problems set for CHAPTER 7 & CHAPTER 8

7. {1, 3, 7, 13(a), 23(a,b,c,h), 25} 22

8. {1, 3, 4, 8, 11, 25}.

problem 7.1 The first five points of the eight-point DFT of a real valued sequence are $\{0.25,\ 0.125 - j0.3018;\ 0,\ 0.125 - j0.0518,\ 0\}$

Determine the remaining three points.

Real sequence implies these symmetries.

- real part of DFT is even-symmetric
- imaginary part of DFT is odd-symmetric

remaining points: $\{0.125 + j0.0518,\ 0,\ 0.125 + j0.3018\}$

problem 7.3 let $X(k)$, $0 \le k \le N-1$, be the N-point DFT of the sequence $x(n)$, $0 \le n \le N-1$, we define

$$\hat{X}(k) = \begin{cases} X(k), & 0 \le k \le k_c,\ N-k_c \le k \le N-1 \\ 0, & k_c < k < N-k_c \end{cases}$$

and we compute the inverse N-point DFT of $\hat{X}(k)$, $0 \le k \le N-1$. what is the effect of this process on the sequence $x(n)$? explain:

consider $\hat{X}(k)$ as the output/result of the DFT sequence $X(k)$

with $H(k) = \begin{cases} 1, & 0 \le k \le k_c,\ N-k_c \le k \le N-1 \\ 0, & k_c < k < N-k_c \end{cases}$

~~Em~~ $H(k)$ acting as an ideal filter removing frequency components between $(k_c+1)\frac{2\pi}{N}$ to $(N-k_c-1)\frac{2\pi}{N}$.

$\hat{x}(n)$ is a filtered version of $x(n)$

$\boxed{\text{problem } 7.7}$ if $X(k)$ is the DFT of the sequence $x(n)$, determine the N-point DFTs of the sequences:

$$x_c(n) = x(n) \cos \frac{2\pi k_0 n}{N}, \quad 0 \le n \le N-1$$

and

$$x_s(n) = x(n) \sin \frac{2\pi k_0 n}{N}, \quad 0 \le n \le N-1$$

in terms of $X(k)$.

$$X_c(k) = \sum_{n=0}^{N-1} x(n) \cdot \frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}} \right] e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi (k-k_0)n}{N}} + \frac{1}{2} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi (k+k_0)n}{N}}$$

$$= \frac{1}{2} X(k-k_0)_{\text{mod } N} + \frac{1}{2} X(k+k_0)_{\text{mod } N}$$

similarly, $$X_s(k) = \sum_{n=0}^{N-1} x(n) \cdot \frac{1}{2j}\left[e^{j\frac{2\pi k_0 n}{N}} - e^{-j\frac{2\pi k_0 n}{N}} \right] e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{2j} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi (k-k_0)n}{N}} - \frac{1}{2j} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi (k+k_0)n}{N}}$$

$$= \frac{1}{2j} X(k-k_0)_{\text{mod } N} - \frac{1}{2j} X(k+k_0)_{\text{mod } N}$$

$\boxed{\text{problem } 7.13(a)}$ let $x_p(n)$ be a periodic sequence with fundamental period N. Consider the following DFTs:

$$x_p(n) \xleftrightarrow[N]{DFT} X_1(k)$$

$$x_p(n) \xleftrightarrow[3N]{DFT} X_3(k)$$

(a) What is the relationship between $X_1(k)$ and $X_3(k)$?

$$X_1(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$X_3(k) = \sum_{n=0}^{3N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi kn}{3N}} + \sum_{n=2N}^{3N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}} + \sum_{n=0}^{N-1} x(n-N)\, e^{-j\frac{2\pi k(n-N)}{3N}} + \sum_{n=0}^{N-1} x(n-2N)\, e^{-j\frac{2\pi k(n-2N)}{3N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}}\, e^{+j\frac{2\pi kN}{3N}} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}\, e^{+j\frac{2\pi k2N}{3N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{3N}} \left[1 + e^{+j\frac{2\pi k(-N)}{3}} + e^{+j\frac{4\pi k(-N)}{3}} \right]$$

$$= X(k) \left[1 + e^{+j\frac{2\pi k}{3}} + e^{-j\frac{4\pi k}{3}} \right]$$

if we use $W_N^{kn} = e^{-j\frac{2\pi kn}{N}}$

$$X_3(k) = X(k) \left[1 + W_3^{k} + W_3^{2k} \right]$$

$\square$

$\boxed{\text{problem } 7.23}$ a,b,c,h

Compute the N-point DFT of the signals:

a) $x(n) = \delta(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} = 1 \quad , \quad 0 \le k \le N-1$$

b) $x(n) = \delta(n-n_0) \quad , \quad 0 < n_0 < N$

$$X(k) = \sum_{n=0}^{N-1} \delta(n-n_0)\, e^{-j\frac{2\pi kn}{N}} = e^{-j\frac{2\pi k n_0}{N}} \quad , \quad 0 \le k \le N-1$$

c) $x(n) = a^n \quad , \quad 0 \le n \le N-1$

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} \left(a e^{-j\frac{2\pi k}{N}}\right)^n$$

$$= \frac{1 - \left(a e^{-j\frac{2\pi k}{N}}\right)^N}{1 - a e^{-j\frac{2\pi k}{N}}} = \frac{1-a}{1 - a e^{-j\frac{2\pi k}{N}}}$$

h) $x(n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \quad , \quad 0 \le n \le N-1$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \quad , \text{ assume } N \text{ is odd.}$$

$$= 1 + e^{-j\frac{2\pi}{N}2k} + e^{-j\frac{2\pi}{N}4k} + e^{-j\frac{2\pi}{N}6k} + \cdots + e^{-j\frac{2\pi}{N}(N-1)k}$$

$$= \frac{1 - \left(e^{-j\frac{2\pi}{N}2k}\right)^{\frac{N+1}{2}}}{1 - e^{-j\frac{4\pi}{N}k}} = \frac{1 - e^{-j\frac{2\pi}{N}k}}{1 - e^{-j\frac{4\pi k}{N}}}$$

problem 7.25

(a) Determine the Fourier Transform $X(\omega)$ of the signal

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1, 0\}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= e^{j2\omega} + 2\underbrace{e^{j\omega} + 3 + 2e^{-j\omega}} + e^{-j2\omega}$$

$$= 3 + 4\cos(\omega) + 2\cos(2\omega)$$

(b) Compute the six-point DFT $V(k)$ of the signal.

$$v(n) = \{3, 2, 1, 0, 1, 2\}$$

$$V(k) = \sum_{n=0}^{5} v(n) e^{-j\frac{2\pi k n}{6}}$$

$$= 3 + 2e^{-j\frac{2\pi k}{6}} + e^{-j\frac{2\pi 2k}{6}} + 0 + e^{-j\frac{2\pi}{6}\cdot 4k} + 2e^{-j\frac{2\pi 5k}{6}}$$

$+6$ $+6$

$$= 3 + 2e^{-j\frac{2\pi k}{6}} + \underbrace{e^{-j\frac{2\pi}{6}2k} + e^{+j\frac{2\pi}{6}2k} + 2e^{j\frac{2\pi}{6}k}}$$

$$= 3 + 4\cos\left(\frac{2\pi k}{6}\right) + 2\cos\left(\frac{2\pi}{3}k\right)$$

c) Is there any relation between $X(\omega)$ and $V(k)$? explain.

$$V(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{6} = \frac{\pi k}{3}}$$

$v(n)$ is one period of a periodic sequence of repeating $x(n)$.

problem 7.28 Frequency domain sampling, consider the following discrete-time signal

$$x(n) = \begin{cases} a^{|n|}, & |n| \leq L \\ 0, & |n| > L \end{cases}$$

where: $a = 0.95$ & $L = 10$

a) Compute and plot the signal $x(n)$.

(b) Show that $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = x(0) + 2 \sum_{n=1}^{L} x(n) \cos \omega n$.

and plot $X(\omega)$ by computing it at $\omega = \frac{\pi k}{100}$, $k = 0, 1, \ldots, 100$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = x(0) + \sum_{-L}^{1} a^{-n} e^{-j\omega n} \cancel{e^{-j\omega n}} + \sum_{1}^{L} a^{n} e^{-j\omega n}$$

$$= a + \sum_{1}^{L} a^{n} e^{j\omega n} + \sum_{1}^{L} a^{n} e^{-j\omega n}$$

$$= a + \sum_{1}^{L} a^{n} \left(e^{j\omega n} + e^{-j\omega n} \right)$$

$$= a + 2 \sum_{1}^{L} a^{n} \cos(\omega n)$$

$$= a + 2 \sum_{1}^{L} x(n) \cos(\omega n).$$

problem 8.1

Show that each of the numbers $e^{j(\frac{2\pi}{N})k}$, $0 \leq k \leq N-1$ corresponds to an Nth root of unity. Plot these numbers as phasors in the complex plane and illustrate, by means of this figure, the orthogonality property.

$$\sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})kn} e^{-j(\frac{2\pi}{N})ln} = \begin{cases} N, & \text{if } \mathrm{mod}(k-l/N)=0 \\ 0, & \text{o/w} \end{cases}$$

let $\left[e^{j(\frac{2\pi}{N})k}\right]^N = e^{j2\pi k} = 1 \Rightarrow X^N = 1$ i.e $e^{j\frac{2\pi k}{N}}$ is the Nth root of unity.

