Complex Numbers and the Complex Exponential

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1 Numbers and Equations

Numbers have often been invented to solve equations.

For example, by introducing negative numbers the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ can be extended to the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ so that we may solve simple equations such as $x+2 = 0$. Likewise, by introducing integer quotients, the integers can to be extended to the rational numbers $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$ so that we may solve simple equations such as $2x = 1$.

The rationals seem like a very nice set in which to do arithmetic. There is a well-defined addition operation and a well-defined multiplication operation, and $\mathbb{Q}$ is *closed* with respect to these operations. Addition and multiplication are associative (i.e., for all $x, y, z \in \mathbb{Q}$, $x(yz) = (xy)z$ and likewise for addition) and commutative (i.e., for all $x, y \in \mathbb{Q}$, $x + y = y + x$ and likewise for multiplication). Every element $x \in \mathbb{Q}$ has an additive inverse $-x$ (from which we may define a subtraction operation) and every nonzero element $x \in \mathbb{Q}, x \neq 0$, has a multiplicative inverse $1/x$ (from which we may define a division operation). Furthermore, multiplication and addition satisfy the distributive law, i.e., for all $x, y, z \in \mathbb{Q}$, $x(y+z) = xy+xz$. Mathematically speaking, the rational numbers form a *field*. Who could ask for anything more?

The trouble is that certain simple equations such as

$$x^2 - 2 = 0 \tag{1}$$

have *no solutions* in $\mathbb{Q}$, i.e., no rational number x satisfies (1). However, following the progression from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$, we might try to get around this problem by *extending* $\mathbb{Q}$, i.e., by adjoining an element—let's call it θ for now—that satisfies $\theta^2 - 2 = 0$, or, equivalently, $\theta^2 = 2$. We will demand that θ be combinable with ordinary rational numbers (and with itself) via addition and multiplication, while satisfying all of the formal arithmetic properties (such as closure with respect to addition and multiplication, associativity, commutativity, the distributive law, etc.) that we have grown to expect.

If we denote this extended set by $\mathbb{Q}[\theta]$, then certainly $\mathbb{Q}[\theta]$ must contain all numbers of the form $a + b\theta$, where $a, b \in \mathbb{Q}$. Numbers involving higher powers of θ do not arise, since any such higher power can be reduced to a multiple of a lower power, i.e., $\theta^2 = 2$, $\theta^3 = 2\theta$, $\theta^4 = 4$, etc. Indeed, the sum, difference, product and quotient of any two elements of this form is another element of this form (provided that we don't attempt to divide by zero). To see this, observe that

$$(a + b\theta) \pm (c + d\theta) = (a \pm c) + (b \pm d)\theta,$$

and since $a \pm c$ and $b \pm d$ are rational when a, b, c and d are rational, we have another number of the same form. Likewise

$$\begin{aligned}(a + b\theta)(c + d\theta) &= ac + ad\theta + bc\theta + bd\theta^2 \\ &= (ac + 2bd) + (ad + bc)\theta.\end{aligned}$$

Again, since $ac + 2bd$ and $ad + bc$ are rational when a, b, c and d are rational, we have another number of the same form. Finally, to show that we can form quotients, it is enough to show that we can form reciprocals. Note that if a and b are not both zero then

$$\begin{aligned}\frac{1}{a + b\theta} &= \frac{a - b\theta}{(a + b\theta)(a - b\theta)} \\ &= \frac{a - b\theta}{a^2 - b^2\theta^2} \\ &= \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\theta.\end{aligned}$$

Now, $a^2 - 2b^2 \neq 0$ (why?), so $a/(a^2 - 2b^2)$ and $-b/(a^2 - 2b^2)$ are rational when a, b, c and d are rational; thus once again we have another number of the same form. Indeed, we have shown that $\mathbb{Q}[\theta]$ is a field, and unlike $\mathbb{Q}$, in this field (1) does indeed have a solution.

Since every element of $\mathbb{Q}[\theta]$ can be written as $a + b\theta$, with $a, b \in \mathbb{Q}$, we could associate such an element with the *ordered pair* (a, b). We might refer to the first component, a, as the "rational part" of $a + b\theta$ and the second component, b, as the "irrational part" of $a + b\theta$. (Note that the "irrational part" is in fact itself a rational number!) The arithmetic in $\mathbb{Q}[\theta]$ could be defined via operations that occur on such ordered pairs, e.g.,

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \times (c, d) &= (ac + 2bd, ad + bc)\end{aligned}$$

From this viewpoint there is no mention of the "irrational" number θ; indeed, from this viewpoint one can simply regard θ as a "tag", pointing out which of the two rational numbers in question is the designated "irrational part."

