Complex Numbers Review

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Complex Exponential Function

Continuous-time:

$$\begin{aligned} x(t) &= Ae^{j(\omega t+\phi)} \\ &= A(\cos(\omega t+\phi) + j\sin(\omega t+\phi)) \\ &= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi) \end{aligned}$$

Recall, Euler's relation: $e^{j\theta} = \cos\theta + j\sin\theta$

complex function

The complex exponential function has similar properties to sinusoids – e.g., periodicity

Complex Numbers and the Quadratic Equation

- complex numbers are natural solutions to:

$$aw^2 + bw + c = 0 \qquad \text{where } b^2 - 4ac < 0$$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- we define $j \triangleq \sqrt{-1}$

Complex Numbers and Coordinate Systems

Two common types:

1. rectangular: real and imaginary components
2. polar: magnitude and phase components

Rectangular Coordinates

$$w = \underbrace{x}_{\text{real part}} + j \underbrace{y}_{\text{imaginary part}} \qquad x, y \in \mathbb{R}, w \in \mathbb{C}$$

Note: $x = \mathcal{R}e\{w\}$ and $y = \mathcal{I}m\{w\}$

Let $w_1 = x_1 + j\, y_1$ and $w_2 = x_2 + j\, y_2$.

$$\begin{aligned} w_1 \pm w_2 &= (x_1 + j\, y_1) \pm (x_2 + j\, y_2) \\ &= \underbrace{(x_1 \pm x_2)}_{\text{real part}} + j\, \underbrace{(y_1 \pm y_2)}_{\text{imag part}} \\ w_1 \cdot w_2 &= (x_1 + j\, y_1) \cdot (x_2 + j\, y_2) \\ &= x_1 x_2 + x_1 (j\, y_2) + (j\, y_1) x_2 + (j\, y_1)(j\, y_2) \\ &= x_1 x_2 + j x_1 y_2 + j x_2 y_1 + j^2 y_1 y_2 \\ &= \underbrace{(x_1 x_2 - y_1 y_2)}_{\text{real part}} + j\, \underbrace{x_1 y_2 + x_2 y_1)}_{\text{imag part}} \\ \frac{w_1}{w_2} &= \frac{x_1 + j\, y_1}{x_2 + j\, y_2} \cdot \left(\frac{x_2 - j\, y_2}{x_2 - j\, y_2} \right) = \cdots \\ &= \underbrace{\frac{x_1 x_2 + y_1 y 2}{x_2^2 + y_2^2}}_{\text{real part}} + j\, \underbrace{\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}}_{\text{imag part}} \end{aligned}$$

Rectangular Coordinates

Note:

$$w_1^* = x_1 - j\, y_1$$

$$|w_1| = \sqrt{x_1^2 + y_1^2}$$

and

$$w_1 \cdot w_1^* = (x_1 + j\, y_1)(x_1 - j\, y_1) = x_1^2 + y_1^2 = |w_1|^2 \quad \text{REAL}$$

Polar Coordinates

$$w = r e^{j\theta} \qquad r \in \mathbb{R}^+, \theta \in \mathbb{R}, w \in \mathbb{C}$$

Note: $r \equiv$ magnitude and $\theta \equiv$ phase

Let $w_1 = r_1 e^{j\theta_1}$ and $w_2 = r_2 e^{j\theta_2}$.

$$
\begin{aligned}
w_1 \cdot w_2 &= r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = (r_1\ r_2) e^{j(\theta_1\ +\ \theta_2)} \\
\text{magnitude} &\equiv r_1\ r_2 \\
\text{phase} &\equiv \theta_1 + \theta_2 \\
\frac{w_1}{w_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1\ -\ \theta_2)} \\
\text{magnitude} &\equiv \frac{r_1}{r_2} \\
\text{phase} &\equiv \theta_1 - \theta_2 \\
w_1 \pm w_2 &= r_1 e^{j\theta_1} \pm r_2 e^{j\theta_2} = \cdots \\
\text{magnitude} &\equiv \sqrt{(r_1 \cos\theta_1 \pm r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 \pm r_2 \sin\theta_2)^2} \\
\text{phase} &\equiv \arctan\left(\frac{r_1 \sin\theta_1 \pm r_2 \sin\theta_2}{r_1 \cos\theta_1 \pm r_2 \cos\theta_2}\right) \quad \ddot\frown
\end{aligned}
$$

Polar Coordinates

Note:

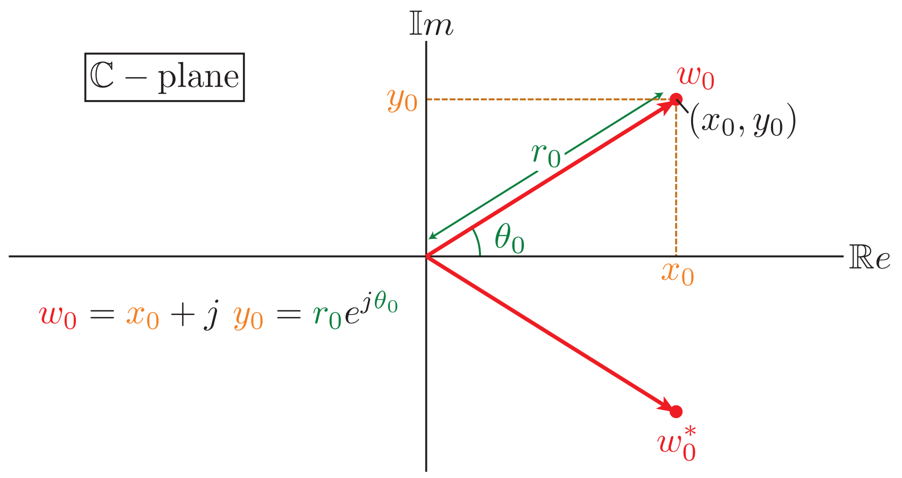
$$w_1^* = r_1 e^{-j\ \theta_1}$$

$$|w_1| = |r_1 e^{j\theta_1}| = |r_1| \cdot |e^{j\theta_1}| = r_1 \cdot 1 = r_1$$

and

$$w_1 \cdot w_1^* = (r_1 e^{+j\ \theta_1})(r_1 e^{-j\ \theta_1}) = r_1^2 = |w_1|^2 \quad \text{REAL}$$

Complex Plane Relations



Rectangular to Polar Conversion:

$$
\begin{aligned}
r_0 &= \sqrt{x_0^2 + y_0^2} \\
\theta_0 &= \arctan\left(\frac{y_0}{x_0}\right)
\end{aligned}
$$

Polar to Rectangular Conversion:

$$
\begin{aligned}
x_0 &= r_0 \cos\theta_0 \\
y_0 &= r_0 \sin\theta_0
\end{aligned}
$$

Note:

$$
\begin{aligned}
z_0 &= x_0 + j\ y_0 = r_0 \cos\theta_0 + j r_0 \sin\theta_0 = r_0 \underbrace{(\cos\theta_0 + j \sin\theta_0)}_{= e^{j\theta_0}\ \text{(EULER)}} \\
&= r_0 e^{j\theta_0}
\end{aligned}
$$

Complex Functions

The same coordinate systems exist for complex functions.

For $x(t) \in \mathbb{C}$:

1. rectangular: $x(t) = x_R(t) + jx_I(t)$
2. polar: $x(t) = |x(t)|e^{j\angle x(t)}$

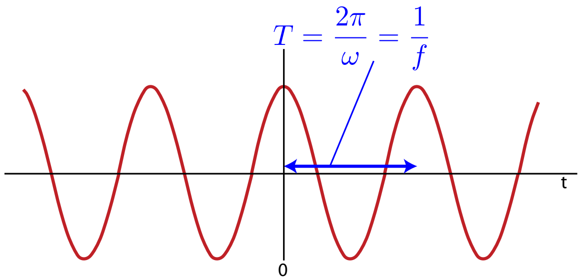
Note: This means that two graphs are required to plot a complex function $x(t)$:
(i) $x_R(t)$ vs. t and $x_I(t)$ vs. t, OR
(ii) $|x(t)|$ vs. t and $\angle x(t)$ vs. t.

Example: $x(t) = Ae^{j(\omega t+\phi)}$

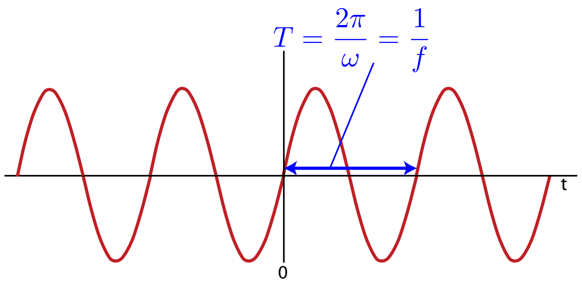
Rectangular:

$$\begin{aligned} x(t) &= Ae^{j(\omega t+\phi)} \\ &= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi) \\ \text{real} = x_R(t) &= A\cos(\omega t+\phi) \\ \text{imaginary} = x_I(t) &= A\sin(\omega t+\phi) \end{aligned}$$

$x_R(t)$:



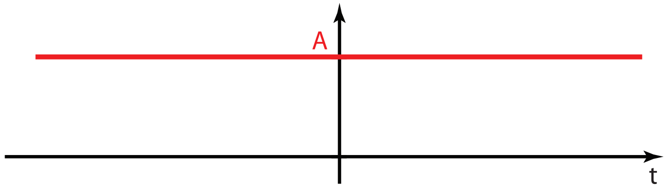
$x_I(t)$:



Polar:

$$\begin{aligned} x(t) &= Ae^{j(\omega t+\phi)} \\ \text{magnitude} = |x(t)| &= A \\ \text{phase} = \angle x(t) &= \omega t + \phi \end{aligned}$$

$|x(t)|$:



$\angle x(t)$ (for $\phi = 0$ and $\omega > 0$):