http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

HOMEWORK #3 - SOLUTIONS

4.13

Recall that the Fourier transform $X(\omega)$ of a signal $x(n)$ is given by:

$$X(\omega) = F[x(n)] = \sum_{n=-\infty}^{\infty} x(n) \cdot exp(-j\omega n)$$

Therefore from example 4.4.2, the Fourier transform of the signal $x(n) = \begin{cases} 1 & -M \le n \le M \\ 0 & otherwise \end{cases}$ is

$X(\omega) = 1 + 2 \cdot \sum_{n=1}^{M} \cos(\omega n)$.

The Fourier transform of the signal $x_1(n) = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}$ is given by:

$$X_1(\omega) = F[x_1(n)] = \sum_{n=-\infty}^{\infty} x_1(n) \cdot exp(-j\omega n) = \sum_{n=0}^{M} exp(-j\omega n) = \sum_{n=0}^{M} [\exp(-j\omega)]^n$$

$$i.e., \qquad X_1(\omega) = \frac{1 - \exp(-j\omega(M+1))}{1 - \exp(-j\omega)}$$

The Fourier transform of the signal $x_2(n) = \begin{cases} 1 & -M \le n \le -1 \\ 0 & otherwise \end{cases}$ is given by:

$$X_2(\omega) = F[x_2(n)] = \sum_{n=-\infty}^{\infty} x_2(n) \cdot exp(-j\omega n) = \sum_{n=-M}^{-1} exp(-j\omega n) = \sum_{n=-M}^{-1} [\exp(-j\omega)]^n = \sum_{n=1}^{M} [\exp(j\omega)]^n$$

$$i.e., \qquad X_2(\omega) = \frac{1 - \exp(j\omega M)}{1 - \exp(j\omega)} \cdot \exp(j\omega)$$

Adding the two Fourier transforms $X_1(\omega)$ and $X_2(\omega)$ we get:

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{[1 - \exp(-j\omega(M+1))] \cdot [1 - \exp(j\omega)] + [\exp(j\omega) - \exp(j\omega(M+1))][1 - \exp(-j\omega)]}{[1 - \exp(-j\omega)] \cdot [1 - \exp(j\omega)]}$$

$$= \frac{[\exp(j\omega M) + \exp(-j\omega M)] - [\exp(j\omega(M+1)) + \exp(-j\omega(M+1))]}{2 - [\exp(j\omega) + \exp(-j\omega)]}$$

$$= \frac{2 \cdot \cos(\omega M) - 2 \cdot \cos(\omega(M+1))}{2 - 2 \cdot \cos(\omega)}$$

Or equivalently:

$$X(\omega)=\frac{\sin\left(\left(M+\frac{1}{2}\right)\cdot\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

4.16

The signal that we have to consider is $a^n u(n)$ with Fourier transform $\frac{1}{1-a\cdot\exp(-j\omega)}$, $|a|<1$.

The differentiation in frequency theorem states that if $x(n)\overset{F}{\leftrightarrow}X(\omega)$, then $n\cdot x(n)\overset{F}{\leftrightarrow}j\frac{dX(\omega)}{d\omega}$.

We want to prove that $\frac{(n+l-1)!}{n!(l-1)!}a^n u(n)\overset{F}{\leftrightarrow}\frac{1}{(1-a\cdot\exp(-j\omega))^l}$

Let's process by induction by first proving that the property is true for *l*=1:

$$\left.\frac{(n+l-1)!}{n!(l-1)!}a^n u(n)\right|_{l=0}=\frac{n!}{n!}a^n u(n)\overset{F}{\leftrightarrow}\frac{1}{(1-a\cdot\exp(-j\omega))}$$

Then suppose that the following property holds:

$$x_k(n)=\frac{(n+k-1)!}{n!(k-1)!}a^n u(n)\overset{F}{\leftrightarrow}\frac{1}{(1-a\cdot\exp(-j\omega))^k}=X_k(\omega)$$

And finally, let's compute $x_{k+1}(n)$:

$$x_{k+1}(n)=\frac{(n+(k+1)-1)!}{n!((k+1)-1)!}a^n u(n)=\frac{(n+k)!}{n!k!}a^n u(n)=\frac{n+k}{k}x_k(n)=x_k(n)+\frac{n}{k}x_k(n)$$

This result and the differentiation in frequency theorem lead us to the following:

$$x_k(n)+\frac{n}{k}x_k(n)\overset{F}{\leftrightarrow}X_k(\omega)+\frac{1}{k}\cdot\left(j\frac{dX_k(\omega)}{d\omega}\right)=\frac{1}{(1-a\cdot\exp(-j\omega))^k}+\frac{a\cdot\exp(-j\omega)}{(1-a\cdot\exp(-j\omega))^{k+1}}$$
$$=\frac{(1-a\cdot\exp(-j\omega))+a\cdot\exp(-j\omega)}{(1-a\cdot\exp(-j\omega))^{k+1}}$$
$$=\frac{1}{(1-a\cdot\exp(-j\omega))^{k+1}}=X_{k+1}(\omega)$$