By the way, a more usual notation for θ is $\sqrt{2}$.

2 Complex Numbers and the Complex Plane

So far we have "extended" $\mathbb{Q}$ by adjoining $\sqrt{2}$, thus taking one small step toward the construction of the real numbers $\mathbb{R}$. Let us now proceed to the point where we have constructed all of $\mathbb{R}$. Certain real numbers (e.g., $\sqrt[3]{3}$) appear as the zeros of a polynomial with rational coefficients and can be adjoined to $\mathbb{Q}$ in the same manner that we used to adjoin $\sqrt{2}$; others (e.g., π) appear as the limit points of certain sequences.

Unfortunately, even in $\mathbb{R}$, many polynomials have no zeros, e.g., there is no real number x that satisfies

$$x^2 + 1 = 0. \tag{2}$$

Emboldened by our experience in extending $\mathbb{Q}$, let us extend $\mathbb{R}$ by introducing a new element, a purely "imaginary" number, j, that satisfies $\mathsf{j}^2 + 1 = 0$. Once again, we will demand that j satisfy all of the formal arithmetic properties (such as closure with respect to addition and multiplication, associativity, commutativity, the distributive law, etc.) that we have grown to expect in $\mathbb{R}$.

Let us denote the extended set by $\mathbb{C}$. Certainly, as we compute all possible sums and products involving real numbers and j, we find that $\mathbb{C}$ must contain all numbers of the form $a + b\,\mathsf{j}$, where $a, b \in \mathbb{R}$. Numbers involving higher powers of j do not arise, since any such higher power can be reduced to a multiple of a lower power, i.e., $\mathsf{j}^2 = -1$, $\mathsf{j}^3 = -\,\mathsf{j}$, $\mathsf{j}^4 = 1$, etc. We have, for all a, b, c and d in $\mathbb{R}$,

$$\begin{aligned}(a + b\,\mathsf{j}) \pm (c + d\,\mathsf{j}) &= (a \pm c) + (b \pm d)\,\mathsf{j} \\ (a + b\,\mathsf{j}) \times (c + d\,\mathsf{j}) &= (ac - bd) + (ac + bd)\,\mathsf{j}\end{aligned}$$

and, provided a and b are not both zero,

$$\frac{1}{a + b\,\mathsf{j}} = \frac{a - b\,\mathsf{j}}{(a + b\,\mathsf{j})(a - b\,\mathsf{j})} = \frac{a - b\,\mathsf{j}}{a^2 + b^2}.$$

This shows that the sum, difference, product and quotient of any two elements of this form is another element of this form (provided that we don't attempt to divide by zero), and so $\mathbb{C}$ is a field. The elements of $\mathbb{C}$ are called *complex numbers* and $\mathbb{C}$ is referred to as the complex field.

We usually denote individual complex numbers with a single symbol, such as z. Now, since every such element can be written in the form $z = a + b\,\mathsf{j}$, with $a, b \in \mathbb{R}$, we can associate such an element with the *ordered pair* (a, b) of real numbers. We refer to the first component, a, as the "real part" of z (written $\mathsf{Re}(z)$) and the second component, b, as the "imaginary part" of z (written $\mathsf{Im}(z)$). Thus for all $z \in \mathbb{C}$, we have

$$z = \mathsf{Re}(z) + \mathsf{Im}(z)\,\mathsf{j}\,.$$

Note that the "imaginary part" is in fact itself a real number! If $\mathsf{Im}(z) = 0$, then z is referred to as a (purely) real number; likewise if $\mathsf{Re}(z) = 0$, then z is referred to as a purely imaginary number. Thus, in particular, j itself is purely imaginary.

Two complex numbers z_1 and z_2 are *equal* if and only if their real and imaginary parts agree, i.e.,

$$z_1 = z_2 \text{ if and only if } [\mathsf{Re}(z_1) = \mathsf{Re}(z_2)] \text{ and } [\mathsf{Im}(z_1) = \mathsf{Im}(z_2)].$$

Thus every complex equality can always be regarded as a *pair* of real equalities.

The *complex conjugate* z^* of a complex number $z = \mathsf{Re}(z) + \mathsf{Im}(z)\,\mathsf{j}$ is the complex number

$$z^* = \mathsf{Re}(z) - \mathsf{Im}(z)\,\mathsf{j}\,.$$

Note that $z^* = z$ if and only if $\mathsf{Im}(z) = 0$, i.e., if and only if z is real.

