ECE 1502 — Information Theory
November 30, 2007

9.3 *Output power constraint.* Since Z is independent of X, we have

$$EY^2 = EX^2 + EZ^2 = EX^2 + \sigma^2,$$

and by the output power constraint, we have

$$EX^2 \le P - \sigma^2.$$

In the following, we assume $P > \sigma^2$, since otherwise the problem is uninteresting, since the output power constraint would be violated by the noise alone. Now, for a maximum expected output power P, the entropy of Y is maximized when $Y \sim \mathcal{N}(0, P)$, which is achieved when $X \sim \mathcal{N}(0, P - \sigma^2)$. Therefore, the channel is equivalent to one with an input power constraint $EX^2 \le P - \sigma^2$, and it follows that the capacity is

$$C = \frac{1}{2}\log_2\left(1 + \frac{P - \sigma^2}{\sigma^2}\right) = \frac{1}{2}\log_2\left(\frac{P}{\sigma^2}\right).$$

9.5 *Fading Channel.* We have

$$\begin{aligned} I(X;Y|V) &= H(X|V) - H(X|Y,V) \\ &= H(X) - H(X|Y,V) \\ &\ge H(X) - H(X|Y) \\ &= I(X;Y), \end{aligned}$$

where the second equality follows since X and V are independent, and the inequality follows since conditioning reduces entropy. Therefore, as is intuitively reasonable, knowledge of the fading factor improves capacity.

9.6 *Parallel channels and waterfilling.* By the result of Section 10.4, it follows that we will put all the signal power into the channel with less noise until the total power of noise + signal in that channel equals the noise power in the other channel. After that, we will split any additional power evenly between the two channels.

Thus the combined channel begins to behave like a pair of parallel channels when the signal power is equal to the difference of the two noise powers, i.e., when $2P = \sigma_1^2 - \sigma_2^2$.

9.7 *Multipath Gaussian channel.*

(a) The channel output is

$$Y = 2X + Z_1 + Z_2 = 2X + Z,$$

where $Z = Z_1 + Z_2$ is the sum of two Gaussian random variables, and is thus itself Gaussian, and

$$EZ^2 = EZ_1^2 + 2EZ_1Z_2 + EZ_2^2 = 2\sigma^2(1 + \rho),$$

therefore $Z \sim \mathcal{N}(0, 2\sigma^2(1+\rho))$. Similarly, since Y is itself the sum of two Gaussian random variables ($2X$ and Z), we have $Y \sim \mathcal{N}(0, 4P + 2\sigma^2(1+\rho))$. Therefore,

$$\begin{aligned} C &= \frac{1}{2}\log_2\left(\frac{4P + 2\sigma^2(1+\rho)}{2\sigma^2(1+\rho)}\right) \\ &= \frac{1}{2}\log_2\left(1 + \frac{2P}{\sigma^2(1+\rho)}\right) \end{aligned}$$

(b) When $\rho = 0$, we have $Z \sim \mathcal{N}(0, 2\sigma^2)$, and thus the noise power is doubled (relative to the case of a single noise source), while the source power is (independent of ρ) increased by a factor of 4, since two copies of X are coherently added at the receiver. Therefore,

$$C = \frac{1}{2}\log_2\left(1 + \frac{2P}{\sigma^2}\right).$$

When $\rho = 1$, we have $Z \sim \mathcal{N}(0, 4\sigma^2)$, and thus the noise power is multiplied by a factor of 4 (relative to the case of a single noise source), since $Z_1 = Z_2$ and they are added coherently at the receiver. Therefore,

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma^2}\right).$$

When $\rho = -1$, the noise is perfectly canceled, since $Z_1 = -Z_2$. Therefore, for any non-zero P, the capacity is infinitely large.

9.9 *Vector Gaussian channel.* For the vector Gaussian channel, we have

$$C = \max_{K_X} \frac{1}{2}\log_2\frac{|K_X + K_Z|}{|K_Z|},$$

where the maximization is over all K_X satisfying $\mathrm{tr}[\mathrm{K_X}] \leq \mathrm{P}$. Now, for the given noise covariance matrix, we have $|K_Z| = 0$. Furthermore, from the water-filling argument in the case of parallel channels with coloured noise, the optimal K_X satisfies $|K_X + K_Z| > 0$ for any $P > 0$. Therefore, the capacity of this channel is infinitely large. Why is this so? It turns out that the given Z is of the form $Z = (Z_1, Z_2, Z_1 + Z_2)$, where $Z_1 \sim \mathcal{N}(0,1)$ and $Z_2 \sim \mathcal{N}(0,1)$ are independent, and the value of Z_3 is specified exactly by knowledge of Z_1 and Z_2. Therefore, the optimal coding scheme is to set $X_1 = X_2 = 0$, from which the receiver detects the values of Z_1 and Z_2, then uses these to cancel the noise in Y_3, thus providing a noise-free channel from X_3 to Y_3, which clearly permits an infinite rate of error-free communication for any $P > 0$.