The property was proven to be true for l=1 then was supposed true for l=k and was proven to be true for l=k+1, we can then infer that it is true for all l.

$$\frac{(n+l-1)!}{n!(l-1)!}a^n u(n)\overset{F}{\leftrightarrow}\frac{1}{(1-a\cdot\exp(-j\omega))^l}$$

4.17

Recall the Fourier transform, $X(\omega)$, of a signal $x(n)$ is given by $X(\omega)=F[x(n)]=\sum_{n=-\infty}^{\infty}x(n)\cdot exp(-j\omega n)$

(a) The signal to consider is $x^*(n)$.

$$F[x^*(n)]=\sum_{n=-\infty}^{\infty}x^*(n)\cdot\exp(-j\omega n)=\left(\sum_{n=-\infty}^{\infty}x(n)\cdot\exp(-j(-\omega)n)\right)^*=X^*(-\omega)$$

$$\Rightarrow x^*(n)\overset{F}{\leftrightarrow}X^*(-\omega)$$

(b) The signal to consider is $x^*(-n)$.

$$F[x^*(-n)]=\sum_{n=-\infty}^{\infty}x^*(-n)\cdot\exp(-j\omega n)=\left(\sum_{n=-\infty}^{\infty}x(-n)\cdot\exp(-j(-\omega)n)\right)^*$$
$$=\left(\sum_{n'=-\infty}^{\infty}x(n')\cdot\exp(-j(-\omega)(-n'))\right)^*\qquad(with\quad n'=-n)$$
$$=\left(\sum_{n'=-\infty}^{\infty}x(n')\cdot\exp(-j\omega n')\right)^*=X^*(\omega)$$

$$\Rightarrow x^*(-n)\overset{F}{\leftrightarrow}X^*(\omega)$$

(c) The signal to consider is $y(n) = x(n) - x(n-1)$.

$$F[y(n)] = \sum_{n=-\infty}^{\infty} y(n) \cdot \exp(-j\omega n) = \sum_{n=-\infty}^{\infty} [x(n) - x(n-1)] \cdot \exp(-j\omega n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot \exp(-j\omega n) - \sum_{n=-\infty}^{\infty} x(n-1) \cdot \exp(-j\omega n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot \exp(-j\omega n) - \sum_{n=-\infty}^{\infty} x(n) \cdot \exp(-j\omega(n+1))$$

$$= X(\omega) - X(\omega) \cdot \exp(-j\omega)$$

$$= X(\omega) \cdot (1 - \exp(-j\omega)) = Y(\omega)$$

$$\Rightarrow Y(\omega) = X(\omega) \cdot (1 - \exp(-j\omega))$$

(d) We have $y(n) = \sum_{k=-\infty}^{n} x(k)$ and therefore $x(n) = y(n) - y(n-1)$ which means that from (c) $F[x(n)] = X(\omega) = Y(\omega) \cdot (1 - \exp(-j\omega))$ or equivalently:

$$F[y(n)] = Y(\omega) = \frac{X(\omega)}{1 - \exp(-j\omega)} \quad .$$

(e) The signal to consider is $y(n) = x(2n)$.

$$F[y(n)] = \sum_{n=-\infty}^{\infty} y(n) \cdot \exp(-j\omega n) = \sum_{n=-\infty}^{\infty} [x(2n)] \cdot \exp(-j\omega n)$$

$$= \sum_{n'=-\infty}^{\infty} x(n') \cdot \exp\left(-j\omega\left(\frac{n'}{2}\right)\right) \qquad with \quad n' = 2n$$

$$= \sum_{n'=-\infty}^{\infty} x(n') \cdot \exp\left(-j\left(\frac{\omega}{2}\right)n'\right)$$

$$= X\left(\frac{\omega}{2}\right)$$

$$\Rightarrow Y(\omega) = X\left(\frac{\omega}{2}\right)$$

(f) The signal to consider is $y(n) = x\left(\frac{n}{2}\right)$.

$$F[y(n)] = \sum_{n=-\infty}^{\infty} y(n) \cdot \exp(-j\omega n) = \sum_{n=-\infty,n\ even}^{\infty} \left[x\left(\frac{n}{2}\right)\right] \cdot \exp(-j\omega n)$$

$$= \sum_{n'=-\infty}^{\infty} x(n') \cdot \exp(-j\omega(2n')) \qquad with \quad n' = \frac{n}{2}$$

$$= \sum_{n'=-\infty}^{\infty} x(n') \cdot \exp(-j(2\omega)n')$$

$$= X(2\omega)$$

$$\Rightarrow Y(\omega) = X(2\omega)$$

4.18

(a) The signal to consider is $x_1(n) = \{1, \quad 1, \quad \underset{\uparrow}{1}, \quad 1, \quad 1\}$.

$$X_1(\omega) = F[x_1(n)] = \sum_{n=-\infty}^{\infty} x_1(n) \cdot exp(-j\omega n)$$

