. The likelihood function is jointly maximized for all parameters of the Gaussian model. The computational complexity of this method grows exponentially with the number of sources.

Similar to DSPE, an algorithm called DISPARE has been presented for localization of ID sources [6]. In DISPARE, the covariance matrix of the array is approximated by a low-rank model, and then, a spatial spectrum is constructed with peaks associated with spatial parameters of the ID sources.

In [7], an algorithm has been presented for localization of a single uniformly incoherently distributed (UID) source. In this algorithm, the extension width of the source is estimated from the eigenvalues of the correlation matrix. Estimation of the source central angle is based on the properties of eigenvectors of the correlation matrix. It has been shown that the eigenvectors of the correlation matrix are modulated discrete prolate spheroidal sequences (DPSSs) [8]. In [9], the central angle of the UID source is estimated by TLS-ESPRIT [10], and then, the extension width is estimated using the algorithm presented in [7].

In [2], a Taylor series expansion has been used to derive an approximate model called the generalized array manifold (GAM). GAM is based on a linear combination of array location vector and its derivatives. Using GAM, an algorithm is presented to estimate the source *spatial signature* by exploiting a Vandermonde structure. The algorithm can only be applied to uniform linear arrays (ULAs) and uniform CD sources.

In [11] and [12], a distributed source is approximated by two point sources. Then, the directions-of-arrival (DOAs) of the point sources are estimated using MUSIC or ROOT-MUSIC. The angular spread is obtained by using a lookup table that describes the relation between the distance of the two estimated DOAs and the angular spread. In [13], a subspace fitting method has also been proposed for estimating the angular parameters of distributed sources.

In this paper, we propose an algorithm for parameter estimation of distributed sources based on TLS-ESPRIT. For CD sources, using a GAM modeling, we show that a rotational eigenstructure exists approximately for two identical closely spaced subarrays. Thus, the TLS-ESPRIT algorithm is used to estimate the central angles of sources. Extension widths are estimated by constructing a one-dimensional (1-D) DSPE spectrum for each source. In this algorithm, the DSPE spectrum is computed for each source separately. Thus, sources might

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have different distributions. Since the range of extension width is much smaller than the range of central angle, computing the 1-D DSPE spectrum for each source has a small computational cost.

For ID sources, we give an approximation to the covariance matrix by using the GAM. Using a first-order Taylor series expansion, we show that each ID source approximately introduces a two-dimensional (2-D) subspace in the observation space. However, higher order Taylor series might be used to improve the accuracy of approximation. Again, we show that the rotational invariant structure exists for two identical closely spaced subarrays. Hence, TLS-ESPRIT can be used to estimate DOAs—a pair of DOAs for each source. The covariance matrix is formulated by the location vectors and their derivatives as well as the central moments of the distributions. We will show that the distance of the two estimated DOAs is related to the source angular spread.

II. DATA MODEL

Consider an array of $2p$ sensors (p doublets). Assume that the two sensors in each doublet are identical and have the same gain, phase, and sensitivity pattern and are separated by a constant displacement vector $\vec{d}$. The two induced subarrays are denoted by X and Y. Furthermore, it is assumed that q narrowband distributed sources with the same central frequency ω_0 are present in the environment of these subarrays. The complex envelope of the output of ith sensor in subarray X is

$$x_i = \sum_{m=1}^{q} \int_{-\pi/2}^{\pi/2} a_i(\theta) s_m(\theta; \boldsymbol{\psi}_m) \, d\theta + n_{x_i} \tag{1}$$

where
$a_i(\theta)$ response of the ith sensor to a unit energy source emitting at direction θ with respect to the orthogonal direction to the displacement vector $\vec{d}$;
$s_m(\theta; \boldsymbol{\psi}_m)$ angular density of the mth source;
$\boldsymbol{\psi}_m$ mth source location parameter vector;
n_{x_i} additive zero-mean noise at the ith sensor uncorrelated from the signals.

Examples of the parameter vector $\boldsymbol{\psi}_m$ are the two limits of DOA for uniform spatial extension or the angle of maximum power and standard deviation for a Gaussian distribution. Note that spatial overlap of signals is allowed.

The complex envelope of the output of ith sensor in subarray Y is

$$y_i = \sum_{m=1}^{q} \int_{-\pi/2}^{\pi/2} a_i(\theta) e^{j\omega_0 \tau(\theta)} s_m(\theta; \boldsymbol{\psi}_m) \, d\theta + n_{y_i} \tag{2}$$

where n_{y_i} is an additive zero-mean noise at the ith sensor of subarray Y uncorrelated from the signals, and $\tau(\theta)$ is the propagation delay between the identical elements of a doublet in two subarrays for a signal arriving at direction θ. Throughout, we assume that θ in (1) and (2) is measured with respect to a direction orthogonal to $\vec{d}$. Then, we have

$$\tau(\theta) = \frac{d}{c} \sin \theta \tag{3}$$

where c is the wave propagation speed.

In vector representation, (1) and (2) can be written as

$$\mathbf{x} = \sum_{m=1}^{q} \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta) s_m(\theta; \boldsymbol{\psi}_m) \, d\theta + \mathbf{n}_x \tag{4}$$

$$\mathbf{y} = \sum_{m=1}^{q} \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta) e^{j\omega_0 \tau(\theta)} s_m(\theta; \boldsymbol{\psi}_m) \, d\theta + \mathbf{n}_y \tag{5}$$

where
$\mathbf{x}$ and $\mathbf{y}$ output vectors of the subarrays X and Y, respectively;
$\mathbf{n}_x$ and $\mathbf{n}_y$ corresponding noise vectors;
$\mathbf{a}(\theta)$ subarray X location vector for a source at direction θ.

For the subarray X, the covariance matrix can be written as

$$\mathbf{R}_{xx} = \sum_{i=1}^{q} \sum_{j=1}^{q} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta) p_{ij}(\theta, \theta'; \boldsymbol{\psi}_i, \boldsymbol{\psi}_j)$$
$$\cdot \mathbf{a}^H(\theta') \, d\theta \, d\theta' + \mathbf{R}_n \tag{6}$$

where superscript H represents Hermitian transposition, $\mathbf{R}_n$ is the noise correlation matrix, and

$$p_{ij}(\theta, \theta'; \boldsymbol{\psi}_i, \boldsymbol{\psi}_j) = E\{s_i(\theta; \boldsymbol{\psi}_i) s_j^*(\theta'; \boldsymbol{\psi}_j)\} \tag{7}$$

is called the *angular cross-correlation kernel*. In (7), $E\{\cdot\}$ denotes statistical expectation, and $*$ represents complex conjugation.

A. Coherently Distributed Sources

A source is called coherently distributed (CD) if the received signal components of the source at different angles are delayed and scaled replicas of the same signal. In such a case, the angular signal density can be represented by

$$s(\theta; \boldsymbol{\psi}_i) = \gamma_i g_i(\theta; \boldsymbol{\psi}_i) \tag{8}$$

where γ_i is a random variable, and $g_i(\theta; \boldsymbol{\psi}_i)$, which is the *deterministic angular signal density*, is a complex-valued deterministic function of θ. Note that a CD signal is decomposable into random and deterministic components. The deterministic component $g_i(\theta; \boldsymbol{\psi}_i)$ characterizes the spatial distribution of the source, and the random component γ_i reflects the temporal behavior of the source. The index i in $g_i(\theta; \cdot)$ has been used to emphasize different deterministic angular signal densities.

