# Uplink-Downlink Duality via Minimax Duality

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*Abstract* — **The capacity of a Gaussian vector channel remains the same if the input and the output are interchanged. This duality relation, when generalized to multiuser channels, has been instrumental in a recent characterization of the Gaussian vector broadcast channel sum capacity. This paper examines the uplink-downlink duality from a convex optimization viewpoint. The main result is a new and simple derivation of uplink-downlink duality as a special case of a duality between two minimax optimization problems. The new derivation generalizes the previous solution for Gaussian vector broadcast channels under a power constraint to channels with linear covariance input constraints. It also illustrates that the minimax representation of the broadcast channel sum capacity is more general than the duality representation.**

## I. Introduction

There is a curious input-output duality for Gaussian vector channels. Consider a Gaussian vector channel under a power constraint:

$$\mathbf{Y} = H\mathbf{X} + \mathbf{Z}, \tag{1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector-valued input and output respectively, $H$ is the channel matrix, and $\mathbf{Z}$ is the additive i.i.d. Gaussian vector noise. The capacity of the channel remains the same if the input and the output are interchanged, the channel matrix transposed, and the same power constraint applied to the "dual" channel:

$$\mathbf{Y}' = H^T\mathbf{X}' + \mathbf{Z}'. \tag{2}$$

This is true because the capacity of a Gaussian vector channel under a power constraint is computed via a water-filling of total power over the set of singular values of the channel matrix, and the singular values of $H$ and $H^T$ are the same. Note that this holds even when the matrix $H$ is not necessarily a square matrix. In this case $\mathbf{X}'$ and $\mathbf{X}$ do not have the same dimension.

Interestingly and surprisingly, the input-output duality of Gaussian vector channels generalizes to the multiuser setting. Let $\mathbf{X}^T = [\mathbf{X_1}^T \mathbf{X_2}^T]$. Consider the Gaussian vector channel as a multiple access channel where $\mathbf{X_1}$ and $\mathbf{X_2}$ do not cooperate. The capacity region of the multiple access channel is:

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2) \\ R_2 &\leq I(X_2; Y|X_1) \\ R_1 + R_2 &\leq I(X_1, X_2; Y). \end{aligned} \tag{3}$$

[1] This work was supported by the Canada Research Chair program.

On the other hand, let $\mathbf{Y}'^T = [\mathbf{Y_1'}^T \mathbf{Y_2'}^T]$. Consider the Gaussian vector channel as a broadcast channel. Using a coding technique called "writing-on-dirty-paper" [1] [2] [3], the following rate pairs are achievable:

$$\begin{aligned} R_1 &= I(X_1'; Y_1'|X_2') \\ R_2 &= I(X_2'; Y_2') \end{aligned} \tag{4}$$

Or,

$$\begin{aligned} R_1 &= I(X_1'; Y_1') \\ R_2 &= I(X_2'; Y_2'|X_2'), \end{aligned} \tag{5}$$

where $\mathbf{X_1'}$ and $\mathbf{X_2'}$ are independent Gaussian random vectors with $\mathbf{X}' = \mathbf{X_1'} + \mathbf{X_2'}$, and $\mathbf{X}'$ satisfies the power constraint $P$. The union of all such points as expressed in (4) and (5) is the largest known achievable region for the Gaussian vector broadcast channel, but a converse has not been established except in the sum capacity case. In fact, the optimization of the rate region above does not even appear to be a convex optimization problem. However, in a surprisingly result [4] [5], Jindal, Viswanath and Goldsmith showed that the union of these achievable points is precisely the multiple access channel capacity region (3) under a sum power constraint across both $X_1$ and $X_2$. As the multiple access channel is often referred to as the uplink channel and the broadcast channel the downlink channel, this relation is called "uplink-downlink duality". The proof of this duality result depends on a clever choice of the multiple access transmit covariance matrices for each achievable rate point in the broadcast channel capacity region and vice versa.

The objective of this paper is to give a new perspective on uplink-downlink duality for the sum capacity point. The main result of the paper is a minimax duality for a class of Gaussian mutual information optimization problems. Uplink-downlink duality can be derived easily as a special case of minimax duality. This new perspective allows the duality between uplink and downlink to be generalized to broadcast channels under linear covariance constraints. It also illustrates that duality depends critically on the linearity of the input constraint. For a broadcast channel with general convex input constraints, the minimax expression for the sum capacity is more general than the duality expression.

