Incoherently Distributed Source Localization

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Abstract

A new algorithm based on ESPRIT is proposed for the estimation of central angle and angular extension of Incoherently Distributed (ID) sources. The central angles are estimated using TLS-ESPRIT. The covariance matrix is approximated using a finite Taylor series expansion which leads to the formulation of covariance matrix in terms of central moments of the angular power distribution. The extension widths are estimated using the central moments of the distribution. The algorithm can be used for sources with different angular distributions and has a low computational cost.

1   Introduction

Several applications of array processing — such as operating antenna arrays at base stations for mobile communications, passive sonar, and underwater acoustics — require a spatially distributed source modeling; a modeling recently paid much attention in the literature of array processing [1], [2], [3]. Depending on the nature of reflection and scattering in the above examples, signal components arriving from different directions exhibit varying degrees of correlation, ranging from totally incorrelated (incoherent) to fully correlated (coherent) cases. Distributed source modeling suffers from a deficiency, namely seizing the whole observation space by signal components and nullifying the noise subspace. This begets a break-down of the techniques which exploit the orthogonality of signal and noise subspaces, such as MUSIC [4] and its variants.

Several distributed source localization techniques have been proposed in the recent literature. The first attempt for generalization of the signal and noise subspace concepts to distributed sources has been done in [1]. Based on these concepts, an algorithm called the Distributed Source Parameter Estimator (DSPE), has been proposed which is the generalization of MUSIC to distributed sources and can be applied to both Coherently Distributed (CD) and Incoherently Distributed (ID) sources. Since DSPE is essentially a MUSIC-type algorithm, it suffers from intrinsic disadvantages of MUSIC such as array manifold measurements and calibration.

In [5], a maximum likelihood algorithm has been proposed for localization of Gaussian distributed sources. The likelihood function is jointly maximized for all parameters of the Gaussian model. The computational complexity of this method grows exponentially with the number of sources.

Similar to DSPE, an algorithm called DISPARE has been presented for localization of ID sources [6]. In DISPARE, the covariance matrix of the array is approximated by a low-rank model and then a spatial spectrum is constructed with peaks associated to spatial parameters of ID sources .

In [7], an algorithm has been presented for localization of a single Uniformly Incoherently Distributed (UID) source. In this algorithm, extension width of the source is estimated from the eigenvalues of the correlation matrix. Estimation of the source central angle is based on the properties of eigenvectors of the correlation matrix. It has been shown that the eigenvectors of the correlation matrix are modulated Discrete Prolate Spheroidal Sequences (DPSS's) [8]. In [9], the central angle of the UID source is estimated by TLS-ESPRIT [10] and then the extension width is estimated using the algorithm presented in [7].

In [2], a Taylor series expansion has been used to derive an approximate model, called the Generalized Array Manifold (GAM). GAM is based on a linear combination of array location vector and its derivatives. Using GAM, an algorithm is presented to estimate the source *spatial signature* by exploiting a Vandermonde structure. The algorithm can only be applied to Uniform Linear Arrays (ULA) and uniform CD sources.

In [11], a distributed source is approximated by two point sources. Then, the Direction-of-Arrivals (DOA's) of the point sources are estimated using MUSIC or ROOT-MUSIC. The

angular spread is obtained by using a lookup table which describes the relation between the distance of the two estimated DOA's and the angular spread. In [12], a subspace fitting method has also been proposed for estimating the angular parameters of distributed sources which has a high computational cost.

In this paper, we propose an algorithm for parameter estimation of Incoherently Distributed (ID) sources based on TLS-ESPRIT. We give an approximation to the covariance matrix by using the GAM. Using a first order Taylor series expansion, we show that each ID source approximately introduces a two-dimensional subspace in the observation space. However, higher order Taylor series might be used to improve the accuracy of approximation. Again, we show that the rotational invariant structure exists for two identical closely-spaced sub-arrays. Hence, TLS-ESPRIT can be used to estimate DOA's — a pair of DOA's for each source. The covariance matrix is formulated by the location vectors and their derivatives as well as the central moments of the distributions. We will show that the distance of the two estimated DOA's is related to the source angular spread.

2 Data Model

Consider an array of $2p$ sensors (p doublets). Assume that the two sensors in each doublet are identical and have the same gain, phase, and sensitivity pattern and are separated by a constant displacement vector $\vec{d}$. The two induced sub-arrays are denoted by X and Y. Furthermore, it is assumed that q narrowband distributed sources with the same central frequency ω_0 are present in the environment of these sub-arrays. The complex envelope of the output of ith sensor in sub-array X is

$$x_i = \sum_{m=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a_i(\theta) s_m(\theta, \boldsymbol{\psi}_m) d\theta + n_{x_i}, \tag{1}$$

where $a_i(\theta)$ is the response of the ith sensor to a unit energy source emitting at direction θ with respect to the orthogonal direction to the displacement vector $\vec{d}$, $s_m(\theta, \boldsymbol{\psi}_m)$ is the angular density of the mth source, $\boldsymbol{\psi}_m$ is the mth source location parameter vector, and n_{x_i} is the additive zero-mean noise at the ith sensor uncorrelated from the signals. Examples of the parameter vector $\boldsymbol{\psi}_m$ are the two limits of (DOA) for uniform spatial extension, or the angle of maximum power and standard deviation for a Gaussian distribution.

The complex envelope of the output of ith sensor in sub-array Y is

$$y_i = \sum_{m=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a_i(\theta) e^{j\omega_0 \tau(\theta)} s_m(\theta, \boldsymbol{\psi}_m) d\theta + n_{y_i}, \tag{2}$$

where n_{y_i} is an additive zero-mean noise at the ith sensor of sub-array Y uncorrelated from the signals, and $\tau(\theta)$ is the propagation delay between the identical elements of a doublet in two sub-arrays for a signal arriving at direction θ. Throughout, we assume that θ in (1) and (2) is measured with respect to a direction orthogonal to $\vec{d}$. Then, we have

$$\tau(\theta) = \frac{d}{c} \sin \theta, \tag{3}$$

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where c is the wave propagation speed.

In vector representation, (1) and (2) can be written as

$$\mathbf{x} = \sum_{m=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta)s_m(\theta, \boldsymbol{\psi}_m)d\theta + \mathbf{n}_x, \tag{4}$$

$$\mathbf{y} = \sum_{m=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta)e^{j\omega_0\tau(\theta)}s_m(\theta, \boldsymbol{\psi}_m)d\theta + \mathbf{n}_y, \tag{5}$$

where $\mathbf{x}$ and $\mathbf{y}$ are output vectors of the sub-arrays X and Y, respectively, $\mathbf{n}_x$ and $\mathbf{n}_y$ are the corresponding noise vectors, and $\mathbf{a}(\theta)$ is the sub-array X location vector for a source at direction θ.