It is assumed that $g_i(\theta; \boldsymbol{\psi}_i)$ belongs to a set of parametric functions characterized by two parameters: *central angle* and *extension width*. The central angle is the mass center of the distribution $g_i(\theta; \boldsymbol{\psi}_i)$ defined as

$$\theta_{0i} = \int_{-\pi/2}^{\pi/2} \theta g_i(\theta; \boldsymbol{\psi}_i) \, d\theta. \tag{9}$$

For the extension width, several definitions might be proposed. For instance, for a source uniformly distributed over the interval $[\theta_0 - \Delta, \theta_0 + \Delta]$, the extension width can be defined as the angular extent Δ, and for a Gaussian distributed source,

the extension with can be defined as the standard deviation of the distribution.

From (8), the angular cross-correlation kernel for a CD signal is given by

$$p_{ij}(\theta, \theta'; \boldsymbol{\psi}_i, \boldsymbol{\psi}_j) = g_i(\theta; \boldsymbol{\psi}_i) E\{\gamma_i \gamma_j^*\} g_j^*(\theta'; \boldsymbol{\psi}_j). \quad (10)$$

Let $\mathbf{b}(\boldsymbol{\psi}_i)$ be defined as

$$\mathbf{b}(\boldsymbol{\psi}_i) = \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta) g_i(\theta; \boldsymbol{\psi}_i) \, d\theta. \quad (11)$$

Then, (6) can be written as

$$\mathbf{R}_{xx} = \sum_{i=1}^{q} \sum_{j=1}^{q} \mathbf{b}(\boldsymbol{\psi}_i) E\{\gamma_i \gamma_j^*\} \mathbf{b}^H(\boldsymbol{\psi}_j) + \mathbf{R}_n$$
$$= \mathbf{B}(\boldsymbol{\psi}) \boldsymbol{\Gamma} \mathbf{B}^H(\boldsymbol{\psi}) + \mathbf{R}_n \quad (12)$$

where

$$\mathbf{B}(\boldsymbol{\psi}) = [\mathbf{b}(\boldsymbol{\psi}_1) \quad \mathbf{b}(\boldsymbol{\psi}_2) \quad \cdots \quad \mathbf{b}(\boldsymbol{\psi}_q)] \quad (13)$$

and

$$[\boldsymbol{\Gamma}]_{ij} = E\{\gamma_i \gamma_j^*\}. \quad (14)$$

In addition, (4) can be written as

$$\mathbf{x} = \mathbf{B}(\boldsymbol{\psi}) \boldsymbol{\gamma} + \mathbf{n}_x \quad (15)$$

where $\boldsymbol{\gamma} = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_q]^T$. Equation (12) shows that the signal subspace [defined as the column span of $\mathbf{B}(\boldsymbol{\psi})$] is spanned by the generalized eigenvectors of the matrix pencil $(\mathbf{R}_{xx}, \mathbf{R}_n)$ corresponding to the q largest generalized eigenvalues [4].

Note that if the noise is spatially white, i.e., $\mathbf{R}_n = \sigma_n^2 \mathbf{I}$, where σ_n^2 is the unknown noise power, then the signal subspace is spanned by the eigenvectors of $\mathbf{R}_{xx}$ corresponding to the q largest eigenvalues. For simplicity, we assume that the noise is spatially white—nonwhite noise can be handled by prewhitening.

Similarly, it can be shown that

$$\mathbf{y} = \mathbf{C}(\boldsymbol{\psi}) \boldsymbol{\gamma} + \mathbf{n}_y \quad (16)$$

and

$$\mathbf{R}_{yy} = \mathbf{C}(\boldsymbol{\psi}) \boldsymbol{\Gamma} \mathbf{C}^H(\boldsymbol{\psi}) + \sigma_n^2 \mathbf{I} \quad (17)$$

where

$$\mathbf{C}(\boldsymbol{\psi}) = [\mathbf{c}(\boldsymbol{\psi}_1) \quad \mathbf{c}(\boldsymbol{\psi}_2) \quad \cdots \quad \mathbf{c}(\boldsymbol{\psi}_q)] \quad (18)$$

and

$$\mathbf{c}(\boldsymbol{\psi}_i) = \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta) e^{j\omega_0 \tau(\theta)} g_i(\theta; \boldsymbol{\psi}_i) \, d\theta. \quad (19)$$

B. Incoherently Distributed Sources

A source is said to be ID if the signal rays arriving from different directions are uncorrelated, i.e.,

$$E\{s_i(\theta; \boldsymbol{\psi}_i) s_i^*(\theta'; \boldsymbol{\psi}_i)\} = \sigma_{si}^2 \rho_i(\theta; \boldsymbol{\psi}_i) \delta(\theta - \theta') \quad (20)$$

where σ_{si}^2 is the ith source power, and $\rho_i(\theta; \boldsymbol{\psi}_i)$ is the normalized *angular power density* of the source satisfying

$$\int_{-\pi/2}^{\pi/2} \rho_i(\theta; \boldsymbol{\psi}_i) \, d\theta = 1. \quad (21)$$

The index i has been used to emphasize that it is not necessary for the sources to have identical angular power densities. Similar to the CD case, the central angle is defined as the mass center of $\rho_i(\theta; \boldsymbol{\psi}_i)$, and the extension width is defined as the parameter determining the angular extent.

In this case, we assume that different ID sources are uncorrelated. Using this assumption for ID sources, we get

$$\mathbf{R}_{xx} = \mathbf{R}_{yy}$$
$$= \sum_{i=1}^{q} \int_{-\pi/2}^{\pi/2} \sigma_{si}^2 \mathbf{a}(\theta) \rho_i(\theta; \boldsymbol{\psi}_i) \mathbf{a}^H(\theta) \, d\theta + \mathbf{R}_n. \quad (22)$$

III. DSPE ALGORITHM

In [1], the DSPE algorithm was proposed in which the signal and noise subspace concept was generalized to distributed sources. DSPE is essentially a MUSIC-type algorithm and, hence, needs array manifold measurement and calibration. In this section, we review DSPE for both CD and ID source models.

A. CD Source Localizer

In the case of CD sources, columns of $\mathbf{B}(\boldsymbol{\psi})$ and the q eigenvectors of $\mathbf{R}_{xx}$ corresponding to the q largest eigenvalues of $\mathbf{R}_{xx}$ span the same subspace called the signal subspace. The DSPE spectrum for CD sources is defined as

$$P_{\text{DSPE}} = \frac{1}{\|\mathbf{b}^H(\boldsymbol{\psi}) \mathbf{E}_n\|^2} \quad (23)$$

where $\mathbf{E}_n$ is a $p \times (p-q)$ matrix with columns representing the eigenvectors of $\mathbf{R}_{xx}$ corresponding to the $(p-q)$ smallest eigenvalues. For an N-dimensional parameter vector $\boldsymbol{\psi}$, the DSPE spectrum is a functional defined over an N-dimensional space. The parameter vector $\boldsymbol{\psi}$ is estimated by locating the prominent peaks of P_{DSPE}, i.e.,

$$\hat{\boldsymbol{\psi}}_i = \arg\max_{\boldsymbol{\psi}} \frac{1}{\|\mathbf{b}^H(\boldsymbol{\psi}) \mathbf{E}_n\|^2}, \qquad i = 1, \ldots, q. \quad (24)$$

Essentially, the DSPE algorithm is a MUSIC-type algorithm and requires calibration—for each $\boldsymbol{\psi}$, $\mathbf{b}(\boldsymbol{\psi})$ must be measured and stored. Besides, in constructing the DSPE spectrum, it is assumed that the deterministic angular signal density $g(\theta; \boldsymbol{\psi})$ is identical for all sources. This may not be true in practice. Different sources might have different angular densities. The proposed technique in this paper handles the cases with different angular densities.