The rest of the paper is organized as follows. Section II contains the main minimax duality result and its proof. Section III illustrates the application of minimax duality to the Gaussian vector broadcast channel and gives a new derivation of uplink-downlink duality. Section IV contains concluding remarks.

## II. Minimax Duality

Consider a Gaussian mutual information expression

$$C = \max_{S_x} \min_{S_z} \frac{1}{2} \log \frac{|H S_x H^T + S_z|}{|S_z|}, \tag{6}$$

where the objective is to choose a transmit covariance matrix $S_x$ to maximize the mutual information and a noise covariance matrix $S_z$ to minimize the mutual information, each subject to a convex constraint. This may correspond to a compound channel in which the transmitter must construct a codebook to achieves a negligible probability of error for all possible channel realizations, or as shall be seen later, a broadcast channel in which the noise correlation among the receivers may vary arbitrarily. This paper focuses on the minimax problem under linear covariance constraints of the form:

$$\mathrm{tr}(S_x Q_x) \le 1 \tag{7}$$
$$\mathrm{tr}(S_z Q_z) \le 1, \tag{8}$$

where $Q_x$ and $Q_z$ are parameters of the linear constraints. Without loss of generality, $Q_x$ and $Q_z$ can be taken as symmetric. For example, the usual power constraint corresponds to the case where $Q_x$ is an identity matrix. The minimax capacity is a function of $H$, $Q_x$ and $Q_z$, and it is denoted as $C(H, Q_x, Q_z)$.

The main result of this paper is a characterization of the dual of the above minimax problem. This duality is not in the sense of Lagrangian duality in convex optimization. Rather, it is a particular feature of the Gaussian mutual information expression. The derivation is most transparent when $H$ is invertible, which we assume for the rest of the section.

The first step in developing the duality is a characterization of the saddle-point of the minimax problem via its Karush-Kuhn-Tucker (KKT) condition. The KKT condition consists of the usual water-filling condition with respect to the maximization over $S_x$:

$$H^T (H S_x H^T + S_z)^{-1} H = \lambda_x Q_x, \tag{9}$$

and the least favourable noise condition with respect to the minimization over $S_z$:

$$S_z^{-1} - (H S_x H^T + S_z)^{-1} = \lambda_z Q_z, \tag{10}$$

where $\lambda_x$ and $\lambda_z$ are the appropriate Lagrangians. (The coefficient $\frac{1}{2}$ is omitted for simplicity.) The KKT condition is necessary and sufficient for optimality. Now, pre- and post-multiple (10) by $H^T$ and $H$ respectively, substitute (9) into (10), and rearrange the terms, we obtain:

$$H^T S_z^{-1} H = H^T \lambda_z Q_z H + \lambda_x Q_x. \tag{11}$$

Thus, if $H$ is invertible, we get

$$H (H^T \lambda_z Q_z H + \lambda_x Q_x)^{-1} H^T = S_z. \tag{12}$$

This is precisely the water-filling KKT condition with $\lambda_z Q_z$ as the transmit covariance matrix and $\lambda_x Q_x$ as the noise covariance. The above is an explicit solution for $S_z$. Further, substitute (12) into (9) and solve for $S_x$:

$$(\lambda_x Q_x)^{-1} - (H^T \lambda_z Q_z H + \lambda_x Q_x)^{-1} = S_x. \tag{13}$$

This is precisely the least-favourable-noise KKT condition with $\lambda_x Q_x$ as the worst noise and $\lambda_z Q_z$ as the transmit covariance. Define

$$\Sigma_x = \lambda_x Q_x \quad , \quad \Sigma_z = \lambda_z Q_z, \tag{14}$$
$$\Psi_x = \frac{1}{\lambda_x} S_x \quad , \quad \Psi_z = \frac{1}{\lambda_z} S_z. \tag{15}$$

Equations (12) (13) can be re-written as:

$$H (H^T \Sigma_z H + \Sigma_x)^{-1} H^T = \lambda_z \Psi_z \tag{16}$$
$$\Sigma_x^{-1} - (H^T \Sigma_z H + \Sigma_x)^{-1} = \lambda_x \Psi_x. \tag{17}$$