9.12 *Time-varying channel.* Before giving a formal analysis, the basic idea is as follows. In the case of the usual (i.e., non-time-varying) Gaussian channel, the converse to Theorem 9.1.1 confirms the communication rate is maximized when each column of the codebook has the same average power P. Put another way, we should not 'save' power now in order to use it in the future. In the case of the time-varying Gaussian channel, the same type of analysis shows that the codebook should be designed so that each codeword symbol is *received* with the same power; in the present case, this means that later symbols must be transmitted with more power than earlier symbols, while still assuring that an overall average power constraint is met. As it turns out, the resulting average received power goes to zero, and thus arbitrarily reliable communication is not possible for non-zero rates.

More formally, following the proof to Theorem 9.1.1, the analysis is identical up to and including (9.46). Now, since

$$EY_i^2 = \frac{EX_i^2}{i^2} + N = \frac{P_i}{i^2} + N,$$

(9.47) becomes

$$h(Y_i) \leq \frac{1}{2} \log 2\pi e \left(\frac{P_i}{i^2} + N \right).$$

It follows that (9.52) becomes

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \log \left(1 + \frac{P_i}{i^2 N} \right) \leq \frac{1}{2} \log \left(1 + \frac{1}{n} \sum_{i=1}^{n} \frac{P_i}{i^2 N} \right),$$

with equality if and only if $\frac{P_i}{i^2}$ is a constant for all i. Therefore, we assume $\frac{P_i}{i^2}$ is a constant for all i, $A = \frac{P_i}{i^2}$, and we have

$$\frac{A}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \sum_{i=1}^{n} P_i \leq P,$$

by the transmitter power constraint. Re-arranging, and recalling that

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6},$$

we have

$$A \leq \frac{6P}{(2n+1)(n+1)}.$$

Therefore, by Fano's Inequality, we have

$$R \leq \frac{1}{2} \log_2 \left(1 + \frac{6P}{N(2n+1)(n+1)} \right) + n\epsilon_n.$$

Now, as $n \to \infty$, we have $R = 0$, and thus the capacity is equal to zero. Why must n go to infinity? This follows from our definition of capacity, since we require that the error rate goes to zero; for any finite n, it is true that the preceeding bound does not force R to zero, but arbitrarily reliable communication will not be possible.

10.1 *One bit quantization of a Gaussian random variable.* Let $X \sim \mathcal{N}(0, \sigma^2)$ and let the distortion measure be squared error. With one bit quantization, the obvious reconstruction regions are the positive and negative real axes. The reconstruction point is the centroid of each region. For example, for the positive real line, the centroid a is

$$a = \int_0^\infty x \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \, dx \quad (1)$$

$$= \int_0^\infty \sigma \sqrt{\frac{2}{\pi}} e^{-y} \, dy \quad (2)$$

$$= \sigma \sqrt{\frac{2}{\pi}}, \quad (3)$$

using the substitution $y = x^2/2\sigma^2$. The expected distortion for one bit quantization is

$$\begin{aligned} D &= \int_{-\infty}^{0} \left(x + \sigma\sqrt{\frac{2}{\pi}}\right)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx & (4)\\ &\quad + \int_{0}^{\infty} \left(x - \sigma\sqrt{\frac{2}{\pi}}\right)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx & (5)\\ &= 2\int_{-\infty}^{\infty} \left(x^2 + \sigma^2\frac{2}{\pi}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx & (6)\\ &\quad -2\int_{0}^{\infty} \left(-2x\sigma\sqrt{\frac{2}{\pi}}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx & (7)\\ &= \sigma^2 + \frac{2}{\pi}\sigma^2 - 4\frac{1}{\sqrt{2\pi}}\sigma^2\sqrt{\frac{2}{\pi}} & (8)\\ &= \sigma^2\frac{\pi - 2}{\pi}. & (9) \end{aligned}$$

10.2 *Rate Distortion.* We wish to evaluate the rate distortion function

$$R(D) = \min_{p(\hat{x}|x):\sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x}) \le D} I(X;\hat{X}). \tag{10}$$

