[6] P. M. T. Broersen, "Selecting the order of autoregressive models from small samples," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-33, pp. 874–879, 1985.
[7] R. Shibata, "Selection of the order of an autoregressive model by Akaike's information criterion," *Biometrika,* vol. 63, pp. 117–126, 1976.
[8] ——, "A theoretical view of the use of AIC," in *Time Series Analysis, Theory and Praxis 4,* O. D. Anderson, Ed. New York: : Elsevier North-Holland Sci., 1983.
[9] P. M. T. Broersen and H. E. Wensink, "A framework for autoregressive theory," in *Proc. EUSIPCO-92, Brussels, Belgium,* pp. 775–778.

The Optimal Focusing Subspace for Coherent Signal Subspace Processing

S. Valaee and P. Kabal

Abstract—**In this correspondence, we introduce a technique to determine an optimal focusing frequency for the direction-of-arrival estimation of wideband signals using the coherent signal-subspace processing method. We minimize the subspace fitting error to select an optimal focusing frequency. Direct optimization of this criterion can be computationally complex—the complexity increases with the number of frequency samples. An alternative technique is introduced that performs nearly as well as the optimal method. This suboptimal technique is based on minimizing a tight bound to the error. The computational complexity of the suboptimal method is independent of the number of frequency samples. The simulation results show that the proposed method reduces both the bias of estimation and the resolution threshold signal-to-noise ratio (SNR).**

I. Introduction

Array processing techniques can be used to locate wideband signals. A wideband signal has a bandwidth comparable to the center frequency. Several methods for the processing of wideband signals using an array of sensors have been proposed in the literature [11], [1], [6]. The first step in some of these techniques is to obtain samples of the signal in the frequency domain. These samples are found by applying a discrete Fourier transformation to the time samples of the signal or by using a filter bank. The samples of the spectrum can be uniformly or nonuniformly spaced.

Many array processing techniques use the concept of the *signal subspace*. The signal subspace is the span of the location vectors of the array for fixed directions-of-arrival (DOA's). Since each location vector is a function of frequency, the signal subspace depends on the frequency of the observation. For wideband signals, the signal subspaces at different frequencies do not overlap, and as a result, the observation vectors at the frequency bins cannot be directly added to each other. Wang and Kaveh [11] propose *focusing* of the observation vectors. Focusing involves transforming the signal subspaces at different frequency bins into a predefined subspace (called the *focusing subspace*). They choose an arbitrary frequency, say, the center frequency of the spectrum of the signals, and transform the subspace at each frequency bin into the subspace created by the span of the location vectors at the focusing frequency. Then, they use a high-resolution algorithm such as MUSIC [8] to estimate the DOA's of the sources. They show that focusing reduces the resolution threshold signal-to-noise ratio (SNR), which is defined as the SNR for a prescribed probability of resolution. They also show that if the integral of the signal covariance matrix taken over the frequency spectrum is full rank, the method can be applied to coherent signal localization. Hung and Kaveh [5] use a unitary variant of the CSM algorithm to avoid the focusing loss. They use the center frequency for focusing.

Swingler and Krolik [9] prove that for a single-source scenario, it is possible to have an unbiased estimate of the DOA's if the centroid of the source spectrum is selected as the focusing frequency. In [10], we showed that for multiple sources, the CSM algorithm cannot provide unbiased estimates of the DOA's. In this work, we propose a method to select the focusing subspace. The method is based on minimizing a subspace fitting error. The subspace fitting error for each frequency bin is defined as the distance between the focusing matrix and the transformed location matrix. Later, we minimize a tight bound to the error. The simulation results show that using the method proposed here reduces the resolution threshold SNR and the bias of the DOA's estimates.

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S. Valaee is with INRS-Telécommunications, Université du Québec, Verdun, Canada H3E 1H6.
P. Kabal is with the Department of Electrical Engineering, McGill University, Montreal, Canada H3A 2A7.