For the sub-array X, the covariance matrix can be written as

$$\mathbf{R}_{xx} = \sum_{i=1}^{q}\sum_{j=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta)p_{ij}(\theta, \theta'; \boldsymbol{\psi}_i, \boldsymbol{\psi}_j)\mathbf{a}^H(\theta')d\theta d\theta' + \mathbf{R}_n, \tag{6}$$

where superscript H represents Hermitian transposition, $\mathbf{R}_n$ is the noise correlation matrix, and

$$p_{ij}(\theta, \theta'; \boldsymbol{\psi}_i, \boldsymbol{\psi}_j) = E\{s_i(\theta; \boldsymbol{\psi}_i)s_j^*(\theta'; \boldsymbol{\psi}_j)\}, \tag{7}$$

is called the *angular cross-correlation kernel*. In (7), $E\{\cdot\}$ denotes statistical expectation, and $*$ represents complex conjugation.

A source is said to be Incoherently Distributed (ID) if the signal rays arriving from different directions are uncorrelated, i.e.,

$$E\{s_i(\theta; \boldsymbol{\psi}_i)s_i^*(\theta'; \boldsymbol{\psi}_i)\} = \sigma_{si}^2\rho_i(\theta; \boldsymbol{\psi}_i)\delta(\theta - \theta'), \tag{8}$$

where σ_{si}^2 is the ith source power and $\rho_i(\theta; \boldsymbol{\psi}_i)$ is the normalized *angular power density* of the source satisfying

$$\int_{-\pi/2}^{\pi/2} \rho_i(\theta; \boldsymbol{\psi}_i)d\theta = 1. \tag{9}$$

The index i has been used to emphasize that it is not necessary for the sources to have identical angular power densities.

In this paper, we assume that different ID sources are uncorrelated. Using this assumption for ID sources, we get

$$\mathbf{R}_{xx} = \mathbf{R}_{yy} = \sum_{i=1}^{q} \int_{-\pi/2}^{\pi/2} \sigma_{si}^2\mathbf{a}(\theta)\rho_i(\theta; \boldsymbol{\psi}_i)\mathbf{a}^H(\theta)d\theta + \mathbf{R}_n. \tag{10}$$

3   DSPE algorithm

In [1], an algorithm, called the Distributed Source Parameter Estimator (DSPE), was proposed in which the signal and noise subspace concept was generalized to distributed sources. DSPE is essentially a MUSIC-type algorithm and hence needs array manifold measurement and calibration. In DSPE algorithm, it is assumed that the angular power density of ID sources belongs

to the same class of positive definite functions parameterized by a parameter vector $\boldsymbol{\psi}$. This means that $\rho_i(\theta; \boldsymbol{\psi}) = \rho(\theta; \boldsymbol{\psi})$ for all i.

For ID sources, the noise subspace is generally degenerate (equal to the zero vector) and the whole observation space is occupied by signal components. In other words, the noise-free covariance matrix is full rank. However, for several cases of practical interest, most of energy of signal is concentrated in a few eigenvalues of the array covariance matrix. The number of these eigenvalues is referred to as the *effective* dimension of signal subspace and is shown by q_e. Let $\mathbf{E}_n$ be a matrix whose columns are the eigenvectors of covariance matrix corresponding to the smallest $(p - q_e)$ eigenvalues. The DSPE spectrum for ID source localization is defined as [1]

$$P_{\mathrm{DSPE}} = \frac{1}{\mathrm{tr}(\mathbf{E}_n^H \mathbf{H}(\boldsymbol{\psi})\mathbf{E}_n)}, \tag{11}$$

where

$$\mathbf{H}(\boldsymbol{\psi}) = \int_{-\pi/2}^{\pi/2} \mathbf{a}(\theta)\rho(\theta; \boldsymbol{\psi})\mathbf{a}^H(\theta)d\theta, \tag{12}$$

and tr(.) stands for the trace of a matrix.

4 TLS-ESPRIT Localizer

In this section, we propose a distributed source parameter estimator based on TLS-ESPRIT. The algorithm uses the Taylor series approximation of array response vector for different values of DOA. We show that the array covariance matrix can be formulated by the central moments of the source angular power density.

4.1 Single Source Scenario

We assume that a single ID source exists in the environment of the array. This is just for simplicity and we will shortly extend our derivation to a multi-source scenario.

Let the mass center of $\rho(\theta; \boldsymbol{\psi})$ be θ_0. The first order Taylor series expansion of $\mathbf{a}(\theta)$ around θ_0 is

$$\mathbf{a}(\theta) \approx \mathbf{a}(\theta_0) + \mathbf{a}'(\theta_0)(\theta - \theta_0). \tag{13}$$

Thus, (4) can be written as

$$\mathbf{x} \approx \mathbf{a}(\theta_0) \int_{-\pi/2}^{\pi/2} s(\theta, \boldsymbol{\psi})d\theta + \mathbf{a}'(\theta_0) \int_{-\pi/2}^{\pi/2} (\theta - \theta_0)s(\theta, \boldsymbol{\psi})d\theta + \mathbf{n}_x. \tag{14}$$

Define α and β as

$$\alpha = \int_{-\pi/2}^{\pi/2} s(\theta, \boldsymbol{\psi})d\theta, \tag{15}$$

$$\beta = \int_{-\pi/2}^{\pi/2} (\theta - \theta_0)s(\theta, \boldsymbol{\psi})d\theta. \tag{16}$$

Then, (14) can be written as

$$\mathbf{x} \approx [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \mathbf{n}_x. \tag{17}$$

5

The array covariance matrix of the sub-array X is

$$\mathbf{R}_{xx} = [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)] \begin{bmatrix} E\{\alpha\alpha^*\} & E\{\alpha\beta^*\} \\ E\{\beta\alpha^*\} & E\{\beta\beta^*\} \end{bmatrix} [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)]^H + \sigma_n^2 \mathbf{\Sigma}_{n_x} \tag{18}$$

where σ_n^2 is the unknown noise power and $\mathbf{\Sigma}_{n_x}$ is the noise covariance matrix which is assumed to be known. For simplicity, we assume that the noise is spatially white, i.e. $\mathbf{\Sigma}_{n_x} = \mathbf{I}$.