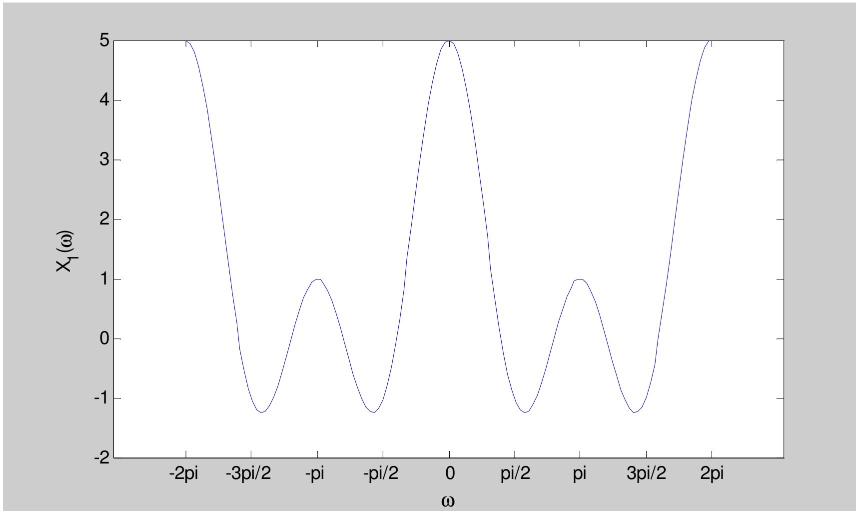
$$= \sum_{n=-2}^{2} exp(-j\omega n)$$

$$= \exp(j2\omega) + \exp(j\omega) + 1 + \exp(-j\omega) + \exp(-j2\omega)$$

$$= 1 + [\exp(j\omega) + \exp(-j\omega)] + [\exp(j2\omega) + \exp(-j2\omega)]$$

$$= 1 + 2 \cdot \cos(\omega) + 2 \cdot \cos(2\omega)$$

$$\Rightarrow X_1(\omega) = 1 + 2 \cdot \cos(\omega) + 2 \cdot \cos(2\omega)$$



(b) The signal to consider is $x_2(n) = \{1, \quad 0, \quad 1, \quad 0, \quad \underset{\uparrow}{1}, \quad 0, \quad 1, \quad 0, \quad 1\}$.

$$X_2(\omega) = F[x_2(n)] = \sum_{n=-\infty}^{\infty} x_2(n) \cdot exp(-j\omega n)$$

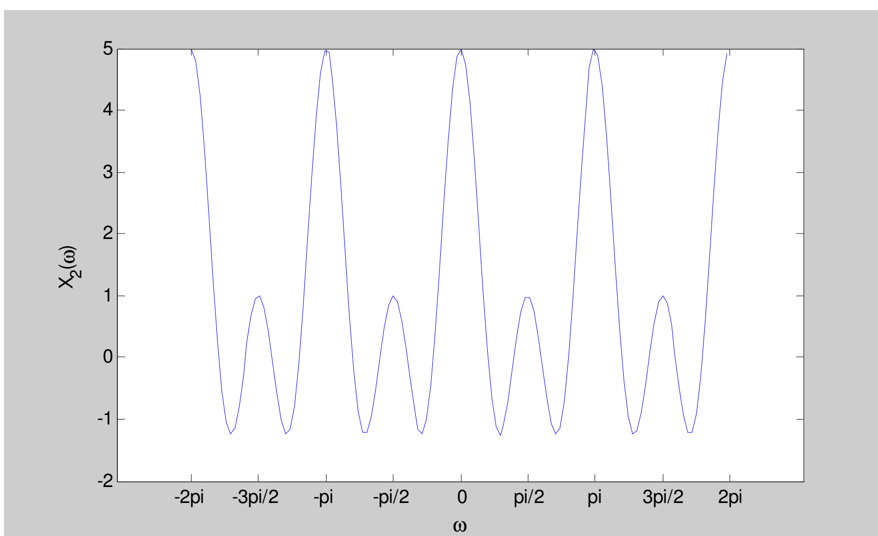
$$= \sum_{n=-4}^{4} x_2(n) \cdot exp(-j\omega n)$$

$$= \exp(j4\omega) + \exp(j2\omega) + 1 + \exp(-j2\omega) + \exp(-j4\omega)$$

$$= 1 + [\exp(j2\omega) + \exp(-j2\omega)] + [\exp(j4\omega) + \exp(-j4\omega)]$$