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II. The CSM Algorithm

Consider an array of p sensors exposed to $q < p$ far-field wideband sources. The output of the sensors in the frequency domain is represented by

$$\mathbf{x}(f) = \mathbf{A}(f, \boldsymbol{\theta})\mathbf{s}(f) + \mathbf{n}(f) \tag{1}$$

where $\mathbf{s}(f)$ and $\mathbf{n}(f)$ are the Fourier transforms of the signal and noise vectors, and $\mathbf{A}(f, \boldsymbol{\theta}) = [\mathbf{a}(f, \theta_1) \cdots \mathbf{a}(f, \theta_q)]$ is the full-rank $p \times q$ matrix of location vectors. It is assumed that the signal and noise samples are independent identically-distributed sequences of complex Gaussian random vectors with unknown covariance matrices $\mathbf{S}(f)$ and $\sigma^2\mathbf{I}$, respectively. With these assumptions, the covariance matrix of the observation vector at the frequency f_j is given by

$$\mathbf{R}(f_j) = \mathbf{A}(f_j, \boldsymbol{\theta})\mathbf{S}(f_j)\mathbf{A}^H(f_j, \boldsymbol{\theta}) + \sigma^2\mathbf{I} \tag{2}$$

where the superscript H represents the Hermitian transposition. For simplicity of notation, we suppress the frequency variable and represent $\mathbf{R}(f_j)$ by $\mathbf{R}_j$, $\mathbf{A}(f_j, \boldsymbol{\theta})$ by $\mathbf{A}_j$, and so on.

The CSM algorithm [11] is based on forming new observation vectors $\mathbf{y}_j$ such that

$$\mathbf{y}_j = \mathbf{T}_j\mathbf{x}_j \tag{3}$$

where the $\mathbf{T}_j$'s are called the *focusing matrices*. In the unitary variant of the CSM algorithm [5], the $\mathbf{T}_j$, $j = 1, \ldots, J$ are selected from

$$\min_{\mathbf{T}_j} \|\mathbf{A}_0 - \mathbf{T}_j\mathbf{A}_j\|, \quad \text{subject to } \mathbf{T}_j^H\mathbf{T}_j = \mathbf{I} \tag{4}$$

where $\|\cdot\|$ is the Frobenius matrix norm [4]. The solution to this minimization is given by [4], [5]

$$\mathbf{T}_j = \mathbf{V}_j\mathbf{W}_j^H \tag{5}$$

where $\mathbf{V}_j$ and $\mathbf{W}_j$ are the left and the right singular vectors of $\mathbf{A}_0\mathbf{A}_j^H$. In (5), $\mathbf{A}_j$ and $\mathbf{A}_0$ are assumed to be known. In practice, a preliminary step is required to determine these matrices. The preprocessing step involves using an ordinary beamformer to estimate the DOA's. These preestimated DOA's are then used to determine the *focusing angles*. The focusing angles are used in the structure of the location matrix to generate $\mathbf{A}_j$ and $\mathbf{A}_0$. An alternative method that avoids the pre-estimation step, but can only be applied to linear arrays, is spatial resampling, as proposed in [2] and [7] (see also [3]).

The transformed observation vectors $\mathbf{y}_i$ are used to form a sample correlation matrix for each frequency bin. An average of these aligned sample correlation matrices gives a universal sample correlation matrix that can be used for detection and DOA estimation. In the past, the focusing frequency in (5) has been chosen to be the center frequency of the spectrum of the signals. This choice is not optimal if the spectrum of the signal is asymmetric around the center frequency or the sampling in the frequency domain is nonuniform. In the following section, we propose a method that finds the optimal focusing frequency.

III. Focusing Frequency Selection

Minimizing the subspace fitting error is an appropriate criterion for focusing frequency selection. In particular, the focusing frequency can be selected from

$$\min_{f_0} \min_{\mathbf{T}_j} \sum_{j=1}^{J} w_j \|\mathbf{A}_0 - \mathbf{T}_j \mathbf{A}_j\|^2, \quad \text{subject to } \mathbf{T}_j^H \mathbf{T}_j = \mathbf{I}, \quad \mathbf{A}_0 \in \mathcal{A}(\boldsymbol{\theta}) \tag{6}$$