The complex conjugate obeys the following properties for all $w, z \in \mathbb{C}$:

$$\begin{aligned}(w \pm z)^* &= w^* \pm z^*;\\ (wz)^* &= (w^*)(z^*);\\ (w/z)^* &= (w^*)/(z^*) \quad (z \neq 0).\end{aligned}$$

Furthermore, for all $z \in \mathbb{C}$ we have

$$\begin{aligned}z + z^* &= 2\,\mathsf{Re}(z);\\ z - z^* &= 2\,\mathsf{j}\,\mathsf{Im}(z);\\ (z^*)^* &= z.\end{aligned}$$

It is possible to show (see the problems) that if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$ is a polynomial with *real coefficients* (i.e., $a_0, a_1, \ldots, a_d \in \mathbb{R}$) and $f(z) = 0$ for some complex value z, then we must also have $f(z^*) = 0$. In other words, if z is a zero of a polynomial with real coefficients then so is its complex conjugate z^*.

Note that, unlike the real numbers, complex numbers are not in general ordered, i.e., it makes no sense to ask which is larger: $2 + 3\,\mathsf{j}$ or $3 + 2\,\mathsf{j}$. However, we can always compare the magnitudes of two complex numbers. The *magnitude* (or *absolute value* or *modulus*) $|z|$ of a complex number z is defined as

$$|z| = \sqrt{zz^*} = \sqrt{\mathsf{Re}^2(z) + \mathsf{Im}^2(z)}.$$

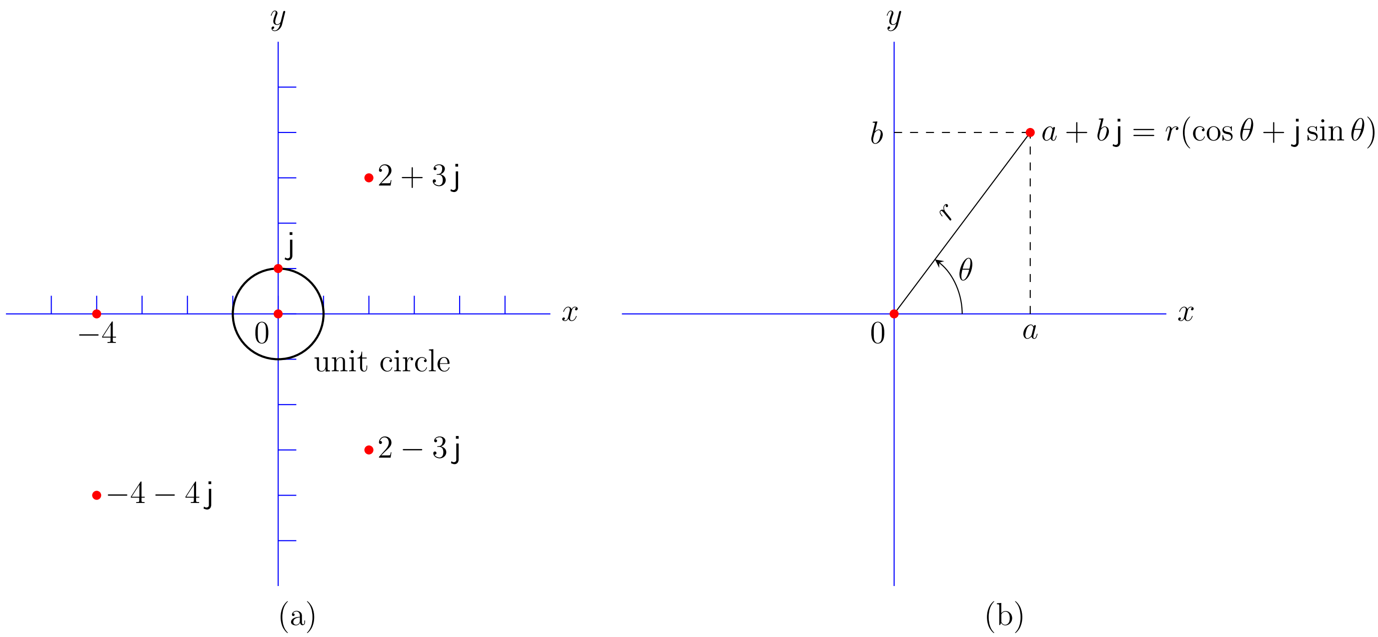

Figure 1: (a) Several points in the complex plane. (b) The polar form of a complex number.