$$= 1 + 2 \cdot \cos(2\omega) + 2 \cdot \cos(4\omega)$$

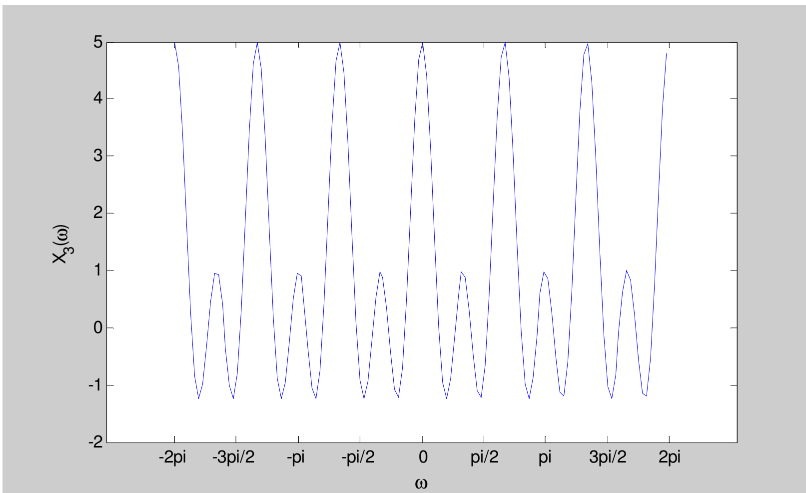
$$\Rightarrow X_2(\omega) = 1 + 2 \cdot \cos(2\omega) + 2 \cdot \cos(4\omega)$$



(c) The signal to consider is $x_3(n) = \{1,\ 0,\ 0,\ 1,\ 0,\ 0,\ \underset{\uparrow}{1},\ 0,\ 0,\ 1,\ 0,\ 0,\ 1\}$.

$$
\begin{aligned}
X_3(\omega) = F[x_3(n)] &= \sum_{n=-\infty}^{\infty} x_3(n)\cdot exp(-j\omega n) \\
&= \sum_{n=-6}^{6} x_3(n)\cdot exp(-j\omega n) \\
&= \exp(j6\omega) + \exp(j3\omega) + 1 + \exp(-j3\omega) + \exp(-j6\omega) \\
&= 1 + [\exp(j3\omega) + \exp(-j3\omega)] + [\exp(j6\omega) + \exp(-j6\omega)] \\
&= 1 + 2\cdot\cos(3\omega) + 2\cdot\cos(6\omega)
\end{aligned}
$$

$$\Rightarrow X_3(\omega) = 1 + 2\cdot\cos(3\omega) + 2\cdot\cos(6\omega)$$



(d)
From the results obtained in (a), (b) and (c), it becomes obvious that $X_1(\omega)$, $X_2(\omega)$ and $X_3(\omega)$ are related by the following equalities:

$$X_2(\omega) = X_1(2\omega) \quad \text{and} \quad X_3(\omega) = X_1(3\omega)$$

(e) Suppose $x_k(n) = \begin{cases} x\left(\dfrac{n}{k}\right) & if\ \dfrac{n}{k} \in Z \\ 0 & otherwise \end{cases}$, then we can write:

$$F[x_k(n)] = \sum_{n=-\infty}^{\infty} x_k(n)\cdot \exp(-j\omega n) = \sum_{n=-\infty,\ \frac{n}{k}\in Z}^{\infty} \left[x\left(\frac{n}{k}\right)\right]\cdot \exp(-j\omega n)$$

$$= \sum_{n'=-\infty}^{\infty} x(n')\cdot \exp(-j\omega(kn')) \quad \textit{with} \quad n = kn'$$