B. ID Source Localizer

For ID sources, the noise subspace is generally degenerate (equal to the zero vector), and the whole observation space is occupied by signal components. In other words, the noise-free covariance matrix is full rank. However, for several cases of

practical interest, most of the energy of the signal is concentrated in a few eigenvalues of the array covariance matrix. The number of these eigenvalues is referred to as the *effective* dimension of signal subspace and is shown by q_e. Let $\mathbf{E}_n$ be a matrix whose columns are the eigenvectors of covariance matrix corresponding to the smallest $(p - q_e)$ eigenvalues. The DSPE spectrum for ID source localization is defined as [1]

$$P_{\text{DSPE}} = \frac{1}{\text{tr}(\mathbf{E}_n^H \mathbf{H}(\boldsymbol{\psi})\mathbf{E}_n)} \tag{25}$$

where

$$\mathbf{H}(\boldsymbol{\psi}) = \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta)\rho(\theta; \boldsymbol{\psi})\mathbf{a}^H(\theta)\, d\theta \tag{26}$$

and tr(.) stands for the trace of a matrix. Note that in constructing DSPE spectrum, it is assumed that the angular power densities of ID sources belong to the same class of positive definite functions parameterized by a parameter vector $\boldsymbol{\psi}$.

IV. TLS-ESPRIT LOCALIZER

In this section, we propose a distributed source parameter estimator based on TLS-ESPRIT. The algorithm uses the Taylor series approximation of array response vector for different values of DOA. We show that for both ID and CD sources, the array covariance matrix can be formulated by the central moments of source distribution.

A. Coherent Sources

We derive an approximately invariant structure between the two subspaces spanned by $\mathbf{b}(\boldsymbol{\psi}_i)$ and $\mathbf{c}(\boldsymbol{\psi}_i)$ for $i = 1, 2, \ldots, q$. Remember that $\tau(\theta)$ is the propagation delay between the signals arriving at two sensors in each doublet for an emitting source at direction θ. Using (3), the exponential term in $\mathbf{c}(\boldsymbol{\psi}_i)$ is

$$e^{j\omega_0\tau(\theta)} = e^{j2\pi(d/\lambda)\sin\theta} \tag{27}$$

where λ is the wavelength at frequency ω_0. We assume $d \ll \lambda$—this is an essential assumption in our approximation.

Now, let θ_{0i} be the central angle of the distribution $g_i(\theta; \boldsymbol{\psi}_i)$. In addition, assume that $g_i(\theta; \boldsymbol{\psi}_i)$ is a normalized symmetric function around θ_{0i}

$$\int_{-\pi/2}^{\pi/2} g_i(\theta; \boldsymbol{\psi}_i)\, d\theta = 1 \tag{28}$$

and

$$\theta_{0i} = \int_{-\pi/2}^{\pi/2} \theta g_i(\theta; \boldsymbol{\psi}_i)\, d\theta. \tag{29}$$

Using the Taylor series expansion of $\mathbf{a}(\theta)$ around $\theta = \theta_{0i}$, we can write (11) as

$$\mathbf{b}(\boldsymbol{\psi}_i) = \int_{-\pi/2}^{\pi/2} \sum_{n=0}^{\infty} \frac{\mathbf{a}^{(n)}(\theta_{0i})}{n!} (\theta - \theta_{0i})^n g_i(\theta; \boldsymbol{\psi}_i)\, d\theta \tag{30}$$

where $\mathbf{a}^{(n)}(\theta_{0i})$ is the nth derivative of $\mathbf{a}(\theta)$ at $\theta = \theta_{0i}$. Then, we have

$$\mathbf{b}(\boldsymbol{\psi}_i) = \sum_{n=0}^{\infty} \frac{\mathbf{a}^{(n)}(\theta_{0i})}{n!} M_{n,i} \tag{31}$$

where $M_{n,i}$, which is the nth moment of $g_i(\theta; \boldsymbol{\psi}_i)$ around θ_{0i}, is defined as

$$M_{n,i} = \int_{-\pi/2}^{\pi/2} (\theta - \theta_{0i})^n g_i(\theta; \boldsymbol{\psi}_i)\, d\theta. \tag{32}$$

For symmetric $g_i(\theta; \boldsymbol{\psi}_i)$, $M_{n,i}$ vanishes for odd values of n, resulting in

$$\mathbf{b}(\boldsymbol{\psi}_i) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{(2k)}(\theta_{0i})}{(2k)!} M_{2k,i}. \tag{33}$$

Similarly, $\mathbf{c}(\boldsymbol{\psi})$ can be expressed as

$$\mathbf{c}(\boldsymbol{\psi}_i) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{\partial^{2k}}{\partial\theta^{2k}} \left(\mathbf{a}(\theta)e^{j2\pi(d/\lambda)\sin\theta}\right)\Bigg|_{\theta=\theta_{0i}} M_{2k,i}. \tag{34}$$

Now, let $f(\theta) \triangleq 2\pi(d/\lambda)\sin\theta$. Then

$$\frac{\partial}{\partial\theta}\left(\mathbf{a}(\theta)e^{jf(\theta)}\right) = \mathbf{a}'(\theta)e^{jf(\theta)} + jf'(\theta)\mathbf{a}(\theta)e^{jf(\theta)} \tag{35}$$

where the superscript $'$ denotes first-order derivative with respect to θ. Note that for $d \ll \lambda$, the derivative $f'(\theta) = 2\pi(d/\lambda)\cos\theta$ is negligible. Hence

$$\frac{\partial}{\partial\theta}\left(\mathbf{a}(\theta)e^{jf(\theta)}\right) \approx \mathbf{a}'(\theta)e^{jf(\theta)}. \tag{36}$$

Furthermore, we have

$$\frac{\partial^2}{\partial\theta^2}\left(\mathbf{a}(\theta)e^{jf(\theta)}\right) \approx \mathbf{a}''(\theta)e^{jf(\theta)} + jf'(\theta)\mathbf{a}'(\theta)e^{jf(\theta)}$$
$$\approx \mathbf{a}''(\theta)e^{jf(\theta)} \tag{37}$$

and similarly

$$\frac{\partial^{2k}}{\partial\theta^{2k}}\left(\mathbf{a}(\theta)e^{jf(\theta)}\right) \approx \mathbf{a}^{(2k)}(\theta)e^{jf(\theta)}. \tag{38}$$

