CHAPTER 5, problem set $\{5.1, 5.3, 5.4\,(a,b,c,d,n), 5.9, 5.11, 5.23, 5.31, 5.65\}$

problem $\boxed{5.1}$ The following input-output pairs have been observed during the operation of various systems; Determine their frequency response (LTI)

(a) $x(n) = \left(\frac{1}{2}\right)^n \xrightarrow{T_1} y(n) = \left(\frac{1}{8}\right)^n$

range of $n \in (-\infty, \infty) \Rightarrow$ fourier transform of $x(n)$ & $y(n)$ does not exist.

also $y(n) = x^3(n) \Rightarrow$ system $H$ is non-linear

(b) $x(n) = \left(\frac{1}{2}\right)^n u(n) \xrightarrow{T_2} y(n) = \left(\frac{1}{8}\right)^n u(n)$.

$$X(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$Y(\omega) = \frac{1}{1 - \frac{1}{8} e^{-j\omega}}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{1 - \frac{1}{8} e^{-j\omega}}}{\frac{1}{1 - \frac{1}{2} e^{-j\omega}}} = \frac{1 - \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{8} e^{-j\omega}}$$

(c) $x(n) = e^{j\pi/5}$ ~~u(n)~~ $\xrightarrow{T_3} y(n) = 3e^{+j\pi/5}$

$H(\omega) = 3$ for all $\omega$,

(d) $x(n) = e^{j\pi/5} u(n) \xrightarrow{T_4} y(n) = 3e^{j\pi/5} u(n)$.

~~$H(\omega) = 3$ for all $\omega$.~~ $H(\omega)$ @ $\omega = \frac{\pi}{5} = 3$ i.e $H(\frac{\pi}{5}) = 3$

(e) $x(n) = x(n + N_1) \xrightarrow{T_5} y(n) = y(n + N_2)$, $N_1 \neq N_2$, $N_1, N_2$ pri

for LTI systems: period of the input signal should be the same as the period of the output signal $\Rightarrow N_1 = N_2$ but

since, $N_1 \neq N_2$, the system is non-linear.

$\boxed{\text{problem 5.3}}$ Consider an LTI system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

(b) Determine and sketch the magnitude and phase spectra for the input and output signals for the following inputs

(a) Determine and sketch the magnitude and phase response $|H(\omega)|$ and $\angle H(\omega)$, respectively.

$$H(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$H(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n, \quad \sum_{n=0}^{\infty} a^n, \ |a| < 1$$

$$= \frac{1}{1 - \frac{1}{2} e^{-j\omega}} = \frac{1}{1 - \frac{1}{2}\cos\omega - \frac{1}{2} j \sin\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{\left(1 - \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}}$$

$$= \frac{1}{\left(1 + \frac{1}{4}\cos^2\omega - 2 \cdot \frac{1}{2}\cos\omega + \frac{1}{4}\sin^2\omega\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1 + \frac{1}{4}(\cos^2\omega + \sin^2\omega) - \cos\omega\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(5/4 - \cos\omega\right)^{1/2}}$$

**problem 5.4** Determine and sketch the magnitude and phase response of the following systems.

(a) $y(n) = \frac{1}{2}[x(n) + x(n-1)]$

$$Y(\omega) = \frac{1}{2}[1 + e^{-j\omega}]\, X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2}(1 + e^{-j\omega}) = \frac{1}{2} \cdot e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)$$

$$= \frac{1}{2} \cdot 2 \cdot \cos\left(\frac{\omega}{2}\right) e^{-j\omega/2} = \cos\left(\frac{\omega}{2}\right) \cdot e^{-j\omega/2}$$

(b) $y(n) = \frac{1}{2}[x(n) - x(n-1)]$

$$Y(\omega) = \frac{1}{2}(1 - e^{-j\omega})\, X(\omega)$$

$$H(\omega) = \frac{1}{2}(1 - e^{-j\omega})$$

$$= \sin\left(\frac{\omega}{2}\right) e^{-j\omega/2}\, e^{j\pi/2}$$

(c) $y(n) = \frac{1}{2}[x(n+1) - x(n-1)]$

$$Y(\omega) = \frac{1}{2}\left[e^{j\omega} - e^{-j\omega}\right] X(\omega)$$

$$H(\omega) = \frac{1}{2}\left(e^{j\omega} - e^{-j\omega}\right) = \sin(\omega)\, \underbrace{e^{j\pi/2}}_{=j}$$

(d) $y(n) = \frac{1}{2}[x(n+1) + x(n-1)]$

$$Y(\omega) = \frac{1}{2}\left[e^{j\omega} + e^{-j\omega}\right] X(\omega)$$

$$H(\omega) = \frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right) = \cos(\omega)$$