Thus, associated with the original minimax problem (6), there is a "dual" minimax problem:

$$C = \max_{\Sigma_z} \min_{\Sigma_x} \frac{1}{2} \log \frac{|H^T \Sigma_z H + \Sigma_x|}{|\Sigma_x|}, \tag{18}$$

with linear covariance constraints:

$$\mathrm{tr}(\Sigma_x \Psi_x) \le 1, \tag{19}$$
$$\mathrm{tr}(\Sigma_z \Psi_z) \le 1. \tag{20}$$

The minimax problems (6) and (18) are duals of each other in the following sense:

- The optimal dual variable $\lambda_x$ in the maximization part of (6) is the optimal dual variable in the minimization part of (18).
- The optimal dual variable $\lambda_z$ in the minimization part of (6) is the optimal dual variable in the maximization part of (18).
- The saddle-point $(S_x, S_z)$ of (6) is related to the constraints $(\Psi_x, \Psi_z)$ of (18) by (15).
- The saddle-point $(\Sigma_z, \Sigma_x)$ of (18) is related to the constraints $(Q_z, Q_x)$ of (18) by (14).
- $C(H, Q_x, Q_z) = C(H^T, \Psi_z, \Psi_x)$.

To verify the last claim, it can be easily seen using (9) (10) and (14) that at the saddle-point:

$$\log \frac{|H^T \Sigma_z H + \Sigma_x|}{|\Sigma_x|} = \log \frac{|H S_x H^T + S_z|}{|S_z|}. \tag{21}$$

To summarize, there is a input-output duality for Gaussian mutual information minimax problems. By interchanging the input and the output, the constraints of the original problem become the variables in the dual problem and vice versa. This minimax duality is different from Lagrangian duality. Minimax duality is based on the manipulation of the KKT conditions of the optimization problem. The current derivation assumes that $H$ is invertible and that the optimal $S_x$ and $S_z$ are full rank. These are technical conditions that can be removed.

## III. Broadcast Channel Sum Capacity

We are motivated to study minimax duality because it arises naturally in the characterization of the sum capacity of Gaussian vector broadcast channels. The broadcast channel capacity is a long-standing open problem in information theory. Recently, the sum capacity for the Gaussian vector broadcast channel has been solved independently using two seemingly different approaches [3] [5]

[6]. In [3], it was shown that the broadcast channel capacity is the solution to a minimax mutual information problem. In [4] [5] and [6], it was shown that the broadcast channel sum capacity can be found by solving for the capacity of a dual multiple-access channel with a sum power constraint. The objective of this section is to unify the two approaches. We show that the duality between the broadcast channel and the multiple-access channel is a special case of minimax duality. Further, we illustrate a subtle difference between the two approaches by showing that minimax is a more general expression for sum capacity than duality for a certain class of channels.

### *A. Sum Capacity as a Minimax Problem*

Consider a Gaussian vector broadcast channel:

$$\mathbf{Y} = H\mathbf{X} + \mathbf{Z} \tag{22}$$

where $\mathbf{Y_1}$ and $\mathbf{Y_2}$ are non-coordinated receivers, $\mathbf{Y}^T = [\mathbf{Y_1}^T \mathbf{Y_2}^T]$, and $\mathbf{Z}$ is the unit variance Gaussian noise. A sum power constraint $\mathbf{E}[\mathbf{X}^T\mathbf{X}] \leq P$ is imposed on the input. A key ingredient in the characterization of the capacity is a connection between the broadcast channel and channels with side information, first published in [2]. In a classic result known as "writing on dirty paper", Costa [1] showed that if a Gaussian channel is corrupted by an interference signal $S$ that is known *non-causally* to the transmitter but not to the receiver, i.e.

$$\mathbf{Y} = \mathbf{X} + \mathbf{S} + \mathbf{Z}, \tag{23}$$

the capacity of the channel is the same as if $\mathbf{S}$ does not exist. Thus, in a broadcast channel, if we let $\mathbf{X} = \mathbf{X_1} + \mathbf{X_2}$ where $\mathbf{X_1}$ and $\mathbf{X_2}$ are Gaussian vectors, $\mathbf{X_1}$ can transmit information to $\mathbf{Y_1}$ as if $\mathbf{X_2}$ does not exist, and $\mathbf{X_2}$ can still transmit to $\mathbf{Y_2}$ regarding $\mathbf{X_1}$ as noise. This precoding strategy turns out to be optimal for sum-capacity in a Gaussian broadcast channel. This is proved for the 2-user 2-antenna case by Caire and Shamai [2], and has since been generalized by several authors [3] [5] [6] using different approaches.