Then, (34) can approximately be written as

$$\mathbf{c}(\boldsymbol{\psi}_i) \approx e^{j2\pi(d/\lambda)\sin\theta_{0i}} \sum_{k=0}^{\infty} \frac{\mathbf{a}^{(2k)}(\theta_{0i})}{(2k)!} M_{2k,i}. \tag{39}$$

Hence, we have

$$\mathbf{c}(\boldsymbol{\psi}_i) \approx \mathbf{b}(\boldsymbol{\psi}_i)e^{j2\pi(d/\lambda)\sin\theta_{0i}} \tag{40}$$

and in matrix form

$$\mathbf{C}(\boldsymbol{\psi}) \approx \mathbf{B}(\boldsymbol{\psi})\boldsymbol{\Phi} \tag{41}$$

where

$$\boldsymbol{\Phi} = \text{diag}\left(e^{j2\pi(d/\lambda)\sin\theta_{01}}, e^{j2\pi(d/\lambda)\sin\theta_{02}}, \ldots \right.$$
$$\left. e^{j2\pi(d/\lambda)\sin\theta_{0q}}\right). \tag{42}$$

Equation (41) shows that the bases of the two signal subspaces are approximately related by a rotation matrix $\mathbf{\Phi}$. This $\mathbf{\Phi}$ can be used to estimate the central angles of sources. ESPRIT-type algorithms such as LS-ESPRIT or TLS-ESPRIT might be applied to estimate the central angle of source distribution. Since (41) is an approximation, TLS-ESPRIT might beget a better performance.

Once the central angles are estimated, one can estimate the extension widths by constructing the DSPE spectrum for each source separately. In computing the DSPE spectrum, we need to compute $\mathbf{b}(\boldsymbol{\psi})$ for different values of extension width.

Since the DSPE spectrum is computed separately for each source in our algorithm, sources might have different deterministic angular densities. This is not possible in the 2-D DSPE spectrum given in [1].

B. Incoherent Sources

1) Single ID Source: We assume that a single ID source exists in the environment of the array. This is just for simplicity, and we will shortly extend our derivation to a multisource scenario.

Let the mass center of $\rho(\theta; \boldsymbol{\psi})$ be θ_0. The first-order Taylor series expansion of $\mathbf{a}(\theta)$ around θ_0 is

$$\mathbf{a}(\theta) \approx \mathbf{a}(\theta_0) + \mathbf{a}'(\theta_0)(\theta - \theta_0). \tag{43}$$

Thus, (4) can be written as

$$\mathbf{x} \approx \mathbf{a}(\theta_0) \int_{-\pi/2}^{\pi/2} s(\theta; \boldsymbol{\psi}) \, d\theta + \mathbf{a}'(\theta_0) \int_{-\pi/2}^{\pi/2} (\theta - \theta_0) s(\theta; \boldsymbol{\psi}) \, d\theta. \tag{44}$$

Define α_0 and α_1 as

$$\alpha_0 = \int_{-\pi/2}^{\pi/2} s(\theta; \boldsymbol{\psi}) \, d\theta \tag{45}$$

$$\alpha_1 = \int_{-\pi/2}^{\pi/2} (\theta - \theta_0) s(\theta; \boldsymbol{\psi}) \, d\theta. \tag{46}$$

Then, (44) can be written as

$$\mathbf{x} \approx [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)] \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} + \mathbf{n}_x. \tag{47}$$

The array covariance matrix of the subarray X is

$$\mathbf{R}_{xx} = [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)] \begin{bmatrix} E\{\alpha_0 \alpha_0^*\} & E\{\alpha_0 \alpha_1^*\} \\ E\{\alpha_1 \alpha_0^*\} & E\{\alpha_1 \alpha_1^*\} \end{bmatrix}$$
$$\times [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)]^H + \sigma_n^2 \mathbf{\Sigma}_{n_x} \tag{48}$$

where σ_n^2 is the unknown noise power, and $\mathbf{\Sigma}_{n_x}$ is the noise covariance matrix, which is assumed to be known. For simplicity, we assume that the noise is spatially white, i.e., $\mathbf{\Sigma}_{n_x} = \mathbf{I}$.

Lemma 1: For ID sources

$$E\{\alpha_0 \alpha_0^*\} = \sigma_s^2 \tag{49}$$
$$E\{\alpha_1 \alpha_1^*\} = \sigma_s^2 M_2 \tag{50}$$
$$E\{\alpha_0 \alpha_1^*\} = E\{\alpha_1 \alpha_0^*\} = 0 \tag{51}$$

where M_2 is the second central moment of $\rho(\theta; \boldsymbol{\psi})$ defined as

$$M_2 = \int_{-\pi/2}^{\pi/2} (\theta - \theta_0)^2 \rho(\theta; \boldsymbol{\psi}) \, d\theta. \tag{52}$$

Proof: See the Appendix.

Using Lemma 1, (48) can be written as

$$\mathbf{R}_{xx} = \mathbf{A} \mathbf{\Lambda}_s \mathbf{A}^H + \sigma_n^2 \mathbf{I} \tag{53}$$

where

$$\mathbf{A} = [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)] \tag{54}$$
$$\mathbf{\Lambda}_s = \text{diag}\left(\sigma_s^2, \sigma_s^2 M_2\right). \tag{55}$$

Similarly, the output of subarray $\mathbf{Y}$ can be approximated as

$$\mathbf{y} \approx [\mathbf{b}(\theta_0) \quad \mathbf{b}'(\theta_0)] \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} + \mathbf{n}_y \tag{56}$$

where $\mathbf{b}(\theta) \triangleq \mathbf{a}(\theta) e^{j\omega_0 \tau(\theta)}$. Using (3), we have

$$\mathbf{b}'(\theta) = \mathbf{a}'(\theta) e^{j\omega_0 \tau(\theta)} + j \frac{d}{\lambda} 2\pi \cos\theta \mathbf{a}(\theta) e^{j\omega_0 \tau(\theta)}. \tag{57}$$

Assume the condition for which $d/\lambda \ll 1$. Then, the second term in (57) is negligible, and we have

$$\mathbf{b}'(\theta) \approx \mathbf{a}'(\theta) e^{j\omega_0 \tau(\theta)}. \tag{58}$$

Therefore, (56) can be written as

$$\mathbf{y} \approx [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)] e^{j\omega \tau(\theta_0)} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} + \mathbf{n}_y$$
$$= \mathbf{A} e^{j\omega \tau(\theta_0)} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} + \mathbf{n}_y. \tag{59}$$