Since $d(0,1) = \infty$, we must have $p(0,1) = 0$ for a finite distortion. Thus, the distortion $D = p(1,0)$, and hence we have the following joint distribution for $(X,\hat{X})$ (assuming $D \le \frac{1}{2}$).

$$p(x,\hat{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ D & \frac{1}{2} - D \end{bmatrix} \tag{11}$$

The mutual information for this joint distribution is

$$\begin{aligned} R(D) = I(X;\hat{X}) &= H(X) - H(X|\hat{X}) & (12)\\ &= H(\frac{1}{2},\frac{1}{2}) - (\frac{1}{2} + D)H\left(\frac{\frac{1}{2}}{\frac{1}{2}+D}, \frac{D}{\frac{1}{2}+D}\right) & (13)\\ &= 1 + \frac{1}{2}\log\frac{\frac{1}{2}}{\frac{1}{2}+D} + D\log\frac{D}{\frac{1}{2}+D}, & (14) \end{aligned}$$

which is the rate distortion function for this binary source if $0 \le D \le \frac{1}{2}$. Since we can achieve $D = \frac{1}{2}$ with zero rate (use $p(\hat{x} = 0) = 1$), we have $R(D) = 0$ for $D \ge \frac{1}{2}$.

10.5 *Rate distortion for uniform source with Hamming distortion.* X is uniformly distributed on the set $\{1, 2, \ldots, m\}$. The distortion measure is

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 1 & \text{if } x \ne \hat{x} \end{cases}$$

Consider any joint distribution that satisfies the distortion constraint D. Since $D = \Pr(X \ne \hat{X})$, we have by Fano's inequality

$$H(X|\hat{X}) \le H(D) + D\log(m-1), \tag{15}$$

and hence

$$I(X;\hat{X}) = H(X) - H(X|\hat{X}) \quad (16)$$
$$\geq \log m - H(D) - D\log(m-1). \quad (17)$$

We can achieve this lower bound by choosing $p(\hat{x})$ to be the uniform distribution, and the conditional distribution of $p(x|\hat{x})$ to be

$$p(\hat{x}|x)\begin{cases} = 1-D & \text{if } \hat{x}=x \\ = D/(m-1) & \text{if } \hat{x}\neq x. \end{cases} \quad (18)$$

It is easy to verify that this gives the right distribution on X and satisfies the bound with equality for $D < 1 - \frac{1}{m}$. Hence

$$R(D)\begin{cases} = \log m - H(D) - D\log(m-1) & \text{if } 0 \leq D \leq 1-\frac{1}{m} \\ 0 & \text{if } D > 1-\frac{1}{m}. \end{cases} \quad (19)$$

10.6 *Shannon lower bound on the rate distortion function.*

(a) We define

$$\phi(D) = \max_{\mathbf{p}:\sum_{i=1}^m p_i d_i \leq D} H(\mathbf{p}). \quad (20)$$

From the definition, if $D_1 \geq D_2$, then $\phi(D_1) \geq \phi(D_2)$ since the maximization is over a larger set. Hence $\phi(D)$ is a monotonic increasing function.

To prove concavity of $\phi(D)$, consider two levels of distortion D_1 and D_2 and let $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ achieve the maxima in the definition of $\phi(D_1)$ and $\phi(D_2)$. Let $\mathbf{p}^{(\ell)}$ be the mixture of the two distributions, i.e.,

$$\mathbf{p}^{(\ell)} = \ell\mathbf{p}^{(1)} + (1-\ell)\mathbf{p}^{(2)}. \quad (21)$$

Then the distortion is a mixture of the two distortions

$$D_\ell = \sum_i p_i^{(\ell)} d_i = \ell D_1 + (1-\ell)D_2. \quad (22)$$

Since entropy is a concave function, we have

$$H(\mathbf{p}^{(\ell)}) \geq \ell H(\mathbf{p}^{(1)}) + (1-\ell)H(\mathbf{p}^{(2)}). \quad (23)$$

Hence

$$\phi(D_\ell) = \max_{\mathbf{p}:\sum p_i d_i = D_\ell} H(\mathbf{p}) \quad (24)$$
$$\geq H(\mathbf{p}^{(\ell)}) \quad (25)$$
$$\geq \ell H(\mathbf{p}^{(1)}) + (1-\ell)H(\mathbf{p}^{(2)}) \quad (26)$$
$$= \ell\phi(D_1) + (1-\ell)\phi(D_2), \quad (27)$$

proving that $\phi(D)$ is a concave function of D.