where w_j is a normalized weighting factor proportional to the SNR at the jth frequency bin with $\sum_{j=1}^{J} w_j = 1$, and $\mathcal{A}(\boldsymbol{\theta})$ is the set of all location matrices for the focusing angles $\boldsymbol{\theta}$ and is defined by

$$\mathcal{A}(\boldsymbol{\theta}) = \{\mathbf{A}(f, \boldsymbol{\theta}) \mid f \in \mathcal{F}\} \tag{7}$$

where $\mathcal{F}$ is an interval for the frequency f. The elements of this set are the location matrices with the DOA's $\boldsymbol{\theta}$ and different frequencies. The following lemma holds for any location matrix.

Lemma 1: The location matrix $\mathbf{A}$ of an array with isotropic sensors located in an environment with uniform planar wavefronts satisfies

$$\|\mathbf{A}\|^2 = \sum_{i=1}^{q} \|\mathbf{a}_i\|^2 = pq. \tag{8}$$

Using Lemma 1 and (5), the subspace fitting error is given by

$$\begin{aligned}\sum_{j=1}^{J} w_j \|\mathbf{A}_0 - \mathbf{T}_j \mathbf{A}_j\|^2 &= \sum_{j=1}^{J} w_j \left[\|\mathbf{A}_0\|^2 + \|\mathbf{A}_j\|^2 - 2\Re\, \mathrm{tr}\left(\mathbf{A}_0 \mathbf{A}_j^H \mathbf{T}_j^H\right)\right] \\ &= 2Jpq - 2\sum_{j=1}^{J}\sum_{i=1}^{q} w_j \sigma_i\left(\mathbf{A}_0 \mathbf{A}_j^H\right)\end{aligned} \tag{9}$$

where the $\sigma_i(\mathbf{B})$, $i = 1, \ldots, q$ are the singular values of the matrix $\mathbf{B}$ arranged in nonincreasing order, $\Re(\cdot)$ represents the real part of a complex number, and $\mathrm{tr}(\cdot)$ is the trace operator. From (9), it is seen that the minimization problem (6) is identical to the maximization

$$\max_{f_0} \sum_{j=1}^{J}\sum_{i=1}^{q} w_j \sigma_i\left(\mathbf{A}_0 \mathbf{A}_j^H\right), \quad \text{subject to } \mathbf{A}_0 \in \mathcal{A}(\boldsymbol{\theta}). \tag{10}$$

Direct maximization of (10) is tedious, and the complexity increases with the number of frequency samples. In the sequel, we present a suboptimal method that is based on maximizing an upper bound to (10).

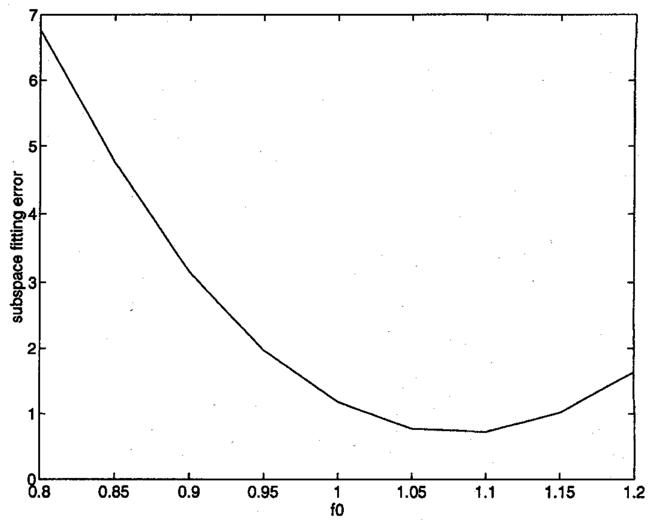

Fig. 1. Subspace fitting error (9) as a function of the focusing frequency for two uncorrelated far-field wideband sources at 10 and 14° arriving at a linear array of eight sensors with a 20 dB SNR.

The following lemma has been adopted from [4].