$$= \sum_{n'=-\infty}^{\infty} x(n')\cdot \exp(-j(k\omega)n')$$

$$= X(\omega k)$$

$$\Rightarrow X_k(\omega) = X(\omega k)$$

4.23

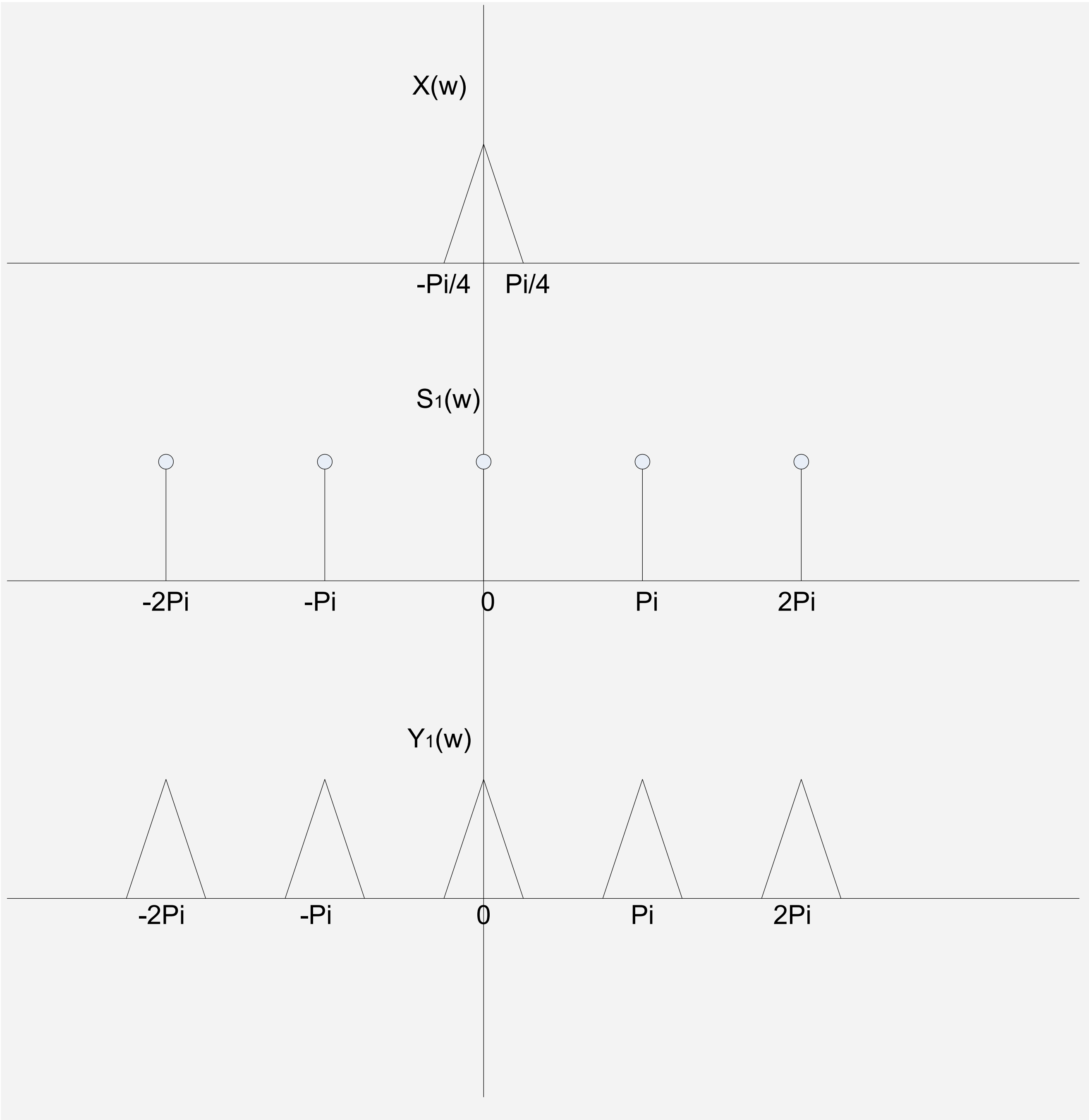
UPDATE: So this is my take (to take with some precaution maybe) on the question asked during the "review" session, hopefully it makes sense and helps understand the problem a little bit better.

From the hint at the end of the problem, let's write $y_1(n) = x(n)s_1(n)$ with $s_1(n) = \{\ldots,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ldots.\}$ which is equivalent to:

$s_1(n) = \sum_{m=-\infty}^{\infty} \delta(n - mT_S) = \sum_{m=-\infty}^{\infty} \delta(n - 2m)$ (T_S corresponds to the interval between two impulses i.e., in that case $T_S = 2$)

Taking the Fourier transform we get $Y_1(\omega) = X(\omega) * S_1(\omega)$ where $S_1(\omega) = \frac{2\pi}{T_S}\sum_{m=-\infty}^{\infty} \delta(\omega - m\cdot\omega_S)$ with $\omega_S = \frac{2\pi}{T_S}$ therefore $Y_1(\omega)$ is a repetition of the signal $X(\omega)$ with period π which means that $Y_1(\omega) \neq X(\omega)$.

Graphically it may be easier to understand so here it is:

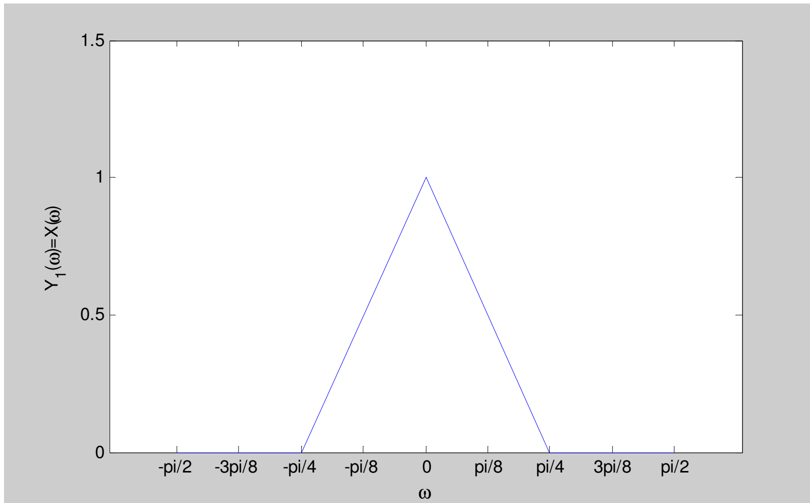


(a) We have $y_1(n) = \begin{cases} x(n) & if\ n\ \ even \\ 0 & otherwise \end{cases}$ which we can rewrite considering (b) and (c) as:

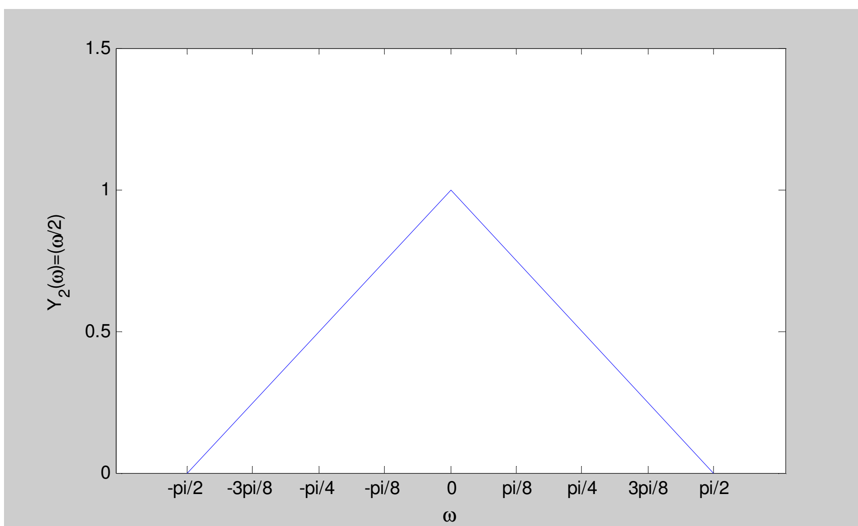
$$y_1(n) = \begin{cases} y_2\left(\frac{n}{2}\right) & if\ n\ \ even \\ 0 & otherwise \end{cases}$$

Therefore the resulting Fourier transform becomes:

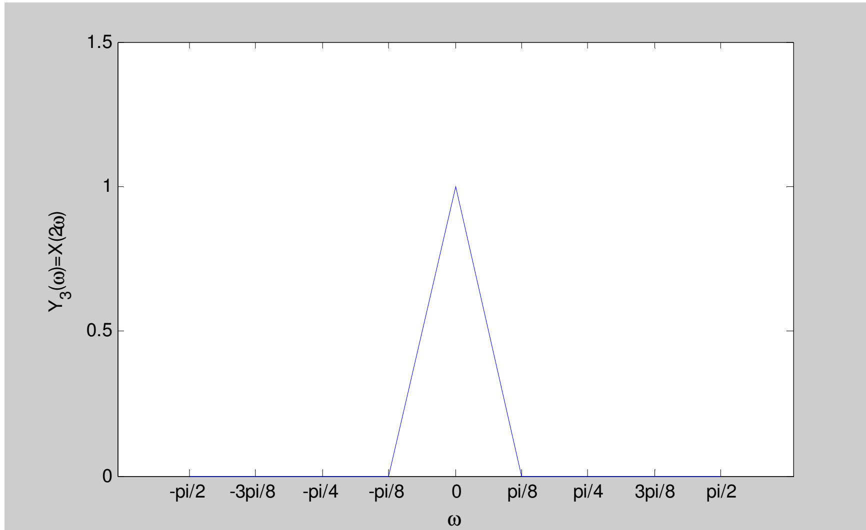
$$Y_1(\omega) = Y_2(2\omega)$$



(b) From problem 4.17 (e) we know that $Y_2(\omega) = X\left(\frac{\omega}{2}\right)$.



(c) From problem 4.17 (f) we know that $Y_3(\omega) = X(2\omega)$.


1.5
1
0.5
0
-pi/2
-3pi/8
-pi/4
-pi/8
0
pi/8
pi/4
3pi/8
pi/2
ω
Y_3(ω)=X(2ω)