Lemma 1 *For ID sources,*

$$E\{\alpha\alpha^*\} = \sigma_s^2, \tag{19}$$

$$E\{\beta\beta^*\} = \sigma_s^2 M_2, \tag{20}$$

$$E\{\alpha\beta^*\} = E\{\beta\alpha^*\} = 0, \tag{21}$$

where M_2 is the second central moment of $\rho(\theta; \boldsymbol{\psi})$ defined as

$$M_2 = \int_{-\pi/2}^{\pi/2} (\theta - \theta_0)^2 \rho(\theta; \boldsymbol{\psi}) d\theta. \tag{22}$$

Proof: See Appendix A.

Using Lemma 1, (18) can be written as

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{\Lambda}_s\mathbf{A}^H + \sigma_n^2 \mathbf{I}, \tag{23}$$

where

$$\mathbf{A} = [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)], \tag{24}$$

$$\mathbf{\Lambda}_s = \text{diag}(\sigma_s^2, \sigma_s^2 M_2). \tag{25}$$

Similarly, the output of sub-array Y can be approximated as

$$\mathbf{y} \approx [\mathbf{b}(\theta_0) \quad \mathbf{b}'(\theta_0)] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \mathbf{n}_y, \tag{26}$$

where $\mathbf{b}(\theta) \triangleq \mathbf{a}(\theta)e^{j\omega_0\tau(\theta)}$. Using (3), we have

$$\mathbf{b}'(\theta) = \mathbf{a}'(\theta)e^{j\omega_0\tau(\theta)} + j\frac{d}{\lambda}2\pi\cos\theta\mathbf{a}(\theta)e^{j\omega_0\tau(\theta)}. \tag{27}$$

Assume the condition for which $\frac{d}{\lambda} \ll 1$. Then, the second term in (27) is negligible and we have

$$\mathbf{b}'(\theta) \approx \mathbf{a}'(\theta)e^{j\omega_0\tau(\theta)}. \tag{28}$$

Therefore, (26) can be written as

$$\begin{aligned} \mathbf{y} &\approx [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0)]e^{j\omega\tau(\theta_0)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \mathbf{n}_y \\ &= \mathbf{A}e^{j\omega\tau(\theta_0)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \mathbf{n}_y. \end{aligned} \tag{29}$$

4.2 Multi-source scenario

Now, consider q uncorrelated narrowband ID sources. Assume that ψ_i, $s_i(\theta; \psi_i)$, $\rho_i(\theta; \psi_i)$, and θ_{0i} are, respectively, the parameter vector, the angular signal density, the angular power density, and the central angle of the angular power density of the ith source. It is also assumed that the sources are uncorrelated. Then, (17) can be modified as

$$\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{n}_x, \tag{30}$$

with

$$\mathbf{A} = [\mathbf{a}(\theta_{01}), \cdots, \mathbf{a}(\theta_{0q}), \ \mathbf{a}'(\theta_{01}), \cdots, \mathbf{a}'(\theta_{0q})], \tag{31}$$

$$\mathbf{s} = [\alpha_1, \ \cdots, \ \alpha_q, \ \beta_1, \ \cdots, \ \beta_q]^T, \tag{32}$$

where

$$\alpha_i = \int_{-\pi/2}^{\pi/2} s_i(\theta; \psi_i) d\theta, \tag{33}$$

$$\beta_i = \int_{-\pi/2}^{\pi/2} (\theta - \theta_{0i}) s_i(\theta; \psi_i) d\theta, \tag{34}$$

for $i = 1, 2, \ldots, q$. Note that

$$E\{\alpha_i \alpha_i^*\} = \sigma_{si}^2, \tag{35}$$

$$E\{\beta_i \beta_i^*\} = \sigma_{si}^2 M_{2,i}, \tag{36}$$

$$E\{\alpha_i \beta_i^*\} = E\{\beta_i \alpha_i^*\} = 0, \tag{37}$$

where σ_{si}^2 is the power of the ith signal and $M_{2,i}$ is the second central moment of the angular power density of the ith source. Since the sources are uncorrelated, we have

$$E\{\alpha_i \alpha_j^*\} = E\{\beta_i \beta_j^*\} = E\{\alpha_i \beta_j^*\} = E\{\beta_i \alpha_j^*\} = 0. \tag{38}$$

The covariance matrix $\mathbf{R}_{xx}$ can be written as

$$\mathbf{R}_{xx} = \mathbf{A}\boldsymbol{\Lambda}_s \mathbf{A}^H + \sigma_n^2 \mathbf{I}, \tag{39}$$

where

$$\boldsymbol{\Lambda}_s = \text{diag}(\sigma_{s1}^2, \ldots, \sigma_{sq}^2, \sigma_{s1}^2 M_{2,1}, \ldots, \sigma_{sq}^2 M_{2,q}). \tag{40}$$

Similarly, $\mathbf{y}$ can be written as

$$\mathbf{y} \approx \mathbf{A}\boldsymbol{\Phi}\mathbf{s} + \mathbf{n}_y, \tag{41}$$

where

$$\boldsymbol{\Phi} = \text{diag}(e^{j\omega_0 \tau(\theta_{01})}, \ldots, e^{j\omega_0 \tau(\theta_{0q})}, \ e^{j\omega_0 \tau(\theta_{01})}, \ldots, e^{j\omega_0 \tau(\theta_{0q})}). \tag{42}$$

Now, let $\mathbf{z}$ be defined as

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \tag{43}$$

and let $\mathbf{E}$ be a $2p \times 2q$ matrix with columns representing the eigenvectors of covariance matrix $\mathbf{R}_{zz} = E\{\mathbf{z}\mathbf{z}^H\}$ corresponding to the $2q$ largest eigenvalues. Then, $\mathbf{E}$ spans the column space of $\bar{\mathbf{A}}$ given by

$$\bar{\mathbf{A}} = \left[\begin{array}{c} \mathbf{A} \\ \mathbf{A}\mathbf{\Phi} \end{array} \right]. \tag{44}$$