Clearly $|z|$ is a non-negative real number, and $|z| = 0$ if and only if $z = 0$. Note that $|z^*| = |z|$, i.e., a complex number and its complex conjugate have the same magnitude.

It is very convenient to visualize $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$, i.e., as a *plane*. Naturally, this plane is referred to as the *complex plane*. The number $a + b\,\mathrm{j}$ corresponds to the point with coordinates (a, b) in the complex plane, as shown in Fig. 1(a).

In Figure 1(a), the two axes have been labelled x and y. Since each point on the x-axis represents a real number, this axis is called the *real axis*. Similarly, the y-axis is called the *imaginary axis*. These two axes intersect at 0, which is the only complex number that is simultaneously purely real and purely imaginary.

The magnitude $|z|$ of a complex number z has a geometric interpretation in the complex plane: $|z|$ measures the (Euclidean) distance between the origin 0 and z. Or if z is regarded as a vector in the complex plane, then $|z|$ is the length of this vector. Similarly, z^* represents the geometric reflection of z in the real axis.

3 Polar Form

The set of points at unit distance from the origin in the complex plane, corresponding to the complex numbers z with $|z| = 1$, form a circle of unit radius centered at the origin. This circle is called the *unit circle* in the complex plane. Every point on the unit circle can be represented in the form $z = \cos\theta + \mathsf{j}\sin\theta$, where (from now on) θ represents an *angle*.

More generally, as illustrated in Fig. 1(b), a complex number $z = a + b\,\mathsf{j}$ can be represented in the *polar form*

$$z = r(\cos\theta + \mathsf{j}\sin\theta), \tag{3}$$

where $r = |z|$ and, if $z \neq 0$, $\cos\theta = \mathsf{Re}(z)/|z|$ and $\sin\theta = \mathsf{Im}(z)/|z|$. The angle θ is called the *argument* or *phase* of z, and is denoted $\mathsf{arg}(z)$. The phase of $z = 0$ is not defined.

Now, since $\cos(x)$ and $\sin(x)$ are periodic with period 2π, it is clear that if θ_1 is a value of $\mathsf{arg}(z)$, then so are $\theta_1 \pm 2\pi$, $\theta_1 \pm 4\pi$, $\theta_1 \pm 6\pi$, and so on. Thus $\mathsf{arg}(z)$ does not specify a *unique* angle, but rather it specifies an infinite class of *equivalent angles*, each differing from the others by some integer multiple of 2π. For example,

$$\begin{aligned}\mathsf{arg}(\mathsf{j}) &= \frac{\pi}{2} + k2\pi,\ k \in \mathbb{Z} \\ &= \{\ldots, -7\pi/2, -3\pi/2, \pi/2, 5\pi/2, \ldots\}.\end{aligned}$$

So that every complex number has a *unique* phase, we can select an angle from any half-open interval I of the real numbers of length 2π, so that it is impossible for more than one angle from each equivalence class to fall in I. By convention, this interval is usually chosen as $(-\pi, \pi]$, and the corresponding angle is referred to as the *principle value of the argument*, denoted $\mathsf{Arg}(z)$, or $\angle z$. Note the capital letter: whereas $\mathsf{arg}(z)$ denotes an infinite set of equivalent angles, $\mathsf{Arg}(z)$ specifies a unique angle in the range $(-\pi, \pi]$. Thus, for example, $\mathsf{Arg}(1) = 0$, $\mathsf{Arg}(\mathsf{j}) = \pi/2$, $\mathsf{Arg}(-1) = \pi$, and $\mathsf{Arg}(-\mathsf{j}) = -\pi/2$.

Although polar form is an inconvenient representation for complex addition, it is great for multiplication. Indeed, if we multiply $z_1 = r_1(\cos\theta_1 + \mathsf{j}\sin\theta_1)$ with $z_2 = r_2(\cos\theta_2 + \mathsf{j}\sin\theta_2)$, we get

$$\begin{aligned}z_1 z_2 &= r_1(\cos\theta_1 + \mathsf{j}\sin\theta_1) r_2(\cos\theta_2 + \mathsf{j}\sin\theta_2) \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \mathsf{j}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + \mathsf{j}\sin(\theta_1 + \theta_2)].\end{aligned}$$

We see that under complex multiplication, *magnitudes multiply* and *phases add*, i.e.,

$$|z_1 z_2| = |z_1| \cdot |z_2|$$

and

$$\mathsf{arg}(z_1 z_2) = \mathsf{arg}(z_1) + \mathsf{arg}(z_2),$$

where the latter equality is interpreted in the following way: if we substitute a specific angle for $\mathsf{arg}(z_1)$ and a specific angle for $\mathsf{arg}(z_2)$, their sum is a member of the equivalence class of angles of $z_1 z_2$.