Lemma 2: If $\mathbf{A}, \mathbf{B} \in \mathbf{M}_{m,n}$ are given matrices with ordered singular values $\sigma_1(\mathbf{A}) \geq \cdots \geq \sigma_q(\mathbf{A}) \geq 0$ and $\sigma_1(\mathbf{B}) \geq \cdots \geq \sigma_q(\mathbf{B}) \geq 0$, where $q = \min\{m, n\}$, then

$$\|\mathbf{A} - \mathbf{B}\|^2 \geq \sum_{i=1}^{q} [\sigma_i(\mathbf{A}) - \sigma_i(\mathbf{B})]^2. \tag{11}$$

Application of Lemma 2 to (9) results in the following lemma.

Lemma 3: For every $\mathbf{A}_0, \mathbf{A}_j \in M_{p,q}$ and $\sum_{i=1}^{q} w_i = 1$

$$\sum_{j=1}^{J}\sum_{i=1}^{q} w_j \sigma_i\left(\mathbf{A}_0 \mathbf{A}_j^H\right) \leq \sum_{j=1}^{J}\sum_{i=1}^{q} w_j \sigma_i(\mathbf{A}_0) \sigma_i\left(\mathbf{A}_j^H\right). \tag{12}$$

The proposed method is based on maximizing the right-hand side of (12). Define

$$\mu_i \triangleq \sum_{j=1}^{J} w_j \sigma_i(\mathbf{A}_j). \tag{13}$$

The maximization problem is represented by

$$\max_{f_0} \sum_{i=1}^{q} \mu_i \sigma_i(\mathbf{A}_0), \quad \text{subject to } \mathbf{A}_0 \in \mathcal{A}(\boldsymbol{\theta}). \tag{14}$$

This is a one-variable maximization problem that can be solved by searching for the best f_0. The computational complexity for the maximization (14) is independent of the number of frequency samples.

The simulation studies, which are reported in the following section, show that in the vicinity of the maximum point, the bound is tight. The tightness of the bound at the maximum point indicates that the method achieves a near-optimum subspace fit.

IV. Simulation Results

Assume that a uniform linear array of eight sensors is exposed to two far-field wideband sources arriving from 10° and 14°. The

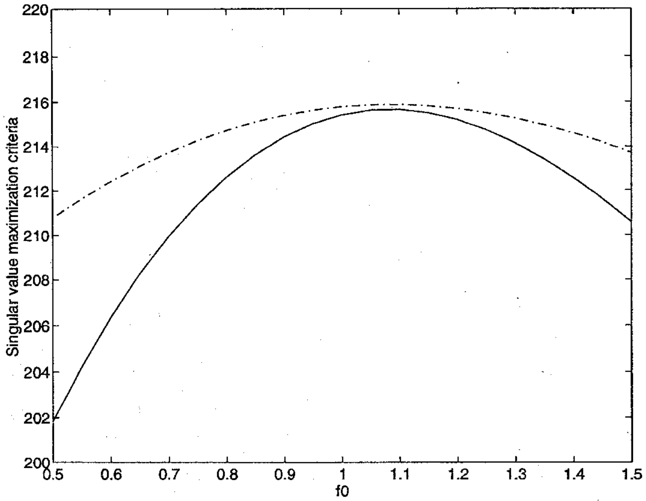

Fig. 2. Left- and right-hand-sides of (12) as functions of the focusing frequency for two uncorrelated far-field wideband sources at 10 and 14° arriving at a linear array of eight sensors with a 20 dB SNR.

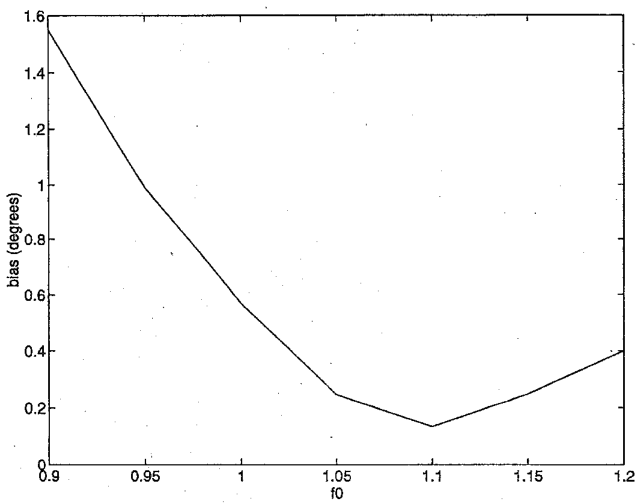

Fig. 3. Norm of the bias vector for the DOA estimation of two uncorrelated far-field wideband sources arriving at a linear array of eight sensors with a 20 dB SNR.