This means that there is an invertible $2q \times 2q$ matrix $\mathbf{T}$ such that

$$\bar{\mathbf{A}} = \mathbf{E}\mathbf{T}. \tag{45}$$

Let $\mathbf{E}_x$ and $\mathbf{E}_y$ be the upper and the lower $p \times 2q$ half matrix of $\mathbf{E}$, respectively, corresponding to the sub-arrays X and Y. From (45) we have

$$\mathbf{A} = \mathbf{E}_x\mathbf{T}, \tag{46}$$
$$\mathbf{A}\mathbf{\Phi} = \mathbf{E}_y\mathbf{T}. \tag{47}$$

Hence

$$\mathbf{E}_y = \mathbf{E}_x\mathbf{T}\mathbf{\Phi}\mathbf{T}^{-1}. \tag{48}$$

Using the definition $\mathbf{\Psi} \triangleq \mathbf{T}\mathbf{\Phi}\mathbf{T}^{-1}$, we have

$$\mathbf{E}_y = \mathbf{E}_x\mathbf{\Psi}. \tag{49}$$

Equation (49) can be solved by the Total Least Squares (TLS) method to find $\mathbf{\Psi}$ whose eigenvalues (diagonal elements of $\mathbf{\Phi}$) are related to the central angles. Note that according to the definition of $\mathbf{\Phi}$, all eigenvalues of $\mathbf{\Psi}$ are repeated with order 2. Hence, averaging should be employed to ascertain each source central angle from the estimates of eigenvalues of $\mathbf{\Psi}$.

To estimate the extension widths, we use the relation

$$\mathbf{\Lambda}_s = \mathbf{A}^\dagger(\mathbf{R}_{xx} - \hat{\sigma}_n^2\mathbf{I})\mathbf{A}^{H\dagger} \tag{50}$$

where $\mathbf{A}^\dagger$ denotes the pseudo-inverse of $\mathbf{A}$, and $\hat{\sigma}_n^2$ is the estimated noise power. The average of the $2p - 2q$ smallest eigenvalues of $\mathbf{R}_{zz}$ can be used as an estimate of the noise power. Note that for angular power densities which are parameterized by two parameters (central angle and extension width), the second central moments can be used to obtain the extension width. Hence $\mathbf{\Lambda}_s$ can be used to estimate the extension widths of different sources; it contains central moment information.

5   Second order Taylor Approximation

In this section, we use higher terms of Taylor series expansion to approximate the array response vector. We rewrite (13) by a second order approximation of Taylor series as

$$\mathbf{a}(\theta) \approx \mathbf{a}(\theta_0) + \frac{\mathbf{a}'(\theta_0)}{1!}(\theta - \theta_0) + \frac{\mathbf{a}''(\theta_0)}{2!}(\theta - \theta_0)^2. \tag{51}$$

Then, (14) can be written as

$$
\begin{aligned}
\mathbf{x} &\approx \mathbf{a}(\theta_0) \int_{-\pi/2}^{\pi/2} s(\theta, \boldsymbol{\psi}) d\theta + \mathbf{a}'(\theta_0) \int_{-\pi/2}^{\pi/2} (\theta - \theta_0) s(\theta, \boldsymbol{\psi}) d\theta \\
&\quad + \frac{\mathbf{a}''(\theta_0)}{2} \int_{-\pi/2}^{\pi/2} (\theta - \theta_0)^2 s(\theta, \boldsymbol{\psi}) d\theta + \mathbf{n}_x,
\end{aligned}
\tag{52}
$$

and in matrix notation we have

$$
\mathbf{x} \approx [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0) \quad \tfrac{1}{2}\mathbf{a}''(\theta_0)] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \mathbf{n}_x,
\tag{53}
$$

where γ is defined as

$$
\gamma = \int_{-\pi/2}^{\pi/2} (\theta - \theta_0)^2 s(\theta, \boldsymbol{\psi}) d\theta.
\tag{54}
$$

Lemma 2 *For an ID source*

$$
\begin{aligned}
E\{\alpha\gamma^*\} &= \sigma_s^2 M_2, \tag{55} \\
E\{\beta\gamma^*\} &= \sigma_s^2 M_3, \tag{56} \\
E\{\gamma\gamma^*\} &= \sigma_s^2 M_4. \tag{57}
\end{aligned}
$$

where M_3 and M_4 are the third and fourth central moments of the angular power density of the source.

Proof: See Appendix B.

Using Lemma 2, the array covariance matrix can be approximated as

$$
\mathbf{R}_{xx} = \mathbf{A}\boldsymbol{\Lambda}_s\mathbf{A}^H + \sigma_n^2 \boldsymbol{\Sigma}_{n_x},
\tag{58}
$$

where

$$
\boldsymbol{\Lambda}_s = \sigma_s^2 \begin{bmatrix} 1 & 0 & M_2 \\ 0 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{bmatrix},
\tag{59}
$$

and

$$
\mathbf{A} = [\mathbf{a}(\theta_0) \quad \mathbf{a}'(\theta_0) \quad \tfrac{1}{2}\mathbf{a}''(\theta_0)].
\tag{60}
$$

As it has been shown in the previous section, for a multi ID source scenario, in which the sources are uncorrelated, (58) can be used as an approximation of the covariance matrix. It is sufficient to modify the definition of $\boldsymbol{\Lambda}_s$ and $\mathbf{A}$ as

$$
\begin{aligned}
\boldsymbol{\Lambda}_s &= \mathrm{diag}(\boldsymbol{\Lambda}_{s1}, \boldsymbol{\Lambda}_{s2}, \cdots, \boldsymbol{\Lambda}_{sq}), \tag{61} \\
\mathbf{A} &= [\mathbf{a}(\theta_{01}) \quad \mathbf{a}'(\theta_{01}) \quad \tfrac{1}{2}\mathbf{a}''(\theta_{01}) \quad \cdots \quad \mathbf{a}(\theta_{0q}) \quad \mathbf{a}'(\theta_{0q}) \quad \tfrac{1}{2}\mathbf{a}''(\theta_{0q})], \tag{62}
\end{aligned}
$$

where $\boldsymbol{\Lambda}_{si}$ is defined as

$$
\boldsymbol{\Lambda}_{si} = \sigma_{si}^2 \begin{bmatrix} 1 & 0 & M_{2,i} \\ 0 & M_{2,i} & M_{3,i} \\ M_{2,i} & M_{3,i} & M_{4,i} \end{bmatrix},
\tag{63}
$$

This means that each source has been represented by a matrix $\mathbf{\Lambda}_{si}$ in the observation space. Using the same procedure and assumptions such as in the previous section, we can show that

$$\mathbf{E}_y = \mathbf{E}_x \mathbf{\Psi} \tag{64}$$

where $\mathbf{E}_x$ and $\mathbf{E}_y$ are the lower and the upper $p \times 3q$ sub-matrix of $\mathbf{E}$, respectively. $\mathbf{\Psi}$ is a $3q \times 3q$ matrix whose eigenvalues are functions of the central angles. Since the eigenvalues of $\mathbf{\Psi}$ are repeated of order 3, it is necessary to do an averaging on the related eigenvalues to estimate the central angles.