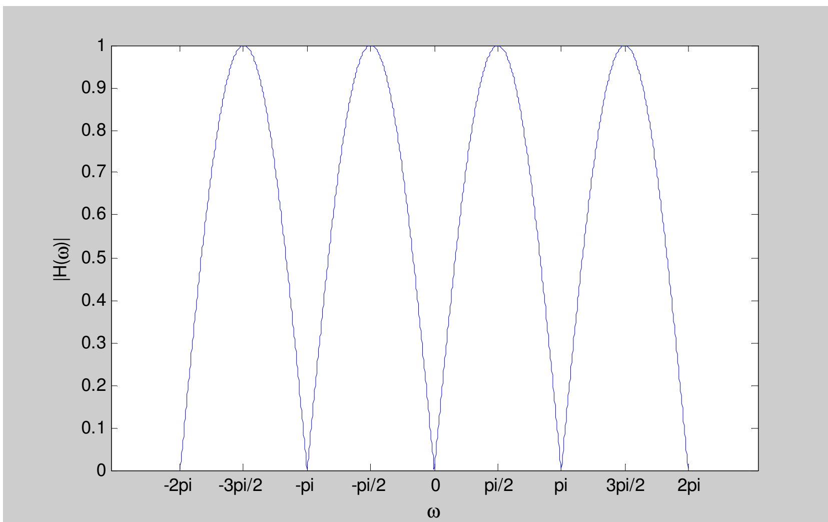
5.4

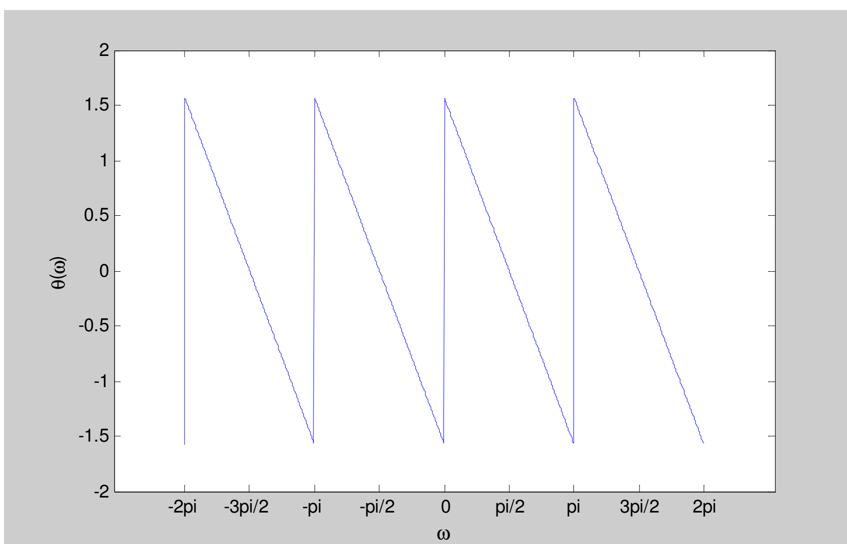
(f) The Fourier transform of the signal $y(n) = \frac{1}{2} \cdot [x(n) - x(n-2)]$ is

$Y(\omega) = \frac{1}{2} \cdot [X(\omega) - X(\omega) \cdot \exp(-2j\omega)] = \frac{1}{2} \cdot [X(\omega) \cdot (1 - \exp(-2j\omega))]$ and the corresponding filter is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2} \cdot (1 - \exp(-2j\omega)) = j \cdot \exp(-j\omega) \cdot \frac{\exp(j\omega) - \exp(-j\omega)}{2j} = \sin(\omega) \cdot \exp\left(-j\left(\omega - \frac{\pi}{2}\right)\right)$$

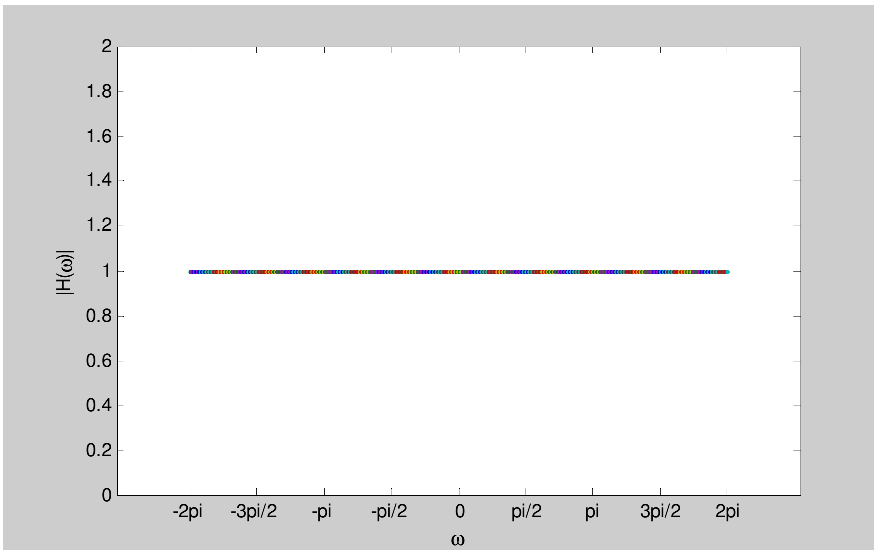
Therefore $$|H(\omega)| = |\sin(\omega)| \text{ and } \theta(\omega) = -\left(\omega - \frac{\pi}{2}\right)$$

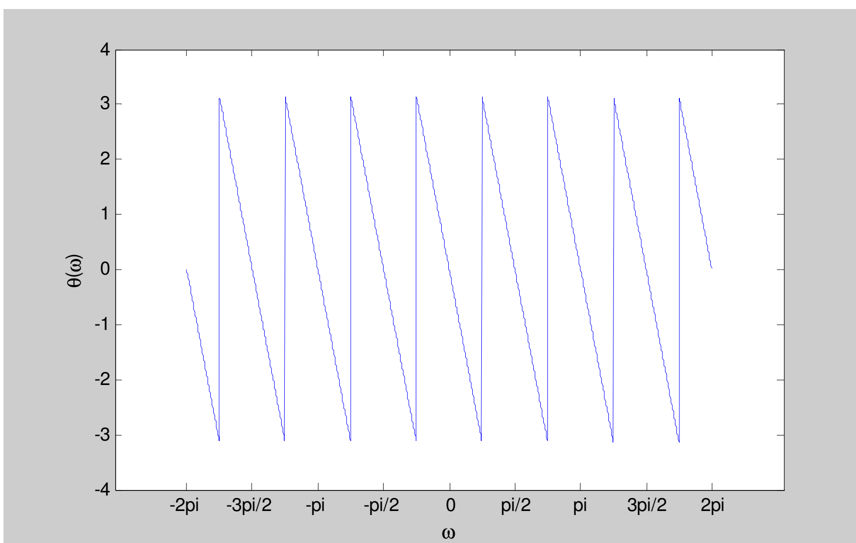




(l) The Fourier transform of the signal $y(n) = x(n-4)$ is $Y(\omega) = \exp(-j4\omega) \cdot X(\omega)$ and the corresponding filter is $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \exp(-j4\omega)$