$N=8$

⊛ now consider:

$$\sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})kn} e^{-j(\frac{2\pi}{N})ln}$$

→ if $k \neq l$, the sum represents the N equally spaced roots on the unit circle, the roots sum to zero.

→ if $k=l$, sum $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(\cancel{k-l})n} = N$

problem 8.3

let $x(n)$ be a real-valued N-point ($N = 2^{\nu}$) sequence. Develop a method to compute an N-point DFT $X'(k)$, which contains only the odd harmonics [i.e. $X'(k) = 0$ if k is even] by using only a real N/2-point DFT.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad , \quad 0 \le k \le N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn} \quad ,$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{r=0}^{\frac{N}{2}-1} x(r+\tfrac{N}{2}) W_N^{(r+\frac{N}{2})k} \quad ,$$

So to keep odd harmonics.

let $X'(k') = X(2k+1)$, $0 \le k' \le \frac{N}{2}-1$

$$\Rightarrow X'(k') = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{(2k'+1)n} + \sum_{r=0}^{\frac{N}{2}-1} x(r+\tfrac{N}{2}) W_N^{(r+\frac{N}{2})(2k'+1)}$$

knowing $W_N^{2k'n} = W_{\frac{N}{2}}^{k'n}$, $W_N^N = 1$

$$\Rightarrow X'(k') = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) W_N^{n} W_{\frac{N}{2}}^{k'n} + x(n+\tfrac{N}{2}) W_{\frac{N}{2}}^{k'n} W_N^{n} \underbrace{W_N^{\frac{N}{2}}}_{} \right]$$

$$W_N^{N/2} = e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}} = -1$$

$$\boxed{X'(k') = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x(n+\tfrac{N}{2}) \right] W_N^{n} W_{\frac{N}{2}}^{k'n}}$$

problem 8.4 A designer has available a number of eight-po FFT chips. Show explicitly how he would interconnect three such chips in order to compute the 24-point DFT

$$X(k) = \sum_{n=0,3,6,\ldots}^{21} x(n)\, W_N^{kn} + \sum_{n=1,4,7,\ldots}^{22} x(n)\, W_N^{kn} + \sum_{n=2,5,8,\ldots}^{23} x(n)\, W_N^{kn}$$

$$= \sum_{r=0}^{7} x(3r)\, W_{\frac{N}{3}}^{kr} + \sum_{r=0}^{7} x(3r+1)\, W_N^{(3r+1)k} + \sum_{r=0}^{7} x(3r+2)\, W_N^{(3r+2)k}$$

$$= \sum_{r=0}^{7} x(3r)\, W_{\frac{N}{3}}^{kr} + \sum_{r=0}^{7} x(3r+1)\, W_{\frac{N}{3}}^{kr}\, W_N^{k} + \sum_{r=0}^{7} x(3r+2)\, W_{\frac{N}{3}}^{kr}\, W_N^{2k}$$

$$\equiv F_1(k) + F_2(k)\, W_N^{k} + F_3(k)\, W_N^{2k}$$

$e^{-j\frac{2\pi}{4}\cdot\frac{N}{2}}$

$e^{-j\pi}$

$\boxed{\text{problem } 8.8}$ Compute the eight-point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & \text{ow} \end{cases}$$

by using the decimation-in-frequency FFT algorithm described in text.

following fig. 8.1.11

1st stage

1st stage → $\{2, 2, 2, 2, 0, 0, 0, 0\}$.

2nd stage → $\{4, 4, 0, 0, 0, 0, 0, 0\}$

3rd stage → $\{8, 0, 0, 0, 0, 0, 0, 0\}$

reordering → remains $\{8, 0, 0, 0, 0, 0, 0, 0\}$.

$\boxed{\text{problem 8.81}}$ Compute the 8-point DFT of the sequence

$$x(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0 \right\}$$

using the in-place ~~2~~ radix-2 decimation-in-time & radix-2 decimation in-frequency algorithms. Follow exactly the corresponding signal flow graphs and keep track of all the intermediate quantities by putting them on the diagrams.