6   Model Ambiguity

Since the different ID sources are assumed to be uncorrelated, each ID source can be split into two uncorrelated ID sources — non overlapping for simplicity. The TLS-ESPRIT estimator approximately selects the central angles of the induced partial sources. In fact, if the angular power distribution is split into the following functions

$$\rho^+(\theta; \boldsymbol{\psi}) \quad \triangleq \quad 2\rho(\theta; \boldsymbol{\psi})u(\theta - \theta_0), \tag{65}$$

$$\rho^-(\theta; \boldsymbol{\psi}) \quad \triangleq \quad 2\rho(\theta; \boldsymbol{\psi})u(-\theta + \theta_0), \tag{66}$$

where $u(\cdot)$ is the unit step function and coefficient 2 normalizes the area under $\rho^+(\theta; \boldsymbol{\psi})$ and $\rho^-(\theta; \boldsymbol{\psi})$ to unity, then, such an estimator selects θ_0^+ and θ_0^-, the mass centers of $\rho^+(\theta; \boldsymbol{\psi})$ and $\rho^-(\theta; \boldsymbol{\psi})$, respectively. It is clear that $(\theta_0^+ - \theta_0^-)$ has a direct relationship to the source extension width. For instance, for a UID source with central angle θ_0 and extension width 2Δ, we have

$$\theta_0^+ - \theta_0^- = \Delta, \tag{67}$$

and for a Gaussian distributed source with central angle θ_0 and standard deviation Δ, we have

$$\theta_0^+ - \theta_0^- = 2\sqrt{\frac{2}{\pi}}\Delta. \tag{68}$$

Hence, an alternative approach to uncorrelated ID source localization may be as follows.

- Estimate the central angles using TLS-ESPRIT in which a dimension of two is assumed for each source; the number of estimated DOA's is $2q$.

- Sort $2q$ estimated DOA's and ascertain the source central angles by averaging each pair of closely-spaced DOA's.

- Estimate the extension widths by using the difference between adjacent DOA's.

Using a dimension of 3 or higher for each source is possible, however, there are two limitations. First, increasing the number of dimensions associated to each source results in increasing the smallest value of threshold SNR for which the new sources can be detected; induced sources have a fraction of the original source power. Second, by increasing the number of DOA's associated

to each source, the probability of resolution decreases; the distance between related DOA's decreases. Simulation results show that increasing the number of dimensions associated to each source may be useful just for high values of SNR or if the true covariance matrix is known which may not be practical.

7 Simulation Results

In the previous sections, we have proposed two algorithms for localization of ID sources with different power distributions. In both algorithms, the central angles are estimated using TLS-ESPRIT, in which each source is modeled by a subspace of dimension 2, but the extension widths are estimated in two different ways. The estimation of extension widths in one of the two algorithms is based on the estimation of the moments of angular power distribution while the other algorithm is based on the differences between two DOA's corresponding to each source.

In order to simulate the proposed algorithms, we have assumed two narrowband ID sources to be as signal emitters whose signals arrive at the two sub-arrays X and Y. The two sub-arrays consist of 16 sensors with an inter-element spacing of half the wavelength. The distance between identical sensors is $d = \lambda/10$.

The central angles of two sources are $\theta_{01} = 10^o$ and $\theta_{02} = 30^o$. The source at $\theta_{01} = 10^o$ has a uniform angular power distribution as

$$\rho_1(\theta; \boldsymbol{\psi}_1) = \begin{cases} \frac{1}{2\Delta_1} & |\theta - \theta_{01}| \leq \Delta_1 \\ 0 & \text{otherwise} \end{cases} \tag{69}$$

where $\Delta_1 = 1.5^o$. The source at $\theta_{02} = 30^o$ has a Gaussian power angular distribution as

$$\rho_2(\theta; \boldsymbol{\psi}_2) = \frac{1}{\sqrt{2\pi}\Delta_2} \exp(-\frac{(\theta - \theta_{02})^2}{2\Delta_2^2}), \tag{70}$$

where $\Delta_2 = 1^o$. The sources are assumed to be equipower and uncorrelated.

A Monte-Carlo simulation with 20 independent runs and 500 snapshots for each trial was performed for different SNR's. Fig. 1 and Fig. 2 show the bias and the standard deviation of the central angle estimator. Fig. 3 and Fig. 4 show the estimation bias of extension width and Fig. 5 and Fig. 6 show the standard deviation of extension width for different algorithms. In order to show the effect of using higher order derivatives of location matrix in the moment based algorithm, we have implemented the algorithm using both first and second order Taylor series. As it can be seen, a first order approximation has a better performance. Note that in a ULA, using higher order derivatives of location vector causes the estimation error of central angle be magnified as the sensor index increases. Using a low-error estimation algorithm for central angle estimation, can improve the performance of the algorithm with increasing the number of implemented derivatives.

A   Appendix A

In this section, we show that

$$E\{\alpha\alpha^*\} = \sigma_s^2 \tag{71}$$

$$E\{\beta\beta^*\} = \sigma_s^2 M_2 \tag{72}$$

$$E\{\alpha\beta^*\} = E\{\beta\alpha^*\} = 0. \tag{73}$$

Using the definition of α in (15), we can write

$$
\begin{aligned}
E\{\alpha\alpha^*\} &= E\left\{\int\int s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})d\theta d\theta'\right\} \\
&= \int\int E\{s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})\}d\theta d\theta' \\
&= \int\int \sigma_s^2\rho(\theta;\boldsymbol{\psi})\delta(\theta-\theta')d\theta d\theta' \\
&= \sigma_s^2\int \rho(\theta;\boldsymbol{\psi})d\theta \\
&= \sigma_s^2.
\end{aligned}
\tag{74}
$$