Therefore $$|H(\omega)| = 1 \quad \text{and} \quad \theta(\omega) = -4\omega$$





5.11

The Fourier transform of the signal $y(n) = x(n) + x(n-M)$ is $Y(\omega) = X(\omega) + \exp(-jM\omega) \cdot X(\omega)$ and the corresponding filter is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + \exp(-jM\omega) = \exp\left(-j\frac{M}{2}\omega\right) \cdot \left[\exp\left(j\frac{M}{2}\omega\right) + \exp\left(-j\frac{M}{2}\omega\right)\right]$$

$$= \exp\left(-j\frac{M}{2}\omega\right) \cdot 2 \cdot \cos\left(\frac{M}{2}\omega\right)$$

Therefore $$|H(\omega)| = \left|2 \cdot \cos\left(\frac{M}{2}\omega\right)\right| \text{ and } \theta(\omega) = -\frac{M}{2}\omega$$

Because $H(\omega)$ is function of a cosine, it equals 0 when $\frac{M}{2}\omega = (2k+1)\frac{\pi}{2}$ with $k \in Z$ or

$$\omega = \frac{(2k+1)\pi}{M} \text{ with } k \in Z.$$

5.65

(a) We are given the signal $y(n) = x(n) - 0.95x(n-6)$ that has the Z-transform:

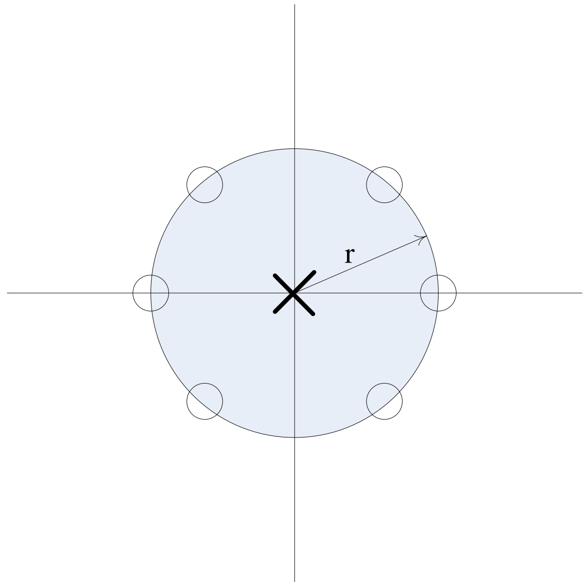
$$Z[y(n)] = Y(z) = X(Z) \cdot (1 - 0.95z^{-6}).$$

Therefore the corresponding filter becomes $H(z) = \frac{Y(z)}{X(z)} = 1 - 0.95z^{-6}$ that is equivalent to:

$$H(z) = \frac{z^6 - 0.95}{z^6}$$

There exists a zero for this filter for $z^6 = 0.95$ or $z = (0.95)^{\frac{1}{6}} \cdot \exp\left(j\frac{2\pi k}{6}\right)$, $k \in Z$

There also exists a pole for $z^6 = 0$ i.e. $z = 0$ is a 6th order pole for this filter.



$r = (0.95)^{\frac{1}{6}}$

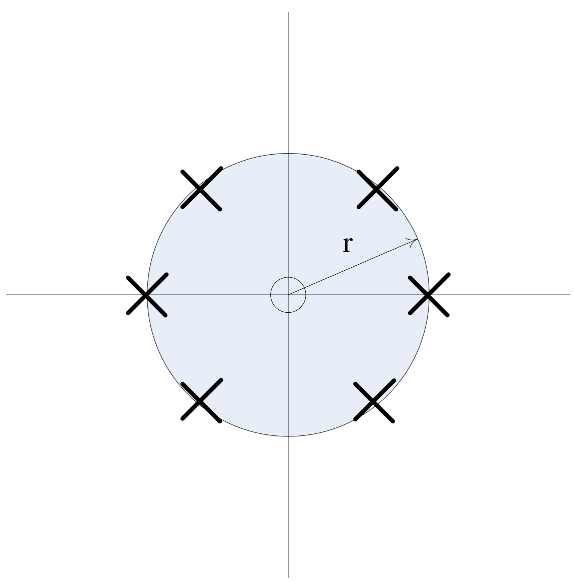
(c) The inverse system corresponds to:

$$H_{inv}(z) = \frac{z^6}{z^6 - 0.95}$$

r

This time, there exists a pole for this filter for $z^6 = 0.95$ or $z = (0.95)^{\frac{1}{6}} \cdot \exp\left(j\frac{2\pi k}{6}\right),\ k \in Z$

There also exists a zero for $z^6 = 0$ i.e. $z = 0$ is a 6th order zero for this filter.



$r = (0.95)^{\frac{1}{6}}$