signals are uncorrelated and have a 40% bandwidth relative to the center frequency. The spectrum for each of the signals is given by

$$S(f) = \begin{cases} 5f - 4 & 0.8 \leq f \leq 1.2 \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The spectrum is sampled using a 16-point FFT algorithm. The data at each frequency bin contain 100 snapshots. Using an ordinary beamformer, a single source is detected at 13°. We add two extra DOA's at 9° and 17° as the focusing angles. The subspace fitting error (9) is shown in Fig. 1. It is seen that the error is minimized at the frequency 1.1, which is 10% higher than the center frequency and 3% higher than the centroid frequency. The left- and the right-hand sides of (12) are compared in Fig. 2. Note that in the vicinity of the optimum point, the bound is tight. For a fixed SNR at 20 dB, we have found the bias of the DOA estimates for different focusing frequencies. The results are depicted in Fig. 3. The bias of the DOA estimate for the centroid frequency is 0.1246 compared with 0.0487 for the optimum focusing frequency.

TABLE I
RESOLUTION OF THE CSM ALGORITHM FOR TWO CLOSELY SPACED WIDEBAND SIGNALS AS A FUNCTION OF THE FOCUSING FREQUENCY

	SNR (dB)			
f_0	−5	0	5	10
0.90	0	6	36	64
0.95	0	22	86	100
1.00	2	41	99	100
1.05	1	71	100	100
1.10	6	89	100	100
1.15	9	99	100	100
1.20	11	100	100	100

TABLE II
NORM OF THE BIAS VECTOR FOR TWO CLOSELY SPACED WIDEBAND SIGNALS AS A FUNCTION OF THE FOCUSING FREQUENCY

	SNR (dB)			
f_0	−5	0	5	10
0.90	–	1.22	1.48	1.55
0.95	–	1.13	1.10	0.98
1.00	1.57	0.88	0.68	0.57
1.05	0.59	0.76	0.36	0.24
1.10	1.08	0.54	0.22	0.13
1.15	0.58	0.42	0.25	0.24
1.20	0.79	0.37	0.39	0.40

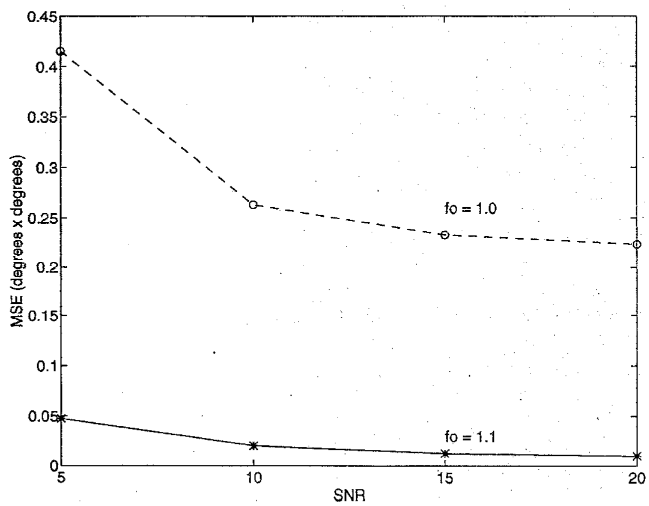

Fig. 4. Mean-square error of the CSM algorithm for the two focusing frequencies 1 and 1.1 and for two uncorrelated far-field wideband sources at 10° and 14° arriving at a linear array of eight sensors.