2) Multisource Scenario: Now, consider q uncorrelated narrowband ID sources. Assume that $\boldsymbol{\psi}_i$, $s_i(\theta; \boldsymbol{\psi}_i)$, $\rho_i(\theta; \boldsymbol{\psi}_i)$, and θ_{0i} are, respectively, the parameter vector, the angular signal density, the angular power density, and the central angle of the angular power density of the ith source. It is also assumed that the sources are uncorrelated. Then, (47) can be modified as

$$\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{n}_x \tag{60}$$

with

$$\mathbf{A} = [\mathbf{a}(\theta_{01}) \quad \cdots \quad \mathbf{a}(\theta_{0q}) \quad \mathbf{a}'(\theta_{01}) \quad \cdots \quad \mathbf{a}'(\theta_{0q})] \tag{61}$$
$$\mathbf{s} = [\alpha_{01} \quad \cdots \quad \alpha_{0q} \quad \alpha_{11} \quad \cdots \quad \alpha_{1q}]^T \tag{62}$$

where

$$\alpha_{0i} = \int_{-\pi/2}^{\pi/2} s_i(\theta; \boldsymbol{\psi}_i) \, d\theta \tag{63}$$

$$\alpha_{1i} = \int_{-\pi/2}^{\pi/2} (\theta - \theta_{0i}) s_i(\theta; \boldsymbol{\psi}_i) \, d\theta \tag{64}$$

for $i = 1, 2, \ldots, q$. Note that

$$E\{\alpha_{0i} \alpha_{0i}^*\} = \sigma_{si}^2 \tag{65}$$
$$E\{\alpha_{1i} \alpha_{1i}^*\} = \sigma_{si}^2 M_{2, i} \tag{66}$$
$$E\{\alpha_{0i} \alpha_{1i}^*\} = E\{\alpha_{1i} \alpha_{0i}^*\} = 0 \tag{67}$$

where σ_{si}^2 is the power of the ith signal, and $M_{2,i}$ is the second central moment of the angular power density of the ith source. Since the sources are uncorrelated, we have

$$E\{\alpha_{0i}\alpha_{0j}^*\} = E\{\alpha_{1i}\alpha_{1j}^*\}$$
$$= E\{\alpha_{0i}\alpha_{1j}^*\}$$
$$= E\{\alpha_{1i}\alpha_{0j}^*\} = 0. \tag{68}$$

The covariance matrix $\mathbf{R}_{xx}$ can be written as

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{\Lambda}_s\mathbf{A}^H + \sigma_n^2\mathbf{I} \tag{69}$$

where

$$\mathbf{\Lambda}_s = \text{diag}\left(\sigma_{s1}^2, \ldots, \sigma_{sq}^2, \sigma_{s1}^2 M_{2,1}, \ldots, \sigma_{sq}^2 M_{2,q}\right). \tag{70}$$

Similarly, $\mathbf{y}$ can be written as

$$\mathbf{y} \approx \mathbf{A}\mathbf{\Phi}\mathbf{s} + \mathbf{n}_y \tag{71}$$

where

$$\mathbf{\Phi} = \text{diag}\left(e^{j\omega_0\tau(\theta_{01})}, \ldots, e^{j\omega_0\tau(\theta_{0q})}\right.$$
$$\left. e^{j\omega_0\tau(\theta_{01})}, \ldots, e^{j\omega_0\tau(\theta_{0q})}\right). \tag{72}$$

Now, let $\mathbf{z}$ be defined as

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \tag{73}$$

and let $\mathbf{E}$ be a $2p \times 2q$ matrix with columns representing the eigenvectors of covariance matrix $\mathbf{R}_{zz} = E\{\mathbf{z}\mathbf{z}^H\}$ corresponding to the $2q$ largest eigenvalues. Then, $\mathbf{E}$ spans the column space of $\overline{\mathbf{A}}$ given by

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\mathbf{\Phi} \end{bmatrix}. \tag{74}$$

This means that there is an invertible $2q \times 2q$ matrix $\mathbf{T}$ such that

$$\overline{\mathbf{A}} = \mathbf{E}\mathbf{T}. \tag{75}$$

Let $\mathbf{E}_x$ and $\mathbf{E}_y$ be the upper and the lower $p \times 2q$ half matrix of $\mathbf{E}$, respectively, corresponding to the subarrays X and Y. From (75), we have

$$\mathbf{A} = \mathbf{E}_x\mathbf{T} \tag{76}$$
$$\mathbf{A}\mathbf{\Phi} = \mathbf{E}_y\mathbf{T}. \tag{77}$$

Hence

$$\mathbf{E}_y = \mathbf{E}_x\mathbf{T}\mathbf{\Phi}\mathbf{T}^{-1}. \tag{78}$$

Using the definition $\mathbf{\Psi} \triangleq \mathbf{T}\mathbf{\Phi}\mathbf{T}^{-1}$, we have

$$\mathbf{E}_y = \mathbf{E}_x\mathbf{\Psi}. \tag{79}$$

Equation (79) can be solved by the total least squares (TLS) method to find $\mathbf{\Psi}$ whose eigenvalues (diagonal elements of $\mathbf{\Phi}$) are related to the central angles. Note that according to the definition of $\mathbf{\Phi}$, all eigenvalues of $\mathbf{\Psi}$ are repeated with order 2. Hence, averaging should be employed to ascertain each source central angle from the estimates of eigenvalues of $\mathbf{\Psi}$.

To estimate the extension widths, we use the relation

$$\mathbf{\Lambda}_s = \mathbf{A}^\dagger\left(\mathbf{R}_{xx} - \hat{\sigma}_n^2\mathbf{I}\right)\mathbf{A}^{H\dagger} \tag{80}$$

where $\mathbf{A}^\dagger$ denotes the pseudo-inverse of $\mathbf{A}$, and $\hat{\sigma}_n^2$ is the estimated noise power. The average of the $2p - 2q$ smallest eigenvalues of $\mathbf{R}_{zz}$ can be used as an estimate of the noise power. Note that for angular power densities that are parameterized by two parameters (central angle and extension width), the second central moments can be used to obtain the extension width. Hence, $\mathbf{\Lambda}_s$ can be used to estimate the extension widths of different sources; it contains central moment information.

V. SECOND-ORDER TAYLOR APPROXIMATION

In this section, we use higher terms of Taylor series expansion to approximate the array response vector. We rewrite (43) by a second-order approximation of Taylor series as

$$\mathbf{a}(\theta) \approx \mathbf{a}(\theta_0) + \frac{\mathbf{a}'(\theta_0)}{1!}(\theta - \theta_0) + \frac{\mathbf{a}''(\theta_0)}{2!}(\theta - \theta_0)^2. \tag{81}$$

Then, (44) can be written as

$$\mathbf{x} \approx \mathbf{a}(\theta_0)\int_{-\pi/2}^{\pi/2}s(\theta;\boldsymbol{\psi})\,d\theta$$
$$+ \mathbf{a}'(\theta_0)\int_{-\pi/2}^{\pi/2}(\theta - \theta_0)s(\theta;\boldsymbol{\psi})\,d\theta$$
$$+ \frac{\mathbf{a}''(\theta_0)}{2}\int_{-\pi/2}^{\pi/2}(\theta - \theta_0)^2 s(\theta;\boldsymbol{\psi})\,d\theta + \mathbf{n}_x \tag{82}$$

and in matrix notation, we have

$$\mathbf{x} \approx \begin{bmatrix} \mathbf{a}(\theta_0) & \mathbf{a}'(\theta_0) & \frac{1}{2}\mathbf{a}''(\theta_0) \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \mathbf{n}_x \tag{83}$$

where α_2 is defined as

$$\alpha_2 = \int_{-\pi/2}^{\pi/2}(\theta - \theta_0)^2 s(\theta, \boldsymbol{\psi})\,d\theta. \tag{84}$$

Lemma 2: For an ID source

$$E\{\alpha_0\alpha_2^*\} = \sigma_s^2 M_2 \tag{85}$$
$$E\{\alpha_1\alpha_2^*\} = \sigma_s^2 M_3 \tag{86}$$
$$E\{\alpha_2\alpha_2^*\} = \sigma_s^2 M_4 \tag{87}$$

where M_3 and M_4 are the third and fourth central moments of the angular power density of the source.

