University of Toronto ECE 1502S Department of Electrical
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Final Examination — April 17, 2006

Instructions: You have approximately 2 hours of "in-class" time, followed by two days of "take-home" time to complete this test. Complete as much as possible during the in-class time; your grade will be computed as a weighted average of your "in-class" grade and your "take-home" grade. (Weights to be determined later.) Answer **all** five [5] questions. All questions have equal value. Show all steps and present all results clearly. **Please start each question on a new page.** Take-home due date: **Wednesday, April 19, 2006, 5:00 p.m.**, or earlier. Please hand in to the instructor (BA4132), to his assistant, Margaret Hewer, or to the teaching assistant, Karen Su. All work is to be done independently. Consultation with others is **not** permitted. Good luck!

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1. **Short Snappers**—The parts of this question all have short answers that require a minimum of computation, though some may rely on the assigned homework. In all cases, justify your answer briefly.

   (a) True or false?: a variable-length source code that satisfies the Kraft inequality is uniquely decodable.

   (b) True or false?: if $H(X) = H(Y)$ then $H(X|Y) = H(Y|X)$.

   (c) True or false?: to achieve the capacity of a weakly symmetric discrete memoryless channel, it is **necessary** to use a channel input distribution that makes the channel output distribution uniform.

   (d) True or false?: to achieve the capacity of a weakly symmetric discrete memoryless channel, it is **necessary** to use a uniform channel input distribution.

   (e) True or false?: a continuous random variable with finite nonzero variance has finite differential entropy.

   (f) True or false: if (R_1, D_1) and (R_2, D_2) are achievable rate-distortion pairs for a given source and distortion measure, then so is $((R_1 + R_2)/2, (D_1 + D_2)/2)$.

   (g) True or false?: all rates less than the capacity of a binary symmetric channel can be achieved using binary block codes in which each codeword is constrained to have an even number of ones.

   (h) A discrete memoryless source X produces letters in the alphabet $\{a, b, c, d, e\}$ with $P_X(a) = 1/2$ and $P_X(b) = P_X(c) = P_X(d) = P_X(e) = 1/8$. Find an optimum binary prefix code for this source.

   (i) Find the capacity of the channel with channel transition matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

   (j) Find the capacity region of the noiseless binary-adder multiple-access channel, where user 1 has channel input $X_1 \in \{0, 1\}$, user 2 has channel input $X_2 \in \{0, 1\}$, and the receiver observes $Y = X_1 \oplus X_2$, the modulo-two sum of X_1 and X_2.

2. **Entropy of Functions of Random Variables**—Let X be a discrete random variable and let $Y = f(X)$ be a deterministic function of X.

(a) Show that $H(X) = H(Y) + H(X|Y)$. Under what condition is $H(X) = H(Y)$?

(b) For any positive integer n, let $\mathsf{H}(q_1, \ldots, q_n)$ denote the entropy function defined by $\mathsf{H}(q_1, \ldots, q_n) = -\sum_{i=1}^{n} q_i \log q_i$. Let X have probability mass function with values $(p_1, p_2, \ldots, p_m)$. Devise an appropriate function f to prove the grouping property of the entropy function:

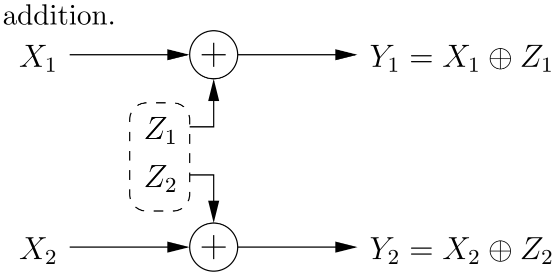
$$\mathsf{H}(p_1, p_2, \ldots, p_m) = \mathsf{H}(z_1, z_2) + z_1 \mathsf{H}(p_1/z_1, p_2/z_1, \ldots, p_r/z_1) + z_2 \mathsf{H}(p_{r+1}/z_2, p_{r+2}/z_2, \ldots, p_m/z_2)$$

where $1 \le r < m$, and

$$z_1 = \sum_{i=1}^{r} p_i, \quad z_2 = \sum_{i=r+1}^{m} p_i.$$

(c) Let $Y = X_1 + X_2$, where X_1 and X_2 are discrete random variables. Under what conditions is $H(Y) = H(X_1) + H(X_2)$? (Give an example where this condition holds when X_1 and X_2 each take on two possible values.)

3. **A Correlated Noise Channel**—Consider two binary symmetric channels with crossover probability $p \le 1/2$ operating in parallel as shown in the figure below. Here, for $i \in \{1, 2\}$, the ith channel has input X_i, noise Z_i and output $Y_i = X_i \oplus Z_i$ where $X_i, Y_i, Z_i \in \{0, 1\}$ and where $\oplus$ denotes modulo-two addition.



The noise in the two channels is not necessarily independent; indeed, the noise pair (Z_1, Z_2) has the joint distribution $P_{Z_1,Z_2}(z_1, z_2)$ given according to the following table, in which a is a parameter satisfying $0 \le a \le p$.

$P(Z_1 = z_1, Z_2 = z_2)$

	$z_2 = 0$	$z_2 = 1$
$z_1 = 0$	$1 + a - 2p$	$p - a$
$z_1 = 1$	$p - a$	a

(a) Let p be fixed. For what value of a does the channel behave as two independent binary symmetric channels in parallel? What is the corresponding channel capacity?

(b) Determine the channel capacity as a function of a and p.

(c) Consider the special case where $a = p = 1/2$. Devise a simple capacity-achieving coding scheme.

(d) Let p be fixed. For what value of a is the channel capacity minimized? For what value(s) of a is the channel capacity maximized? Comment on this result.

4. **Rate-Distortion**—Consider a memoryless source uniformly distributed on the set $\{1, 2, \ldots, m\}$. Find the rate distortion function for this source with Hamming distortion, i.e.,

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}; \\ 1 & \text{if } x \neq \hat{x}. \end{cases}$$

5. **A Periodic Jammer**—Consider a discrete-time additive white Gaussian channel subject to disturbance by a periodic Gaussian jammer, who alternates between periods of transmission (ON) and periods of non-transmission (OFF). The duty cycle of the jammer is 1/3, i.e., the jammer follows the pattern ON, OFF, OFF, ON, OFF, OFF, ... When the jammer is OFF, additive noise in the channel is zero-mean Gaussian with variance $\sigma_0^2 > 0$; when the jammer is ON, the additive noise is zero-mean Gaussian with variance $\sigma_1^2 > \sigma_0^2$. You have an average power constraint P, i.e., if $X_1, X_2, \ldots$ represents a sequence of channel inputs, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E[X_i^2] \leq P.$$

   (a) Suppose that you follow the strategy of *never* transmitting when the jammer is ON, i.e, $X_1 = X_4 = X_7 = \cdots = 0$. Determine the maximum achievable rate of reliable information transmission as a function of P and σ_0^2 in this case.

   (b) Suppose that you follow the strategy of *always* transmitting at power P, i.e., $E[X_1^2] = E[X_2^2] = E[X_3^2] = \cdots = P$. Determine the maximum achievable rate of reliable information transmission as a function of P, σ_0^2 and σ_1^2 in this case.

   (c) Determine the *optimum* transmission strategy and the corresponding maximum achievable rate of reliable information transmission as a function of P, σ_0^2 and σ_1^2. Generalize to the case where the duty cycle of the jammer is an arbitrary positive rational number.