To compare the resolution capability of the CSM algorithm, we performed 100 independent trials and counted the number of times that the two DOA's were resolved as two peaks in the spectrum of the MUSIC algorithm. The results for different focusing frequencies are compared in Table I. The resolution threshold SNR at the optimum focusing frequency is lower than that for the center frequency. There are two factors that affect the resolution SNR. First, the optimum focusing frequency has a smaller subspace fitting error, and second, it is higher than the center frequency. The separation between the

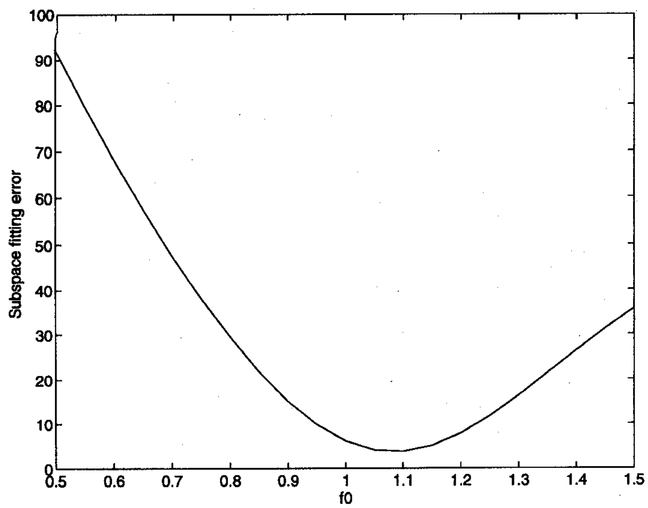

Fig. 5. Resolution of the CSM algorithm for two closely spaced wideband signals as a function of the focusing frequency.

TABLE III
RESOLUTION OF THE CSM ALGORITHM FOR FOUR WIDEBAND SIGNALS AS A FUNCTION OF THE FOCUSING FREQUENCY

	SNR (dB)					
f_0	−5	0	5	10	15	20
0.90	0	0	0	0	0	0
0.95	0	0	0	0	0	0
1.00	0	0	1	9	7	0
1.05	0	0	19	76	99	100
1.10	0	1	72	100	100	100
1.15	0	9	92	100	100	100
1.20	0	17	100	100	100	100

location vectors for higher frequencies is larger, which results in a better resolution performance. The resolution at 1.2 is better than the other frequencies. However, the results of Table II and Fig. 3 show that for SNR's above the resolution threshold, the bias is minimum at the optimum focusing frequency. The mean-square error of the CSM algorithm for the two focusing frequencies 1.0 and 1.1 is compared in Fig. 4.

We have also studied a multigroup scenario. In the second example, the signal of four sources located at 10°, 14°, 33°, and 37° arrive at the same array as in the previous example. The focusing angles are 9°, 13°, 17°, 31°, 35°, and 39°. The spectrum of the signals are the same as the first example. The subspace fitting error (9) is depicted in Fig. 5. The optimum focusing frequency is again 1.1. Table III represents the resolution of the CSM algorithm for different SNR's and focusing frequencies. Note that at the center frequency 1.0, detection is not possible even at high SNR's. If the number of sensors is increased, the array can detect signals using the center frequency for focusing. However, the resolution threshold for the optimum frequency is much smaller than that for the center frequency. The bias and the mean-square error are given in Tables IV and V. As these tables show, the bias and the mean-square error for the optimum frequency are significantly smaller than the corresponding values for the center frequency. Thus, at $f_0 = 1.1$, we simultaneously get good resolution capability (Table III) and low bias (Table IV) over a range of SNR's.

TABLE IV
NORM OF THE BIAS VECTOR FOR FOUR WIDEBAND SIGNALS AS A FUNCTION OF THE FOCUSING FREQUENCY

	SNR (dB)					
f_0	−5	0	5	10	15	20
0.90	–	–	–	–	–	–
0.95	–	–	–	–	–	–
1.00	–	–	2.05	1.66	1.51	–
1.05	–	–	1.37	0.95	0.77	0.67
1.10	–	1.74	0.90	0.44	0.28	0.20
1.15	–	1.28	0.75	0.65	0.65	0.66
1.20	–	1.35	0.96	1.02	1.02	1.04

