University of Toronto
November 6, 2007

Department of Electrical
& Computer Engineering

ECE1502F — Information Theory
Midterm Test

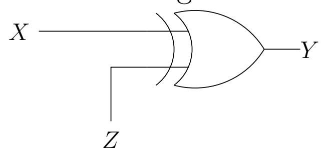
Instructions

- You have one hour and fifty minutes of "in-class" time, followed by three days of "take-home" time to complete this test.
- Complete as much as possible during the in-class time; your grade will be computed as a weighted average of your "in-class" grade and your "take-home" grade. (Weights to be determined later.)
- Answer **all** questions. The value of each question is indicated beside the question. Show all steps and present all results clearly.
- Take-home due date: Friday, November 9, 2007, 10:30 am (in class). Please return a *complete* solution for the take home portion, even if you believe that you answered the question correctly in class.
- All aids are permitted during the take-home portion, but **all work is to be done independently**. (No mutual information!) Consultation with others is **not** permitted.
- Good luck!
- Please note that the notation $X \sim \mathsf{Bern}(p)$ is used to denote a Bernoulli-p random variable X, i.e., a binary random variable X taking values in the set $\{0, 1\}$, and for which $P[X = 1] = p$.

10 marks **1.** (*Noisy Gates*)

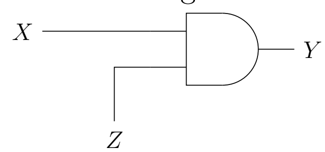
The following diagram shows three types of two-input, one-output logic gates: (a) an XOR (exclusive-OR) gate, (b) an AND gate and (c) an OR gate. The truth table (input/output relation) for each gate is also shown.

(a)
XOR gate



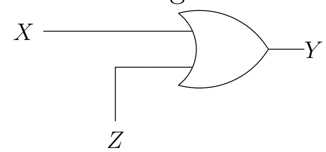
X	Z	Y
0	0	0
0	1	1
1	0	1
1	1	0

(b)
AND gate



X	Z	Y
0	0	0
0	1	0
1	0	0
1	1	1

(c)
OR gate



X	Z	Y
0	0	0
0	1	1
1	0	1
1	1	1

In each case, we regard X as a binary channel input and Y as the corresponding binary channel output, and we regard Z as noise, independent of X. Thus, depending on the distribution of Z, each "noisy gate" defines a binary-input binary-output channel.

(**a**) Determine the capacity of the noisy XOR-gate channel, assuming $Z \sim \mathsf{Bern}(p)$. What is the capacity-achieving channel-input distribution?

(**b**) Determine the capacity of the noisy AND-gate channel, assuming $Z \sim \mathsf{Bern}(1/2)$. What is the capacity-achieving channel-input distribution?

(**c**) Determine the capacity of the noisy OR-gate channel, assuming $Z \sim \mathsf{Bern}(1/2)$. What is the capacity-achieving channel-input distribution?

10 marks **2**. (*Cover's Cutting Contraption*)

You are given a 1m length of steel rod, and are asked to cut it into m pieces, where the ith piece is required to have length $p_i > 0$, $i = 1, 2, \ldots m$, with $\sum_{i=1}^{m} p_i = 1$.

To accomplish this task, you must use "Cover's Cutting Contraption," a machine that can take in a single rod of length $\ell \leq 1$m and cut it into two pieces of length $\alpha\ell$ and $(1 - \alpha)\ell$, $0 < \alpha < 1$, where α can be programmed by adjusting a dial on the machine.

Associated with each use of Cover's Cutting Contraption is a *cost*: the cost of cutting a rod of length ℓ is proportional to ℓ. (For simplicity you may assume that the constant of proportionality is unity.)

Please answer the following questions.

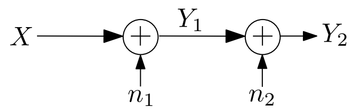
(**a**) Given the desired lengths $p_1, p_2, \ldots, p_m$, how should the cuts be arranged (in general) to minimize the total cost? (You must *prove* in some way—for example, by reducing this problem to an equivalent problem with a known optimal solution—that your sequence is optimal.)

(**b**) Is the optimal sequence of cuts unique?

(**c**) Provide a minimum-cost sequence of cuts that produce the lengths 0.3, 0.3, 0.2, 0.1, 0.1.

10 marks **3**. (*Self-Correcting Channel*)

Consider the channel depicted below, where X is the binary input, n_1 is an additive binary noise signal, and n_2 is an additive binary noise signal whose distribution depends on the value of n_1. All additions are taken modulo 2.



The additive binary noise n_1 is independent of the input, and has distribution $n_1 \sim \mathsf{Bern}(\epsilon)$. The distribution of n_2 depends on the value of n_1 as follows:

- If $n_1 = 1$, then $n_2 \sim \mathsf{Bern}(\alpha_1)$
- If $n_1 = 0$, then $n_2 \sim \mathsf{Bern}(\alpha_2)$

Consider three distinct observers of the channel outputs:

- User A observes Y_1, and its channel has capacity C_a;
- User B observes Y_2, and its channel has capacity C_b;
- User C observes both Y_1 and Y_2, and its channel has capacity C_c.

Please answer the following questions.

(**a**) State conditions on $(\epsilon, \alpha_1, \alpha_2)$ such that $C_a = C_b$, $C_a \geq C_b$, and $C_a \leq C_b$, respectively.

(**b**) Show that $C_c \geq \max(C_a, C_b)$.

(**c**) State conditions on $(\epsilon, \alpha_1, \alpha_2)$ such that $C_c = C_a$.

(**d**) Suppose $0 < \epsilon < 1$, $\alpha_1 = 1$, and $\alpha_2 = 0$. Rank C_a, C_b and C_c in order from smallest to largest.

10 marks **4.** (*AEP*)

In the following, we consider an i.i.d. sequence $x^n = (X_1, X_2, \ldots, X_n)$, where $X_i \sim \mathsf{Bern}(\theta)$. Furthermore, we define the function $L(x^n)$ to be the *Hamming weight* of x^n, i.e., the number of ones in the sequence x^n.

For every $\epsilon > 0$, let $A_\epsilon^{(n)}$ denote the typical set as defined in class and in the textbook.

(**a**) If $\theta = 1/2$, for which values of $\epsilon > 0$ is the all-zero sequence a typical sequence?

(**b**) For general θ, show that knowledge of the Hamming weight $L(x^n)$ of a sequence x^n is sufficient to determine whether x^n is a member of the typical set.

(**c**) For general θ, and n arbitrarily large, are the typical sequences "approximately equiprobable?" Note that we say that the elements of a set of sequences are *approximately equiprobable* if the ratio of the probabilities of the most likely and least likely sequences in the set tends to a constant as n grows.

(**d**) For $\alpha > 0$, $0 \leq p \leq 1$, we define the following set

$$C^{(n)}(\alpha, p) = \{x^n \in \mathcal{X}^n : pn - n\alpha \leq L(x^n) \leq pn + n\alpha\},$$

where $\mathcal{X} = \{0, 1\}$. For $\theta > 1/2$, is it possible to adjust the value of α and p so that $A_\epsilon^{(n)} = C^{(n)}(\alpha, p)$?