Figure 8.1.6 Decimation-in-time FFT

$\rightarrow$ 1st stage output $\rightarrow$ $\left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right\}$

$\rightarrow$ 2nd stage output $\rightarrow$ $\left\{ \underbrace{\left(\frac{1}{2} + \frac{1}{2} \cdot W_8^0\right)}_{1}, \frac{1}{2}(1 + W_8^2), \underbrace{\frac{1}{2} - \frac{1}{2} W_8^0}_{0}, \frac{1}{2}(1 - W_8^2), 1, \frac{1}{2}(1 + W_8^2), 0, \frac{1}{2}(1 - W_8^2) \right\}$

$\rightarrow$ 3rd stage output $\rightarrow$ $\left\{ 2, 0, 0, 0, \frac{1}{2}(1 + W_8^1 + W_8^2 + W_8^3), \frac{1}{2}(1 - W_8^1 + W_8^2 - W_8^3), \frac{1}{2}(1 - W_8^2 + W_8^3 - W_8^5), \frac{1}{2}(1 - W_8^2 - W_8^3 + W_8^5) \right\}$

following Figure 8.1.11 for Decimation-in-frequency FFT

$\rightarrow$ 1st stage: $\left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} W_8^1, \frac{1}{2} W_8^2, \frac{1}{2} W_8^3 \right\}$

$\rightarrow$ 2nd stage: $\left\{ 1, 1, 0, 0, \frac{1}{2} + \frac{1}{2} W_8^2, \frac{1}{2} W_8^1 + \frac{1}{2} W_8^3, \frac{1}{2} + \frac{1}{2} W_8^2, \frac{1}{2} W_8^3 - \frac{1}{2} W_8^5 \right\}$

$\rightarrow$ 3rd stage: $\left\{ 2, 0, 0, 0, \frac{1}{2}(1 + W_8^1 + W_8^2 + W_8^3), \frac{1}{2}(1 - W_8^1 + W_8^2 - W_8^3), \frac{1}{2}(1 - W_8^2 + W_8^3 - W_8^5), \frac{1}{2}(1 - W_8^2 - W_8^3 + W_8^5) \right\}$

problem 8.25 Develop a radix-3 decimation-in-time FFT algorithm for $N = 3^{\nu}$, and draw the corresponding flow graph for $N=9$. What is the number of required complex multiplications? can the operations be performed in place?

$$X(k) = \sum_{n=0}^{8} x(n) W_9^{nk} = \sum_{n=0,3,6} x(n) W_9^{nk} + \sum_{n=1,4,7} x(n) W_9^{nk} + \sum_{n=2,5,8} x(n) W_9^{nk}$$

$$= \sum_{m=0}^{\frac{N}{3}-1=2} x(3m) W_9^{(3m)k} + \sum_{m=0}^{2} x(3m+1) W_9^{(3m+1)k} + \sum_{m=0}^{2} x(3m+2) W_9^{(3m+2)k}$$

$$= \sum_{m=0}^{2} x(3m) W_{\frac{9}{3}}^{mk} + W_9^{k} \sum_{m=0}^{2} x(3m+1) W_3^{km} + W_9^{2k} \sum_{m=0}^{2} x(3m+2) W_3^{mk}$$

$$= F_1(k) + W_9^{k} F_2(k) + W_9^{2k} F_3(k), \quad 0 \le k \le N-1$$

(*) exploiting the periodicity of $F_1(k), F_2(k), F_3(k)$, for $0 \le k \le \frac{N}{3}-1$

$$X(k) = F_1(k) + W_9^{k} F_2(k) + W_9^{2k} F_3(k)$$

$$X(k+\tfrac{N}{3}) = F_1(k) + W_3^{1} W_9^{k} F_2(k) + W_3^{2} W_9^{2k} F_3(k)$$

$$X(k+\tfrac{2N}{3}) = F_1(k) + W_3^{2} W_9^{k} F_2(k) + W_3^{4} W_9^{2k} F_3(k)$$

(*) we have $\log_3 N = 2$ stages.

# of complex multiplications:

in each stage $\Rightarrow$ 6 complex multiplication by twiddle factors + 12 multiplications inside the butterfly's.

$\Rightarrow$ total # of complex multiplications $= 36$.

there are 8 - $W^0 = 1 \Rightarrow$ # of multiplications $= 28$

x(0)
x(3)
x(6)
x(1)
x(4)
x(7)
x(2)
x(5)
x(8)
W_3^0
W_3^1
W_3^2
W_3^4
W_9^0
W_9^1
W_9^2
W_9^0
W_9^2
W_9^4
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)
X(8)