TABLE V
NORM OF THE MEAN-SQUARE-ERROR VECTOR FOR FOUR WIDEBAND SIGNALS AS A FUNCTION OF THE FOCUSING FREQUENCY

	SNR (dB)					
f_0	−5	0	5	10	15	20
0.90	–	–	–	–	–	–
0.95	–	–	–	–	–	–
1.00	–	–	2.78	1.67	1.27	–
1.05	–	–	1.53	0.68	0.40	0.28
1.10	–	2.11	0.64	0.16	0.06	0.03
1.15	–	1.35	0.46	0.32	0.29	0.28
1.20	–	1.49	0.73	0.71	0.67	0.68

V. SUMMARY

In this correspondence, we have proposed a method to determine the optimal focusing frequency for the coherent signal-subspace method with unitary transformations. We have defined a criterion based on the subspace fitting error and optimized a tight upper bound to it. The simulation results show that the method successfully finds the global optimum value and improves the performance of the estimation by reducing the bias, the mean-square error, and the resolution SNR threshold.

REFERENCES

[1] K. M. Buckley and L. J. Griffiths, "Broad-band signal-subspace spatial-spectrum (BASS-ALE) estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 953–964, July 1988.

[2] H. Clergeot and O. Michel, "New simple implementation of the coherent signal subspace method for wideband direction of arrival estimation," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Glasgow, Scotland, May 1989, pp. 2764–2767.

[3] W. Hong and A. H. Tewfik, "Focusing matrices for wideband array processing with no *a priori* angle estimates," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, San Francisco, CA, Mar. 1992, pp. II-493–II-496.

[4] R. A. Horn and C. A. Johnson, *Matrix Analysis*. Cambridge, UK: Cambridge University Press, 1985.

[5] H. Hung and M. Kaveh, "Focusing matrices for coherent signal-subspace processing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 1272–1281, Aug. 1988.

[6] J. Krolik and D. N. Swingler, "Multiple broad-band source location using steered covariance matrices," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 1481–1494, Oct. 1989.

[7] ——, "Focused wide-band array processing by spatial resampling," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 356–360, Feb. 1990.

[8] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antenna Propagat.*, vol. AP-34, pp. 276–280, Mar. 1986.
[9] D. N. Swingler and J. Krolik, "Source location bias in the coherently focused high-resolution broad-band beamformer," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 143–145, Jan. 1989.
[10] S. Valaee and P. Kabal, "Wideband array processing using a two-sided correlation transformation," *IEEE Trans. Signal Processing*, vol. 43, pp. 160–172, Jan. 1995.
[11] H. Hung and M. Kaveh, "Coherent signal-subspace processing for the detection and estimation of angles of arrival of multiple wide-band sources," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, pp. 823–831, Aug. 1985.

Strict Identifiability of Multiple FIR Channels Driven by an Unknown Arbitrary Sequence

Yingbo Hua and Mati Wax

Abstract—**A system of multiple finite impulse response (FIR) channels driven by an unknown input is said to be strictly identifiable if, in the absence of noise, the given channel outputs can only be realized by a unique (up to a constant) system impulse response and a unique input sequence. In this correspondence, we show necessary and sufficient conditions for strict identifiability, and establish a connection among strict identifiability, a cross-relation-based (CR-based) identifiability and a Fisher information-based (FI-based) identifiability.**

I. Introduction

The problem of blind identification of multiple FIR channels driven by a common input arises in a wide range of applications. It has recently received increasing attention in the signal processing community. Much attention has been paid to identifiability-related issues of this problem. Assuming that the input to all channels is white, stationary, and infinitely long, authors of [1]–[3] studied channel identifiability conditions based on second-order statistics of the channel outputs. Assuming that the input is an unknown deterministic sequence, authors of [4] and [5] did a similar study based on a cross-relation (CR) equation in the absence of noise. An M-channel system is said to be CR identifiable if the system impulse response can be uniquely identified by the CR method [5]. In [7], channel identifiability was further analyzed based on a Fisher information (FI) matrix. An M-channel system is said to be FI identifiable if the FI matrix has nullity equal to one. An equivalence between FI identifiability and CR identifiability was established in [7].