(b) For any $(X, \hat{X})$ that satisfy the distortion constraint, we have

$$\begin{aligned}
I(X;\hat{X}) &\stackrel{(a)}{=} H(X) - H(X|\hat{X}) && (28)\\
&\stackrel{(b)}{=} H(X) - \sum_{\hat{x}} p(\hat{x}) H(X|\hat{X}=\hat{x}) && (29)\\
&\stackrel{(c)}{\geq} H(X) - \sum_{\hat{x}} p(\hat{x}) \phi(D_{\hat{x}}) && (30)\\
&\stackrel{(d)}{\geq} H(X) - \phi(\sum_{\hat{x}} p(\hat{x}) D_{\hat{x}}) && (31)\\
&\stackrel{(e)}{\geq} H(X) - \phi(D), && (32)
\end{aligned}$$

where
(a) follows from the definition of mutual information,
(b) from the definition of conditional entropy,
(c) follows from the definition of $\phi(D_{\hat{x}})$ where $D_{\hat{x}} = \sum p(x|\hat{x}) d(x,\hat{x}) = \sum p(x|\hat{x}) d_{i'}$ that $H(p(x|\hat{x})) \leq \phi(D_{\hat{x}})$
(d) follows from Jensen's inequality and the concavity of ϕ, and
(e) follows from the monotonicity of ϕ and the fact that $\sum p(\hat{x}) D_{\hat{x}} = \sum p(x,\hat{x}) d(x,\hat{x}) \leq D$.
Hence, from the definition of the rate distortion function, we have

$$\begin{aligned}
R(D) &= \min_{p(\hat{x}|x): \sum p(x,\hat{x}) d(x,\hat{x}) \leq D} I(X;\hat{X}) && (33)\\
&\geq H(X) - \phi(D), && (34)
\end{aligned}$$

which is the Shannon lower bound on the rate distortion function.

(c) Let $\mathbf{p}^* = (p_1^*, p_2^*, \ldots, p_m^*)$ be the distribution that achieves the maximum in the definition of the $\phi(D)$. Assume that the source has a uniform distribution and that the rows of the distortion matrix are permutations of each other. Let the distortion matrix be $[a_{ij}]$. We can then choose $p(\hat{x})$ to have a uniform distribution and choose $p(x=i|\hat{x}=j) = p_k^*$, if $a_{ij} = d_k$. For this joint distribution,

$$\begin{aligned}
p_x(i) &= \sum_j p_{\hat{x}}(j) p_{x|\hat{x}}(i|j) && (35)\\
&= \sum_j \frac{1}{m} p_k^* && (36)\\
&= \frac{1}{m} && (37)
\end{aligned}$$

since the rows of the distortion matrix are permutations of each other and therefore each element p_k^*, $k = 1, 2, \ldots, m$ occurs once in the above sum. Hence the distribution of x has the desired source distribution. For this joint distribution, we have

$$\sum_{i,j} p_{x,\hat{x}}(i,j) a_{ij} = \sum_j \frac{1}{m} \sum_i p_{x|\hat{x}}(i|j) a_{ij} \qquad (38)$$

$$
\begin{aligned}
&= \sum_j \frac{1}{m} \sum_k p_k^* d_k && (39)\\
&= \sum_j \frac{1}{m} D && (40)\\
&= D, && (41)
\end{aligned}
$$

the desired distortion. The mutual information

$$
\begin{aligned}
I(X;\hat{X}) &= H(X) - H(X|\hat{X}) && (42)\\
&= H(X) - \sum_j \frac{1}{m} H(X|\hat{X}=j) && (43)\\
&= H(X) - \sum_j \frac{1}{m} H(\mathbf{p}^*) && (44)\\
&= H(X) - \sum_j \frac{1}{m} \phi(D) && (45)\\
&= H(X) - \phi(D). && (46)
\end{aligned}
$$

Hence using this joint distribution in the definition of the rate distortion function

$$
\begin{aligned}
R(D) &= \min_{p(\hat{x}|x): \sum p(x,\hat{x}) d(x,\hat{x}) \leq D} I(X;\hat{X}) && (47)\\
&\leq I(X;\hat{X}) && (48)\\
&= H(X) - \phi(D). && (49)
\end{aligned}
$$

Combining this with the Shannon lower bound on the rate distortion function, we must have equality in the above equation and hence we have equality in the Shannon lower bound.