We now briefly review the main result in [3]. The approach in [3] is based on the observation that interference pre-subtraction at the transmitter is identical to a decision-feedback equalizer with feedback "moved" to the transmitter. However, while the decision-feedback structure is capacity achieving for the Gaussian vector channel, it also requires coordination at the receivers because it has a feedforward matrix that operates on both $\mathbf{Y_1}$ and $\mathbf{Y_2}$. Clearly, such coordination is not possible in a broadcast channel. But, precisely because $\mathbf{Y_1}$ and $\mathbf{Y_2}$ cannot coordinate, they are also ignorant of the noise correlation between $\mathbf{Z_1}$ and $\mathbf{Z_2}$. Thus, the sum capacity of the broadcast channel must be bounded by the cooperative capacity with the *least favourable* noise correlation:

$$C_s \leq \min_{S_z} I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2}), \tag{24}$$

where $S_z$ is the covariance matrix for $\mathbf{Z}^T = [\mathbf{Z_1}^T \mathbf{Z_2}^T]$, and the minimization is over all $S_z$ whose block diagonal terms are the covariance matrices of $\mathbf{Z_1}$ and $\mathbf{Z_2}$. This outer bound is due to Sato [7].

The Karush-Kuhn-Tucker (KKT) condition associated with the minimization problem is

$$S_z^{-1} - (HS_xH^T + S_z)^{-1} = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} = \Phi, \tag{25}$$

where $\Phi$ is the dual variable corresponding to the diagonal constraints. Interestingly, $S_z^{-1} - (HS_xH^T + S_z)^{-1}$ also corresponds to the feedforward matrix of the decision-feedback equalizer. So, if the noise covariance is least favourable, the feedforward matrix of the decision-feedback equalizer would be diagonal. Thus, after moving the feedback operation to the transmitter, the entire equalizer de-couples into independent receivers for each user, and no coordination is needed whatsoever. Consequently, the Sato outer bound is achievable. Now, this achievable rate may be further maximized over all $S_x$ subject to the input constraint. Therefore, the sum capacity of a Gaussian vector broadcast channel is:

$$\begin{aligned} \max_{S_x} \min_{S_z} \quad & \frac{1}{2} \log \frac{|HS_xH^T + S_z|}{|S_z|} \\ \text{subject to} \quad & S_z^{(i)} = I, \quad i = 1, 2. \\ & \mathrm{tr}(S_x) \leq P, \\ & S_x, S_z \geq 0, \end{aligned} \tag{26}$$

where $S_z^{(i)}$ refers to the $i$th block-diagonal term of $S_z$.

### *B. Sum Capacity via Minimax Duality*

The sum capacity of a Gaussian vector broadcast channel can also be solved using a different method. In [4], it was observed that under a input power constraint, the achievable region of a broadcast channel using the precoding technique is exactly the same as the capacity region of a dual multiple access channel with the channel matrix transposed and a sum power constraint applied to all inputs. In [6], it was observed that the uplink-downlink duality is closely related to convex Lagrangian duality. Based this observation, [5] and [6] showed that the sum capacity of the broadcast channel is precisely the sum capacity of the dual multiple access channel under a sum power constraint. In this section, we illustrate that uplink-downlink duality can be readily derived from minimax duality.

The starting point of the new derivation is the following KKT condition associated with the minimax optimization problem (26)

$$H^T(HS_xH^T + S_z)^{-1}H = \lambda I \tag{27}$$

$$S_z^{-1} - (HS_xH^T + S_z)^{-1} = \Phi \tag{28}$$

where $\lambda$ is the dual variable associated with the power constraint and $\Phi$ is the dual variable associated with the diagonal constraint. By minimax duality, the minimax optimization problem (26) has a corresponding dual minimax problem with $H^T$ as the channel matrix, $\Phi$ as the input covariance and $\lambda I$ as the noise covariance. In particular, $(\Phi, \lambda I)$ satisfies a water-filling condition:

$$H(H^T\Phi H + \lambda I)^{-1}H^T = S_z. \tag{29}$$

This water-filling condition is the key to the duality between the broadcast channel and the multiple access channel.