In this paper, we study the identifiability of the M-channel FIR system in a strict sense. An M-channel FIR system is said to be strictly identifiable if the given channel outputs can only be realized by a unique system impulse response and a unique input sequence.

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Y. Hua is with the Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, Victoria, Australia (e-mail: yhua@ee.mu.oz.au).
M. Wax is with RAFAEL, Haifa, Israel.

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In contrast to several existing definitions of identifiability, strict identifiability is directly based on a finite length of channel outputs instead of on certain statistics [1]–[3] or certain preprocessing [4], [5] of channel outputs. The word "strict" implies that if an M-channel system is strictly identifiable, then it must be identifiable by some method (e.g., exhaustive search); and, on the other hand, if an M-channel system is strictly not identifiable, then it can not be identifiable by any method. (It will be clear that a system can only be either strictly identifiable or strictly not identifiable. So, "stricly not identifiable" will have the same meaning as "not strictly identifiable.") Surprisingly, however, strict identifiability will be shown to be equivalent to the CR- and FI-based identifiabilities provided that the number of the output samples of each channel is no less than twice the maximum order of the FIR channels (which is a very mild condition).

II. The M-Channel FIR System

For convenience, the M-channel system detailed in [7] is briefly reformulated in this section. For M parallel FIR channels driven by a common input sequence $s(k)$, the output of the ith channel can be written as

$$\boldsymbol{y}_i = \boldsymbol{H}_{(i)}\boldsymbol{s} \tag{2.1}$$

where $\boldsymbol{y}_i$ is the $N \times 1$ output vector of the ith channel; $\boldsymbol{H}_{(i)}$ the $N \times (N+L)$ Sylvester matrix [10] of the impulse response $h_i(k)$ of the ith channel; and s the $(N+L) \times 1$ input vector. Note that N is the total number of output samples from each channel, and L the maximum order of the M channels. Alternatively, we can write

$$\boldsymbol{y}_i = \boldsymbol{S}\boldsymbol{h}_i \tag{2.2}$$

where $\boldsymbol{S}$ is the $N \times (L+1)$ Toeplitz matrix [10] of the input sequence; and $\boldsymbol{h}_i$ the $(L+1) \times 1$ vector of the impulse response of the ith channel. Stacking all channels outputs into one vector yields, from (2.1), the following:

$$\boldsymbol{y} = \boldsymbol{H}_M\boldsymbol{s} \tag{2.3}$$

and from (2.2), the following:

$$\boldsymbol{y} = \boldsymbol{S}_M\boldsymbol{h} \tag{2.4}$$

where $\boldsymbol{y}$ is the stacked vector of $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M\}$, $\boldsymbol{H}_M$ the generalized Sylvester matrix [10] of all channels' impulse responses; $\boldsymbol{S}_M = \text{diag}\{\boldsymbol{S}, \cdots, \boldsymbol{S}\}$; and $\boldsymbol{h}$ the stacked vector of $\{\boldsymbol{h}_1, \cdots, \boldsymbol{h}_M\}$.

III. Strict Identifiability

Definition: The M-channel FIR system is said to be *strictly* identifiable from its output $\boldsymbol{y}$ if there do *not* exist $\boldsymbol{h}'$ and $\boldsymbol{s}'$ where $\boldsymbol{h}'$ is linearly independent of $\boldsymbol{h}$ *or* $\boldsymbol{s}'$ is linearly independent of $\boldsymbol{s}$ such that

$$\boldsymbol{y} = \boldsymbol{H}_M\boldsymbol{s} = \boldsymbol{H}'_M\boldsymbol{s}' \tag{3.1}$$

or equivalently

$$\boldsymbol{y} = \boldsymbol{S}_M\boldsymbol{h} = \boldsymbol{S}'_M\boldsymbol{h}' \tag{3.2}$$

where $\boldsymbol{H}'_M$ and $\boldsymbol{S}'_M$ are defined by $\boldsymbol{h}'$ and $\boldsymbol{s}'$, respectively.