(n) $y(n) = \frac{1}{4}[x(n) - 2x(n-1) + x(n-2)]$

$$Y(\omega) = \frac{1}{4}\left[1 - 2e^{-j\omega} + e^{-j2\omega}\right] X(\omega)$$

$$H(\omega) = \frac{1}{4}\left(1 - e^{j\omega}\right)^2$$

$$= \sin^2\left(\frac{\omega}{2}\right) e^{-j(\omega-\pi)}$$

**problem 5.9** Determine the frequency ~~response~~ content of the outputs of the following systems to the input signal.

$$x(n) = A\cos\frac{\pi}{4}n$$

(a) $y(n) = x(2n) = A\cos\left(\frac{\pi}{2}n\right) \Rightarrow \omega = \frac{\pi}{2}$

(b) $y(n) = x^2(n) = A^2\cos^2\left(\frac{\pi}{4}n\right) = A^2\left(\frac{\cos\left(2\cdot\frac{\pi}{4}n\right)+1}{2}\right)$

$= \frac{1}{2}A^2 + \frac{1}{2}A^2\cos\left(\frac{\pi}{2}n\right) \Rightarrow \omega = 0,\ \omega = \frac{\pi}{2}$

(c) $y(n) = (\cos \pi n)\, x(n)$.

$= A.\cos\left(\frac{\pi}{4}n\right).\cos(\pi n) = \frac{A}{2}.\cos\left(\frac{5\pi}{4}n\right) + \frac{A}{2}\cos\left(\frac{3\pi}{4}n\right)$

$\Rightarrow \omega = \frac{5\pi}{4},\ \frac{3\pi}{4}$

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**problem 5.11** Determine the magnitude and phase response of the multipath channel $y(n) = x(n) + x(n-M)$

at what frequencies does $H(\omega) = 0$

$y(n) = x(n) + x(n-M)$

$Y(\omega) = \left(1 + e^{-j\omega M}\right) X(\omega)$

$H(\omega) = 1 + e^{-j\omega M}$

$H(\omega) = 0 \Rightarrow \omega = \frac{(2k+1)\pi}{M},\ k = 0, 1, \ldots$

odd.

$|H(\omega)| = \left|2\cos\frac{\omega M}{2}\right| \Rightarrow H(\omega) = 0 \Rightarrow \frac{\omega M}{2} = \frac{(2k+1)\pi}{2}$

$\omega = \frac{(2k+1)\pi}{M},\ k = 0, 1, \ldots$

$\boxed{\text{problem 5.23}}$ The frequency response of an ideal bandpass filter is given by

$$H(\omega) = \begin{cases} 0, & |\omega| \leq \frac{\pi}{8} \\ 1, & \frac{\pi}{8} < |\omega| < \frac{3\pi}{8} \\ 0, & \frac{3\pi}{8} \leq |\omega| \leq \pi \end{cases}$$

a) determine the impulse response.

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left[ \int_{-\frac{3\pi}{8}}^{\frac{3\pi}{8}} e^{j\omega n}\, d\omega - \int_{-\frac{\pi}{8}}^{\pi/8} e^{-j\omega n}\, d\omega \right] \leftarrow$$

$$= \frac{1}{\pi n} \left[ \sin \frac{3\pi}{8} n - \sin \frac{\pi}{8} n \right],$$ using $\cos(u)\sin(v) = \frac{1}{2}[\sin(u+v) - \sin(u-$

$$= \frac{2}{\pi n} \sin\left(\frac{\pi}{8} n\right) \cos\left(\frac{\pi}{4} n\right)$$

(b) Show that the impulse response can be expressed as the product of $\cos\left(\frac{n\pi}{4}\right)$ and the impulse response of a lowpass filter.

if $h_1(n) = \dfrac{2\sin\left(\frac{\pi}{8} n\right)}{n\pi}$

then, $H_1(\omega) = \begin{cases} 2, & |\omega| \leq \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |\omega| < \pi \end{cases}$