In convex optimization, the dual variables $\lambda$ and $\Phi$ have the interpretation of being the sensitivity of the saddle-point with respect to the constraints. Let $C_s(P, N_1, N_2)$ denote the sum capacity of the Gaussian vector broadcast channel with power constraint $P$ and noise variance $N_1$, $N_2$ in the receivers. Then,

$$\lambda = \left. \frac{\partial C_s(P, N_1, N_2)}{\partial P} \right|_{(S_x, S_z)} \tag{30}$$

and

$$\Phi_i = - \left. \frac{\partial C_s(P, N_1, N_2)}{\partial N_i} \right|_{(S_x, S_z)} \tag{31}$$

So, if $P$ is increased by $\Delta P$, and $N_1$, $N_2$ are both increased by $\Delta N$ in the proportion:

$$\frac{\Delta P}{\Delta N} = \frac{\sum_i \Phi_i}{\lambda}, \tag{32}$$

then the capacity of the broadcast channel does not change to the first order. On the other hand, from the structure of (26), if $\Delta P$ and $\Delta N$ are in the proportion:

$$\frac{\Delta P}{\Delta N} = \frac{P}{1}, \tag{33}$$

the saddle-point also scales proportionally, and the capacity remains unchanged. For both conditions to be satisfied simultaneously, it must be true that

$$\frac{\sum_i \Phi_i}{\lambda} = P. \tag{34}$$

Now, let $D = \Phi/\lambda$ and re-write (29) as

$$H(H^T D H + I)^{-1} H^T = \lambda S_z. \tag{35}$$

This condition is precisely the KKT condition for a multiple access channel with an input constraint:

$$\mathrm{tr}(DS_z) = \mathrm{tr}(D) = \frac{\sum_i \Phi_i}{\lambda} = P. \tag{36}$$

Thus, the solution to the minimax problem directly corresponds to the solution to a maximization problem:

$$\begin{array}{ll} \max\limits_{D} & \frac{1}{2} \log |H^T D H + I| \\ \text{s.t.} & D \text{ is diagonal} \\ & \mathrm{tr}(D) \leq P, \\ & D \geq 0, \end{array} \tag{37}$$

This establishes the duality of a multiple access channel and a broadcast channel.

### *C. Generalized Duality*

The duality between the Gaussian broadcast channel and the Gaussian multiple access channel is important from a computational perspective. The multiple access channel capacity (37) is considerably easier to compute than the minimax problem (26). Although the duality result as established in [4] [5] and [6] applies only to a broadcast channel with a sum power constraint, it is clear from minimax duality that it also holds when the broadcast channel input constraint is a linear covariance constraint of the form

$$\mathrm{tr}(S_x Q) \leq P, \tag{38}$$

in which case, the noise covariance $I$ in the dual multiple access channel (37) is simply replaced by the covariance matrix $Q$:

$$C = \max_D \frac{1}{2} \log |H^T D H + Q|. \tag{39}$$

The same power constraint applies as before: $\mathrm{tr}(D) \leq P$.

A key requirement for the duality between the broadcast channel and the multiple access channel to hold is the linearity of the constraint. Without linearity, the power constraint derivation (30) - (36) does not follow, and the dual minimax problem does not reduce to a single maximization problem. Thus, for a broadcast channel with an arbitrary convex constraints of the form

$$f(S_x) \leq 0, \tag{40}$$

although minimax duality still exists, the dual noise covariance matrix ($\lambda Q = \lambda f'(\cdot)$) now depends on the dual variables of the minimax problem, which can only be determined after the minimax problem (26) is explicitly solved. Therefore, the minimax expression (26) is a more fundamental characterization of the Gaussian vector broadcast channel than the one that the duality approach provides.

## IV. Conclusions

This paper illustrates an input-output duality for a Gaussian mutual information minimax optimization problem. It is shown that the uplink-downlink duality between a broadcast channel and a multiple access channel can be derived as a special case of the minimax duality. It is also shown that the minimax expression for the sum capacity of the broadcast channel is a more general expression than uplink-downlink duality.

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