Proof: See the Appendix.

Using Lemma 2, the array covariance matrix can be approximated as

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{\Lambda}_s\mathbf{A}^H + \sigma_n^2\mathbf{\Sigma}_{n_x} \tag{88}$$

where

$$\mathbf{\Lambda}_s = \sigma_s^2\begin{bmatrix} 1 & 0 & M_2 \\ 0 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{bmatrix} \tag{89}$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_0) & \mathbf{a}'(\theta_0) & \frac{1}{2}\mathbf{a}''(\theta_0) \end{bmatrix}. \tag{90}$$

As it has been shown in the previous section, for a multi ID source scenario in which the sources are uncorrelated, (88) can be used as an approximation of the covariance matrix. It is sufficient to modify the definition of $\mathbf{\Lambda}_s$ and $\mathbf{A}$ as

$$\mathbf{\Lambda}_s = \mathrm{diag}(\mathbf{\Lambda}_{s1}, \mathbf{\Lambda}_{s2}, \ldots, \mathbf{\Lambda}_{sq}) \tag{91}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_{01}) & \mathbf{a}'(\theta_{01}) & \frac{1}{2}\mathbf{a}''(\theta_{01}) \\ \ldots & \mathbf{a}(\theta_{0q}) & \mathbf{a}'(\theta_{0q}) & \frac{1}{2}\mathbf{a}''(\theta_{0q}) \end{bmatrix} \tag{92}$$

where $\mathbf{\Lambda}_{si}$ is defined as

$$\mathbf{\Lambda}_{si} = \sigma_{si}^2 \begin{bmatrix} 1 & 0 & M_{2,i} \\ 0 & M_{2,i} & M_{3,i} \\ M_{2,i} & M_{3,i} & M_{4,i} \end{bmatrix}. \tag{93}$$

This means that each source has been represented by a matrix $\mathbf{\Lambda}_{si}$ in the observation space. Using the same procedure and assumptions such as in the previous section, we can show that

$$\mathbf{E}_y = \mathbf{E}_x \mathbf{\Psi} \tag{94}$$

where $\mathbf{E}_x$ and $\mathbf{E}_y$ are the lower and the upper $p \times 3q$ submatrix of $\mathbf{E}$, respectively. $\mathbf{\Psi}$ is a $3q \times 3q$ matrix whose eigenvalues are functions of the central angles. Since the eigenvalues of $\mathbf{\Psi}$ are repeated of order 3, it is necessary to do an averaging on the related eigenvalues to estimate the central angles.

VI. MODEL AMBIGUITY

Remember that we have assumed that different ID sources are uncorrelated. Hence, each ID source can be split into two uncorrelated ID sources—nonoverlapping for simplicity. The TLS-ESPRIT estimator selects the central angles of the induced partial sources. In fact, if the angular power distribution is split into the following functions:

$$\rho^+(\theta; \boldsymbol{\psi}) \triangleq 2\rho(\theta; \boldsymbol{\psi})u(\theta - \theta_0) \tag{95}$$

$$\rho^-(\theta; \boldsymbol{\psi}) \triangleq 2\rho(\theta; \boldsymbol{\psi})u(-\theta + \theta_0) \tag{96}$$

where $u(\cdot)$ is the unit step function and coefficient 2 normalizes the area under $\rho^+(\theta; \boldsymbol{\psi})$ and $\rho^-(\theta; \boldsymbol{\psi})$ to unity, then such an estimator selects θ_0^+ and θ_0^-, which are the mass centers of $\rho^+(\theta; \boldsymbol{\psi})$ and $\rho^-(\theta; \boldsymbol{\psi})$, respectively. It is clear that $(\theta_0^+ - \theta_0^-)$ has a direct relationship with the source extension width. For instance, for an UID source with central angle θ_0 and extension width 2Δ, we have

$$\theta_0^+ - \theta_0^- = \Delta \tag{97}$$

and for a Gaussian-distributed source with central angle θ_0 and standard deviation Δ, we have

$$\theta_0^+ - \theta_0^- = 2\sqrt{\frac{2}{\pi}}\,\Delta. \tag{98}$$

Hence, an alternative approach to uncorrelated ID source localization may be as follows.

- Estimate the central angles using TLS-ESPRIT in which a dimension of two is assumed for each source; the number of estimated DOAs is $2q$.
- Sort $2q$ estimated DOAs and ascertain the source central angles by averaging each pair of closely-spaced DOAs.
- Estimate the extension widths by using the difference between adjacent DOAs.

Using a dimension of 3 or higher for each source is possible; however, there are two limitations. First, increasing the number of dimensions associated with each source results in increasing the smallest value of threshold SNR for which the new sources can be detected; induced sources have a fraction of the original source power. Second, by increasing the number of DOAs associated with each source, the probability of resolution decreases; the distance between related DOAs decreases. Simulation results show that increasing the number of dimensions associated with each source may be useful just for high values of SNR or if the true covariance matrix is known, which may not be practical.

VII. SIMULATION RESULTS

A. CD Sources

We have considered two equipower uncorrelated narrowband CD sources as signal emitters. The two subarrays are ULAs, and each one consists of 16 sensors. The distance between the corresponding elements in the two subarrays is $\lambda/10$, and the spacing between adjacent sensors in each subarray is $\lambda/2$ (λ is the wavelength at the operating frequency).

One of the sources is assumed to be uniformly coherently distributed (UCD) with the deterministic angular signal density

$$g_1(\theta; \boldsymbol{\psi}_1) = \begin{cases} \dfrac{1}{2\Delta_1}, & \|\theta - \theta_{01}\| < \Delta_1 \\ 0, & \text{otherwise} \end{cases} \tag{99}$$

where θ_{01} and Δ_1 are the central angle of arrival and the extension width of source, respectively. The other source is assumed to be Gaussian coherently distributed (GCD), that is

$$g_2(\theta; \boldsymbol{\psi}_2) = \frac{1}{\sqrt{2\pi}\Delta_2} \exp\left(-\frac{(\theta - \theta_{02})^2}{2\Delta_2^2}\right) \tag{100}$$

where θ_{02} and Δ_2 are the central angle of arrival and the extension width of the GCD source, respectively.

In our simulation, θ_1 and θ_2 are taken as $10°$ and $20°$ with extension width of $\Delta_1 = 1.5°$ and $\Delta_2 = 1°$. A Monte Carlo simulation of 50 independent runs with 50 snapshots for each trial was performed for different SNRs.

Figs. 1 and 2 show the bias and the standard deviation for the central angle estimator. Figs. 3 and 4 show the bias and the standard deviation for the extension width estimator. As it can be seen, the estimation bias of the central angles are negligible even for low values of SNR. The standard deviation of the central angle estimation is about $0.1°$ for SNR $= 10$ dB.

B. ID Sources

In the previous sections, we have proposed two algorithms for localization of ID sources with different power distributions. In both algorithms, the central angles are estimated using TLS-ESPRIT in which each source is modeled by a subspace of dimension 2, but the extension widths are estimated in two different ways. The estimation of extension widths in one of the two algorithms is based on the estimation of the moments of angular power distribution, whereas the other algorithm is based on the differences between two DOAs corresponding to each source.