$$h(n) = h_1(n) \cos \frac{\pi}{4} n$$

$\boxed{\text{problem 5.31}}$ A causal LTI system excited by the input

$$x(n) = \left(\frac{1}{4}\right)^n u(n) + u(-n-1)$$

produces an output $y(n)$ with z-transform

$$Y(z) = \frac{-\frac{3}{4} z^{-1}}{(1-\frac{1}{4}z^{-1})(1+z^{-1})}$$

(a) determine the system function $H(z)$ and its ROC

$$x(n) = \left(\frac{1}{4}\right)^n u(n) + u(-n-1)$$

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{-1}{1-z^{-1}}, \qquad \text{ROC: } \frac{1}{4} < |z| <$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$X(z) = \frac{(1-z^{-1}) - (1-\frac{1}{4}z^{-1})}{(1-\frac{1}{4}z^{-1})(1-z^{-1})} = \frac{\cancel{1-1} - \frac{3}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-z^{-1})}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\cancel{3/4\, z^{-1}}}{\cancel{(1-\frac{1}{4}z^{-1})}(1+z^{-1})} \times \frac{\cancel{(1-\frac{1}{4}z^{-1})}(1-z^{-1})}{\cancel{-3/4\, z^{-1}}}$$

$$H(z) = \frac{1-z^{-1}}{1+z^{-1}}, \qquad \text{ROC: } |z| < 1$$

(b) Determine the output $y(n)$ of the system (Hint: Pole cancellation increases the original ROC)

$$Y(z) = \frac{-\frac{3}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1+z^{-1})} = \frac{-\frac{3}{5}}{(1-\frac{1}{4}z^{-1})} + \frac{\frac{3}{5}}{(1+z^{-1})}$$

$$\Rightarrow y(n) = -\frac{3}{5}\left(\frac{1}{4}\right)^n u(n) - \frac{3}{5}(-1)^n u(-n-1)$$

problem 5.65 Consider the system $y(n) = x(n) - 0.95\, x(n-6)$

(a) sketch its pole-zero pattern.

(b) sketch its magnitude response using the pole-zero plot.

(c) Determine the system function of its causal inverse system.

(d) Sketch the magnitude response of the inverse system using the pole-zero plot.

(a) $y(n) = x(n) - 0.95\, x(n-6)$

$Y(z) = X(z)\,(1 - 0.95\, z^{-6})$

$\Rightarrow H(z) = 1 - 0.95\, z^{-6} = \dfrac{z^6 - 0.95}{z^6}$

Zeros: $z^6 - 0.95 = 0 \Rightarrow z = (0.95)^{1/6}\, e^{j\frac{2\pi k}{6}}, \; k = 0, 1, \ldots, 5$

poles: $6^{th}$ order pole @ $z = 0$.

(b) sketch its mag.

$\frac{0.5}{3} = \frac{\pi}{3}$

(c) $H_{inv.}(z) = \dfrac{z^6}{z^6 - 0.95}$ , $r = (0.95)^{1/6}$ .

(d)

$$\angle H(\omega) = -\tan^{-1} \frac{\frac{1}{2}\sin\omega}{1-\frac{1}{2}\cos\omega} = \theta(\omega)$$

(b) 1. $x(n) = \cos\frac{3\pi n}{10}, \quad -\infty < n < \infty$

$$= \frac{1}{2}\left[e^{+j\frac{3\pi n}{10}} + e^{-j\frac{3\pi n}{10}}\right]$$

$$\bar{X}(\omega) = \pi\left[\delta\left(\omega - \frac{3\pi}{10}\right) + \delta\left(\omega + \frac{3\pi}{10}\right)\right], \quad |\omega| \le \pi$$

$$Y(\omega) = H(\omega)\,\bar{X}(\omega)$$

$$= H\left(\frac{3\pi}{10}\right)\pi\left[\delta\left(\omega - \frac{3\pi}{10}\right) + \delta\left(\omega + \frac{3\pi}{10}\right)\right]$$

$$y(n) = \left|H\left(\frac{3\pi}{10}\right)\right| \cos\left[\frac{3\pi n}{10} + \theta\left(\frac{3\pi}{10}\right)\right]$$

2. $x(n) = \{\ldots, 1, 0, 0, \underset{\uparrow}{1}, 1, 1, 0, 1, 1, 1, 0, 1, \ldots\}$

Find $x_e(n) = \frac{x(n) + x(-n)}{2} = \{\ldots, \frac{1}{2}, \frac{1}{2}, \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\}$

$x_o(n) = \frac{x(n) - x(-n)}{2} = \{\ldots, \frac{-1}{2}, \frac{-1}{2}, \underset{\uparrow}{0}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \ldots\}$

$$\Rightarrow \bar{X}_R(\omega) = \sum_n x_e(n) e^{-j\omega n}$$

$$\bar{X}_I(\omega) = \sum_n x_o(n) e^{-j\omega n}$$

$$|H(\omega)| = \left[\bar{X}_R^2(\omega) + \bar{X}_I^2(\omega)\right], \quad \theta(\omega) = \tan^{-1}\frac{\bar{X}_I(\omega)}{\bar{X}_R(\omega)}$$

$$\bar{Y}(\omega) = H(\omega) \cdot \bar{X}(\omega).$$