Using the definition of β in (16), we can write

$$
\begin{aligned}
E\{\beta\beta^*\} &= E\left\{\int\int (\theta-\theta_0)s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})(\theta'-\theta_0)d\theta d\theta'\right\} \\
&= \int\int (\theta-\theta_0)E\{s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})\}(\theta'-\theta_0)d\theta d\theta' \\
&= \int\int (\theta-\theta_0)\sigma_s^2\rho(\theta;\boldsymbol{\psi})\delta(\theta-\theta')(\theta'-\theta_0)d\theta d\theta' \\
&= \sigma_s^2\int (\theta-\theta_0)^2\rho(\theta;\boldsymbol{\psi})d\theta \\
&= \sigma_s^2 M_2.
\end{aligned}
\tag{75}
$$

Also we have

$$
\begin{aligned}
E\{\alpha\beta^*\} &= E\left\{\int\int (\theta-\theta_0)s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})d\theta d\theta'\right\} \\
&= \int\int (\theta-\theta_0)E\{s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})\}d\theta d\theta' \\
&= \int\int (\theta-\theta_0)\sigma_s^2\rho(\theta;\boldsymbol{\psi})\delta(\theta-\theta')d\theta d\theta' \\
&= \sigma_s^2\int (\theta-\theta_0)\rho(\theta;\boldsymbol{\psi})d\theta = 0
\end{aligned}
\tag{76}
$$

B   Appendix B

In this appendix, we show that

$$E\{\alpha\gamma^*\} = \sigma_s^2 M_2, \tag{77}$$

$$E\{\beta\gamma^*\} = \sigma_s^2 M_3, \tag{78}$$

$$E\{\gamma\gamma^*\} = \sigma_s^2 M_4. \tag{79}$$

Using the definition of γ in (54) and α in (15), we have

$$
\begin{aligned}
E\{\alpha\gamma^*\} &= E\left\{\int\int s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})(\theta'-\theta_0)^2 d\theta d\theta'\right\} \\
&= \int\int E\{s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})\}(\theta'-\theta_0)^2 d\theta d\theta' \\
&= \int\int \sigma_s^2\rho(\theta;\boldsymbol{\psi})\delta(\theta-\theta')(\theta'-\theta_0)^2 d\theta d\theta' \\
&= \sigma_s^2\int(\theta-\theta_0)^2\rho(\theta;\boldsymbol{\psi})d\theta \\
&= \sigma_s^2 M_2.
\end{aligned}
\tag{80}
$$

Using the definition of β in (16), we can write

$$
\begin{aligned}
E\{\beta\gamma^*\} &= E\left\{\int\int (\theta-\theta_0)s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})(\theta'-\theta_0)^2 d\theta d\theta'\right\} \\
&= \int\int (\theta-\theta_0)E\{s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})\}(\theta'-\theta_0)^2 d\theta d\theta' \\
&= \int\int (\theta-\theta_0)\sigma_s^2\rho(\theta;\boldsymbol{\psi})\delta(\theta-\theta')(\theta'-\theta_0)^2 d\theta d\theta' \\
&= \sigma_s^2\int(\theta-\theta_0)^3\rho(\theta;\boldsymbol{\psi})d\theta \\
&= \sigma_s^2 M_3.
\end{aligned}
\tag{81}
$$

Similarly

$$
\begin{aligned}
E\{\gamma\gamma^*\} &= E\left\{\int\int (\theta-\theta_0)^2 s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})(\theta'-\theta_0)^2 d\theta d\theta'\right\} \\
&= \int\int (\theta-\theta_0)^2 E\{s(\theta;\boldsymbol{\psi})s^*(\theta';\boldsymbol{\psi})\}(\theta'-\theta_0)^2 d\theta d\theta' \\
&= \int\int (\theta-\theta_0)\sigma_s^2\rho(\theta;\boldsymbol{\psi})\delta(\theta-\theta')(\theta'-\theta_0)^2 d\theta d\theta' \\
&= \sigma_s^2\int(\theta-\theta_0)^4\rho(\theta;\boldsymbol{\psi})d\theta \\
&= \sigma_s^2 M_4.
\end{aligned}
\tag{82}
$$