In order to simulate the proposed algorithms, we have assumed two narrowband ID sources to be as signal emitters whose signals
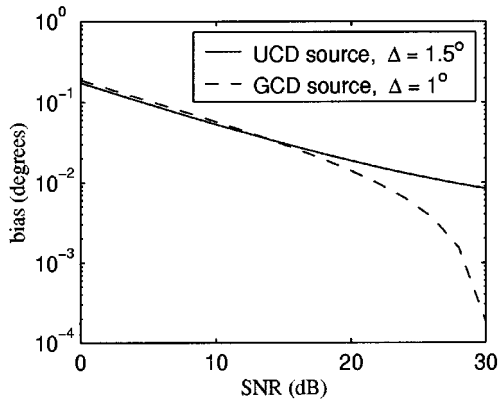
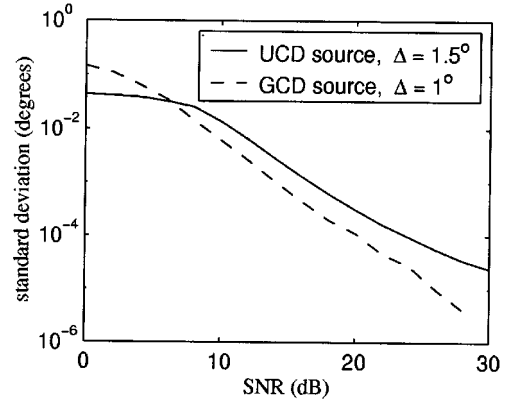Fig. 1.   Estimation bias for the central angle versus SNR for two CD sources.

Fig. 2.   Standard deviation of the central angle estimates versus SNR for two CD sources.
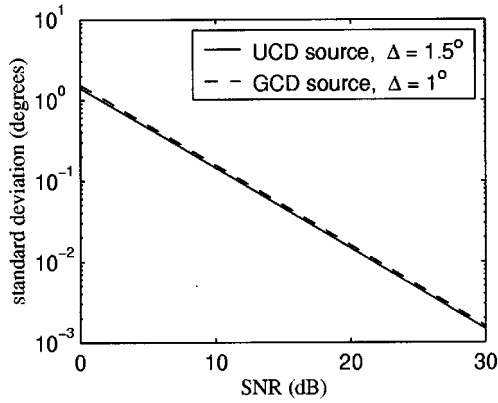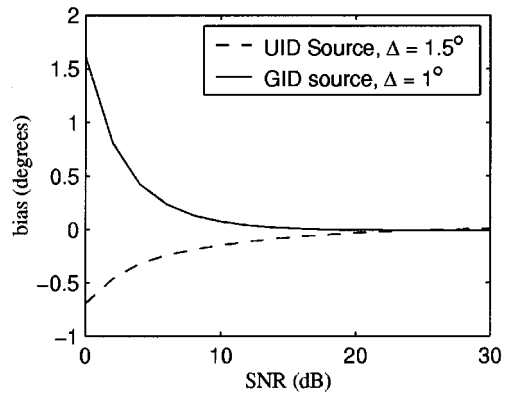

Fig. 3.   Estimation bias of the extension width versus SNR for two CD sources.

Fig. 4.   Standard deviation of the extension width estimates versus SNR for two CD sources.

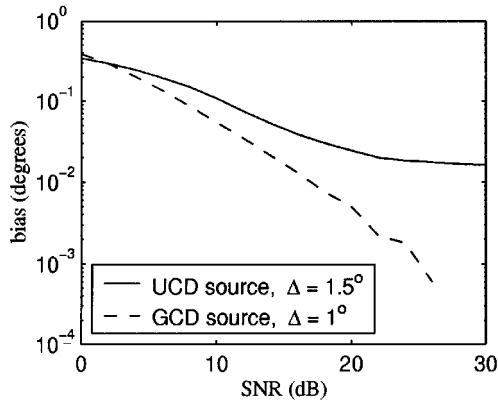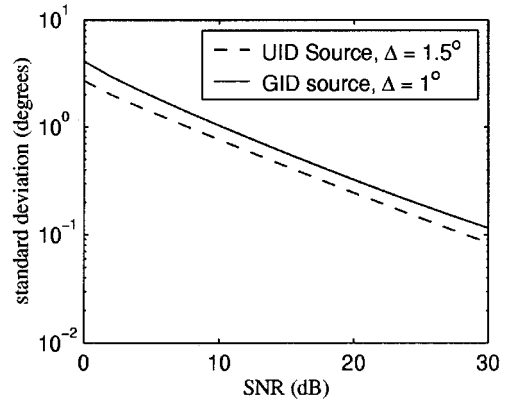Fig. 5.   Estimation bias for the central angle versus SNR for two ID sources.

Fig. 6.   Standard deviations for the central angle estimates versus SNR for two ID sources.

arrive at the two subarrays X and Y. The two subarrays consist of 16 sensors with an interelement spacing of half the wavelength. The distance between identical sensors is $d = \lambda/10$.

The central angles of two sources are $\theta_{01} = 10°$ and $\theta_{02} = 30°$. The source at $\theta_{01} = 10°$ has a uniform angular power distribution as

$$\rho_1(\theta; \boldsymbol{\psi}_1) = \begin{cases} \dfrac{1}{2\Delta_1}, & |\theta - \theta_{01}| \leq \Delta_1 \\ 0, & \text{otherwise} \end{cases} \qquad (101)$$

where $\Delta_1 = 1.5°$. The source at $\theta_{02} = 30°$ has a Gaussian power angular distribution as

$$\rho_2(\theta; \boldsymbol{\psi}_2) = \frac{1}{\sqrt{2\pi}\Delta_2} \exp\left(-\frac{(\theta - \theta_{02})^2}{2\Delta_2^2}\right) \qquad (102)$$

where $\Delta_2 = 1°$. The sources are assumed to be equipower and uncorrelated.

A Monte Carlo simulation with 20 independent runs, and 500 snapshots for each trial was performed for different SNRs. Figs. 5 and 6 show the bias and the standard deviation of the
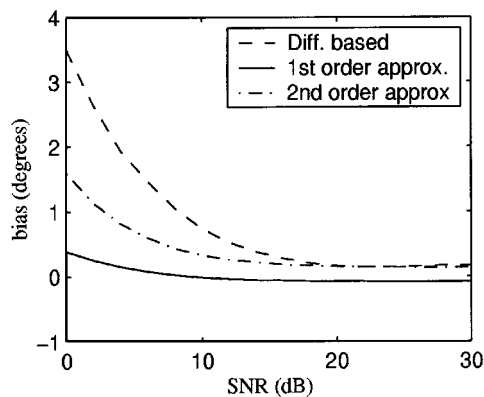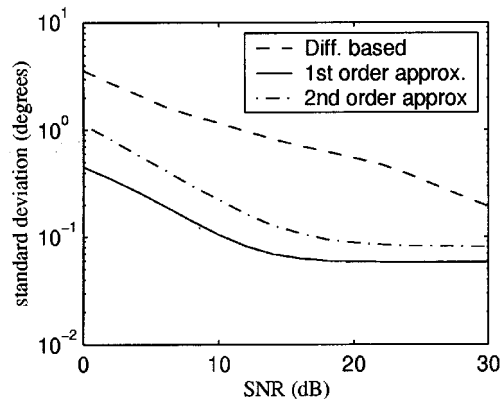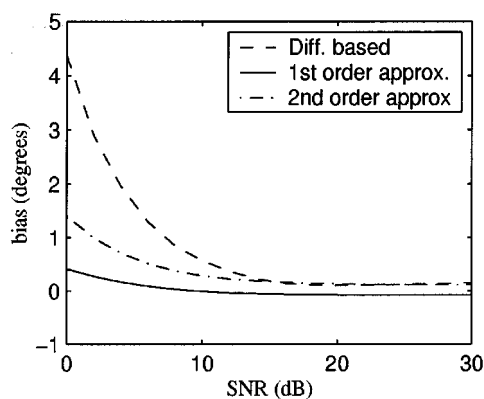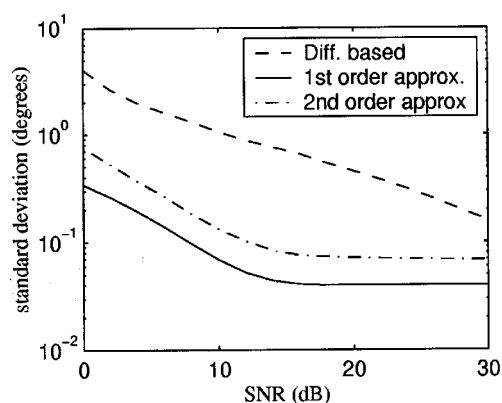
Fig. 7. Estimation bias for the extension width versus SNR for the UID source $\Delta = 1.5°$.

Fig. 8. Estimation bias for the extension width versus SNR for the GID source $\Delta = 1°$.

central angle estimator. Figs. 7 and 8 show the estimation bias of the extension width, and Figs. 9 and 10 show the standard deviation of the extension width for different algorithms. In order to show the effect of using higher order derivatives of location matrix in the moment-based algorithm, we have implemented the algorithm using both first- and second-order Taylor series. As it can be seen, a first-order approximation has a better performance. Note that in a ULA, using higher order derivatives of location vector causes the estimation error of central angle to be magnified as the sensor index increases. Using a low-error estimation algorithm for central angle estimation can improve the performance of the algorithm by increasing the number of implemented derivatives.
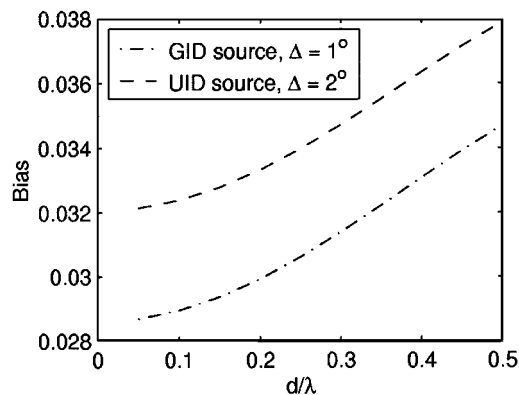
In order to explain the effect of increasing the number of implemented derivatives, we consider the K-term Taylor series expansion of the mth element of the location vector $a_m(\theta)$ as

$$a_m(\theta) = \sum_{k=0}^{K-1} \frac{a_m^{(k)}(\theta_0)}{k!} (\theta - \theta_0)^k + \frac{a_m^{(K)}(\theta_\zeta)}{K!} (\theta - \theta_0)^K$$

$$\text{for some } \theta_\zeta \text{ between } \theta \text{ and } \theta_0. \tag{103}$$

In (103), the last term is the Lagrange residue. For the simulated ULA, we have $a_m(\theta) = e^{jm\pi\sin\theta}$. It is difficult to find a closed form for the residue; however, we can have

$$\frac{a_m^{(K)}(\theta_\zeta)}{K!} = \frac{(jm\pi\sin\theta_\zeta)^K}{K!} e^{jm\pi\sin\theta_\zeta} + \mathcal{O}\left(m^{K-1}\right). \tag{104}$$

Fig. 9. Standard deviation for the extension width estimates versus SNR for the UID source.

Fig. 10. Standard deviation of the extension width estimates versus SNR for the GID source.

Fig. 11. Effect of d/λ on the bias of the central angle estimates.

In (104), the term $(\pi m)^K/K!$ is not a monolithic decreasing function; it increases to a maximum value and decreases thereafter. In order to have a decreasing error, K should be chosen much greater than the value for which the residue has a maximum. This explains why the error increases for $K = 3$ when compared with $K = 2$. Although the accuracy of the estimation is acceptable for $K = 2$, one might use a much larger value for K. Further results on the implication of a large number of derivatives can be found in [14].

Fig. 11 shows the effect of d/λ on the bias of central angle estimates in two different scenarios: a UID source with $\Delta = 2°$

and a GID source with $\Delta = 1°$. In each scenario, the true covariance matrix has been used. As noticed, the bias increases with d/λ. Note, however, that the increase of the bias is negligible even for $d/\lambda = 0.5$.

VIII. CONCLUSION

In this paper, parametric localization of distributed sources have been considered for both coherently and incoherently distributed source models.

In the CD case, the central angle of the sources are estimated by using two closely spaced subarrays. We showed that the rotational invariance exists approximately, and hence, TLS-ESPRIT can be used to estimate the central angles. The extension width can be estimated by constructing a 1-D DSPE spectrum for each source and finding the value for which the spectrum has its global maximum. The proposed algorithm can be applied to a scenario in which different sources have different functional form for angular signal densities. The computational cost of the algorithm is lower than that for the DSPE algorithm.

In the ID case, each source is modeled as a 2-D subspace in the observation space. The source central angles are estimated by TLS-ESPRIT using a dimension of 2 for each source. We have shown that the array covariance matrix can be approximated by the second moment of the source angular power density and the source powers. Hence, the source parameters can be estimated by a least-squares covariance matrix fitting. The proposed algorithm can be applied to a multisource scenario in which different sources may have different parametric angular power densities.

APPENDIX

In this Appendix, we show that

$$E\{\alpha_i \alpha_j^*\} = \sigma_s^2 M_{i+j} \qquad \text{for } i, j = 0, 1, 2. \tag{105}$$

Using the definition of α_is, we can write

$$
\begin{aligned}
E\{\alpha_i \alpha_j^*\} &= E\left\{ \iint (\theta - \theta_0)^i s(\theta; \boldsymbol{\psi}) s^*(\theta'; \boldsymbol{\psi})(\theta' - \theta_0)^j \, d\theta \, d\theta' \right\} \\
&= \iint (\theta - \theta_0)^i \sigma_s^2 \rho(\theta; \boldsymbol{\psi}) \delta(\theta - \theta')(\theta' - \theta_0)^j \, d\theta \, d\theta' \\
&= \sigma_s^2 \int (\theta - \theta_0)^{i+j} \rho(\theta; \boldsymbol{\psi}) \, d\theta \\
&= \sigma_s^2 M_{i+j}.
\end{aligned}
\tag{106}
$$

Note that $M_1 = 0$; hence, $E\{\alpha_0 \alpha_1^*\} = E\{\alpha_1 \alpha_0^*\} = 0$.

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