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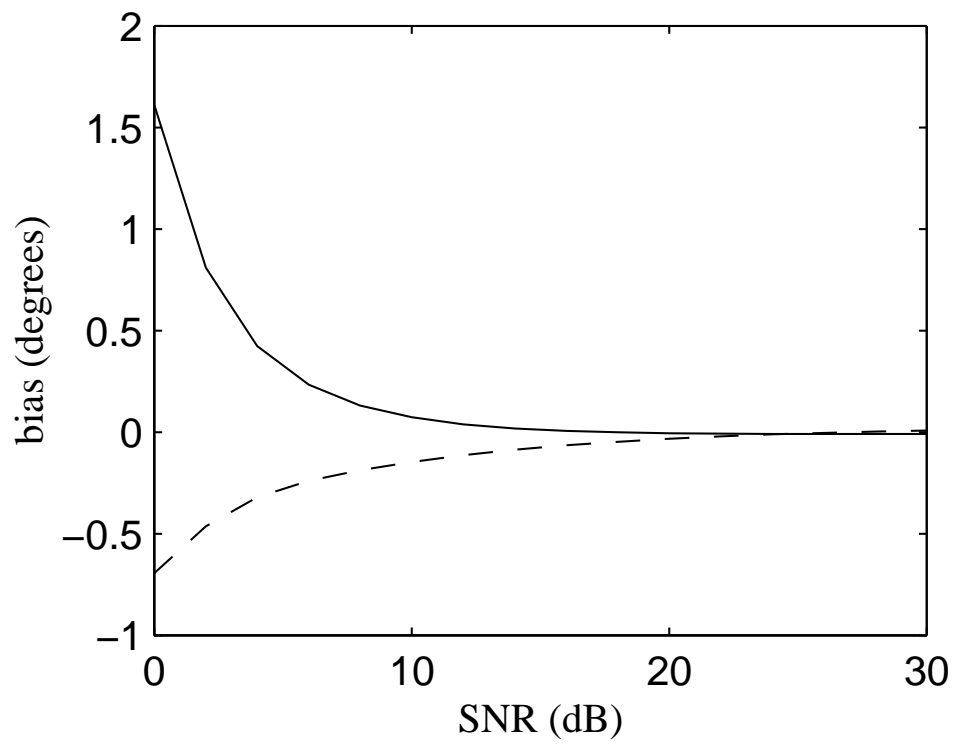
Figure 1: Bias of estimation for central angles versus SNR for two ID sources at 10° (dashed line) and 30° (solid line).
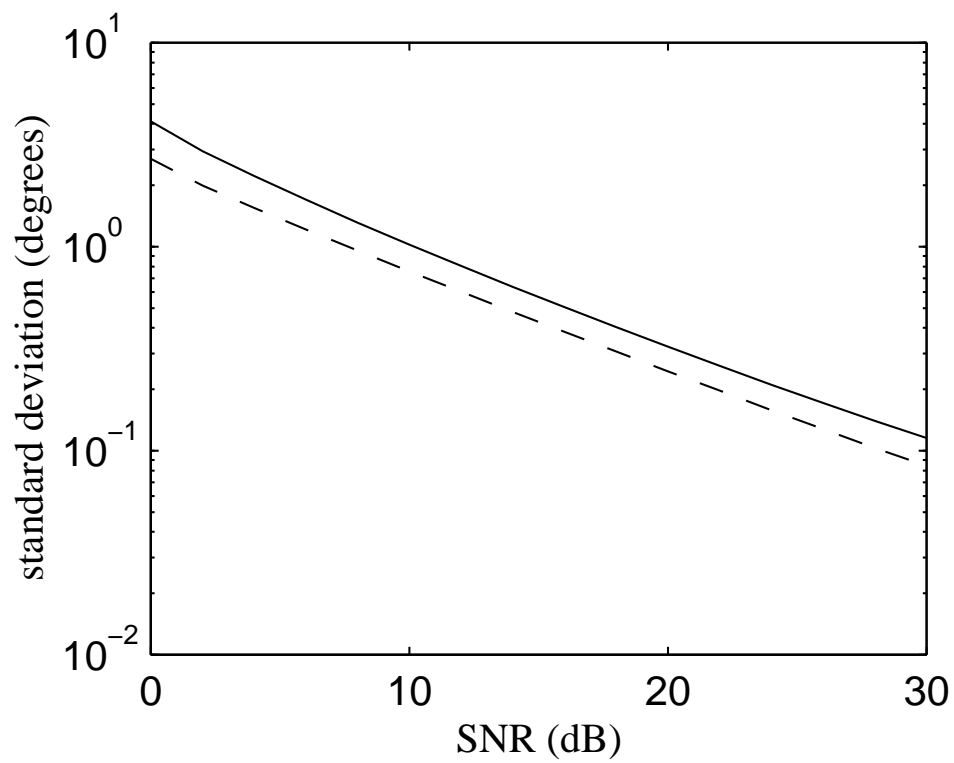
Figure 2: Standard deviations for the central angle estimation versus SNR for two ID source at 10° (dashed line) and 30° (solid line).
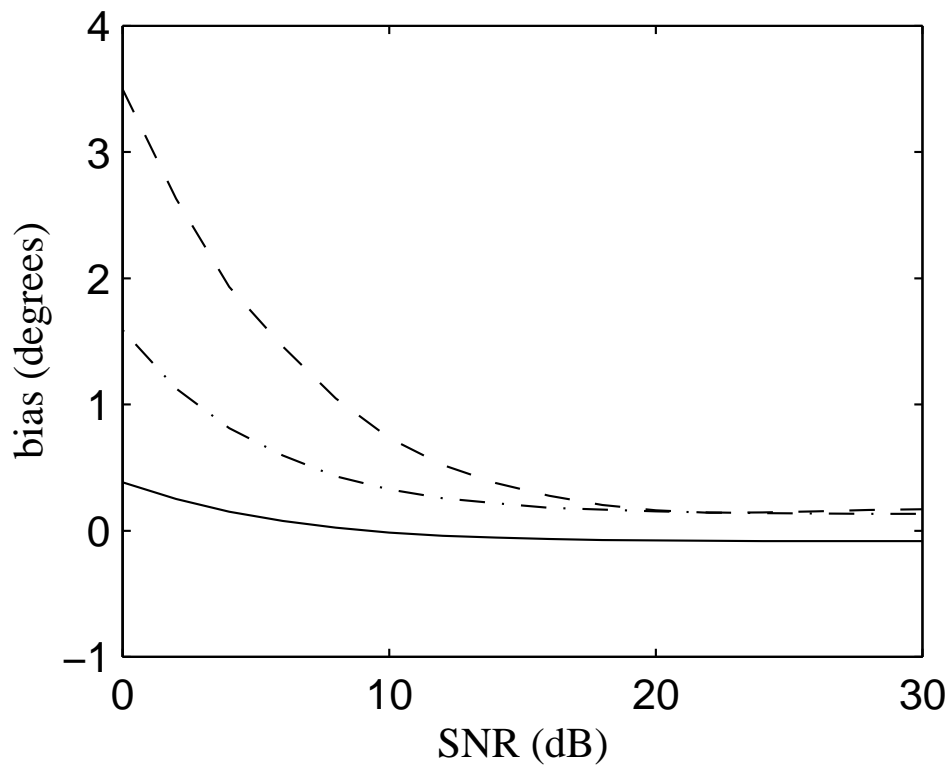
Figure 3: Bias of estimation for extension width versus SNR for UID source at 10° and $\Delta = 1.5^o$ for the difference based method (dashed line), the first order Taylor approximation (solid line), and the second order Taylor approximation (dashed-dotted).
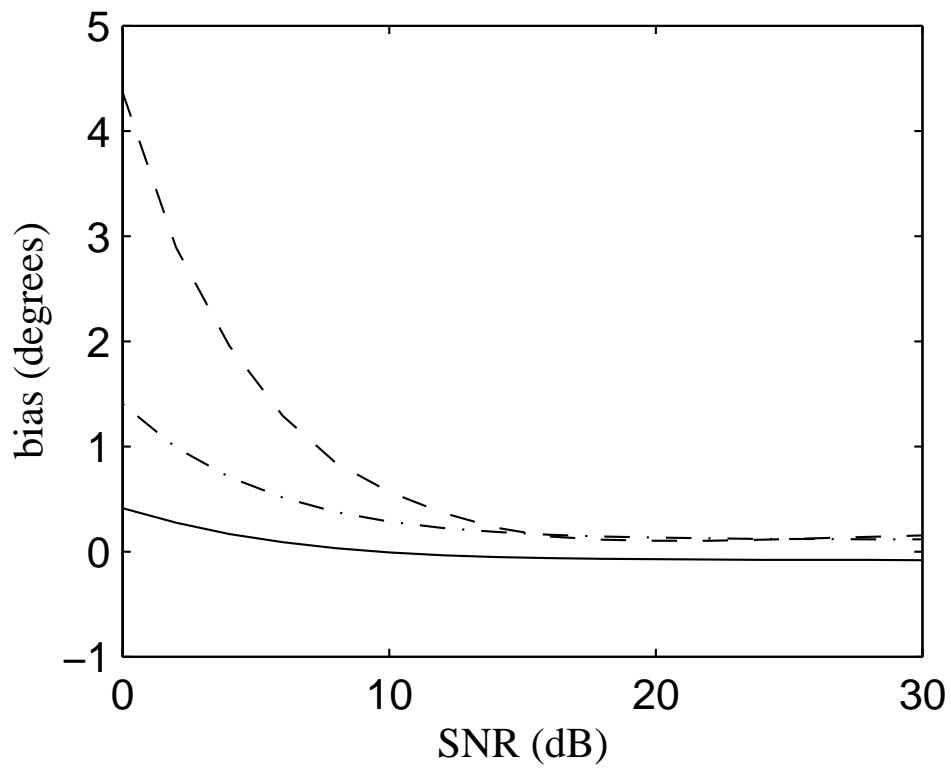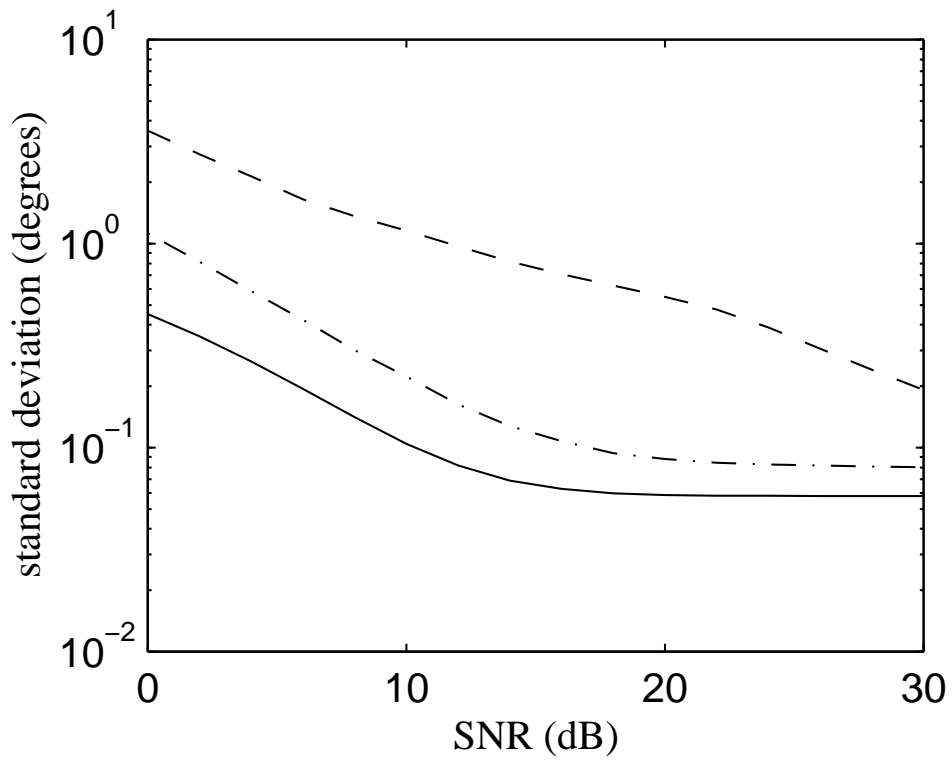
Figure 4: Bias of estimation for extension width versus SNR for GID source at 30° and $\Delta = 1^o$ for the difference based method (dashed line), the first order Taylor approximation (solid line), and the second order Taylor approximation (dashed-dotted).

Figure 5: Standard deviation for extension width estimation versus SNR for UID source at 10° and $\Delta = 1.5^o$ for the difference based method (dash line), the first order Taylor approximation (solid line), second order Taylor approximation (dashed-dotted).
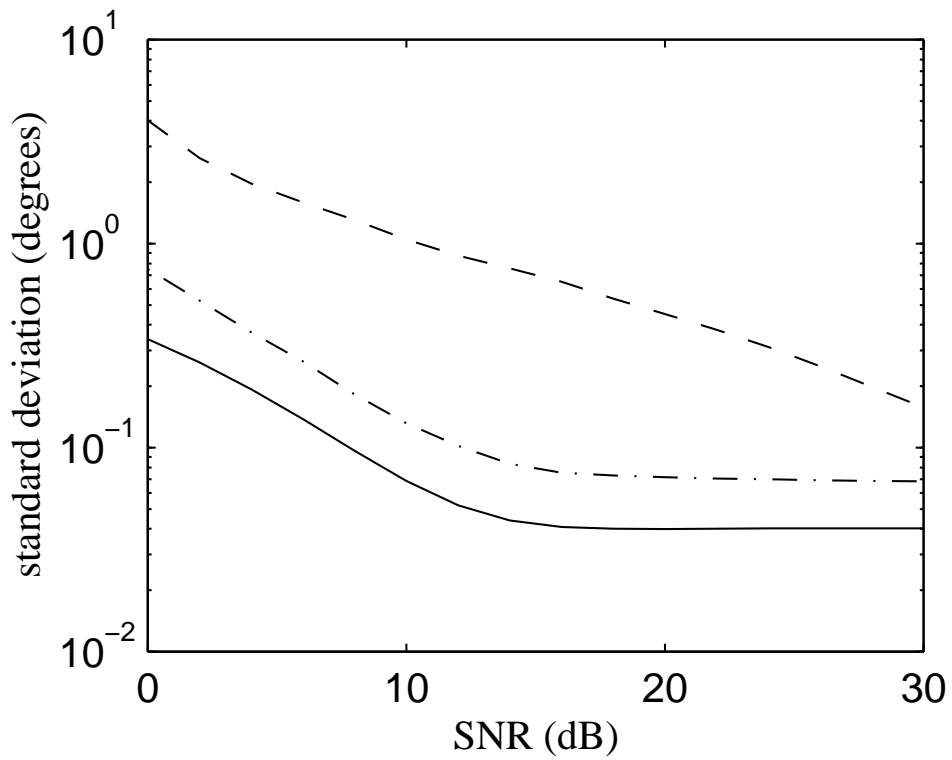
Figure 6: Standard deviation of extension width estimation versus SNR for GID source at 30° and $\Delta = 1^o$ for the difference based method (dashed line), the first order Taylor approximation (solid line), and the second order Taylor approximation (dashed-dotted).