Now, if $z \neq 0$, we have $z(1/z) = 1$; this means that if z has the polar form $z = r(\cos\theta + \mathsf{j}\sin\theta)$, then $1/z$ must have the polar form

$$\frac{1}{z} = \frac{1}{r}[\cos(-\theta) + \mathsf{j}\sin(-\theta)],$$

so that their product has the polar form $1 = 1(\cos 0 + \mathsf{j}\sin 0)$. Thus taking reciprocals in polar form is just as convenient as complex multiplication. It follows from this that if $z_2 \neq 0$

$$\mathsf{arg}(1/z_2) = -\,\mathsf{arg}(z_2),$$
$$|z_1/z_2| = |z_1|/|z_2|,$$

and

$$\mathsf{arg}(z_1/z_2) = \mathsf{arg}(z_1) - \mathsf{arg}(z_2), \text{ provided } z_1 \neq 0.$$

The property that the phase of a product is the sum of the phases is very reminiscent of the rule for multiplying exponentials, where the exponent of a product is the sum of the exponents, i.e., $e^x e^y = e^{x+y}$. As we will see in the next section, where we consider the complex exponential function, this connection is not a coincidence.

4 The Complex Exponential

To define a complex exponential function e^z, we would certainly wish to mimic some of the familiar properties of the real exponential function; e.g., e^z should satisfy, for all z_1, z_2 and z,

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{z_1+z_2} \\ e^{z_1}/e^{z_2} &= e^{z_1-z_2} \\ \frac{d}{dz}e^z &= e^z. \end{aligned}$$

(The last equation requires that we first define what we mean by complex differentiation, something that is beyond the scope of these notes. However, see, e.g., [1, 2].) Here we will introduce the complex exponential in a sneaky way: via its Maclaurin series. (Recall that the Maclaurin series of a function is the Taylor expansion of the function around zero.)

Recall that the real-valued functions $\cos(x)$, $\sin(x)$, and e^x have Maclaurin series given, respectively, by

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \end{aligned}$$

and that each of these series is convergent for every value of $x \in \mathbb{R}$.

It would be natural indeed to define

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{z^i}{i!}, \end{aligned} \tag{4}$$

and this is precisely what we will do. Although we will not prove this here, the Maclaurin series (4) is convergent for every value of $z \in \mathbb{C}$. Clearly this complex-valued exponential agrees with the usual real-valued exponential at every point z on the real-axis in the complex plane.

To see that $e^{z_1} e^{z_2} = e^{z_1 + z_2}$, we multiply the corresponding Maclaurin series. This method of multiplying series is sometimes called the Cauchy product. We want to form

$$e^{z_1} e^{z_2} = \left(\frac{z_1^0}{0!} + \frac{z_1^1}{1!} + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots \right) \times \left(\frac{z_2^0}{0!} + \frac{z_2^1}{1!} + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots \right).$$

Using the distributive law and grouping terms having the same total exponent (sum of z_1 exponent and z_2 exponent), we get

$$e^{z_1} e^{z_2} = \frac{z_1^0 z_2^0}{0!0!} + \left[\frac{z_1^1 z_2^0}{1!0!} + \frac{z_1^0 z_2^1}{0!1!} \right] + \left[\frac{z_1^2 z_2^0}{2!0!} + \frac{z_1^1 z_2^1}{1!1!} + \frac{z_1^0 z_2^2}{0!2!} \right] + \left[\frac{z_1^3 z_2^0}{3!0!} + \frac{z_1^2 z_2^1}{2!1!} + \frac{z_1^1 z_2^2}{1!2!} + \frac{z_1^0 z_2^3}{0!3!} \right] + \cdots$$

This sum can be written as

$$\begin{aligned} e^{z_1} e^{z_2} &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{z_1^{i-j} z_2^j}{(i-j)! j!} \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{i} \frac{i! z_1^{i-j} z_2^j}{(i-j)! j!} \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} z_1^{i-j} z_2^j \\ &= \sum_{i=0}^{\infty} \frac{(z_1 + z_2)^i}{i!} \\ &= e^{z_1 + z_2}, \end{aligned}$$

where, in the second last equality, we have made use of the binomial expansion (which holds in any field).

Now, writing $z = a + \mathsf{j}\, b$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}$, we find that

$$e^z = e^{a + \mathsf{j}\, b} = e^a e^{\mathsf{j}\, b}.$$

Since e^a is a (well understood) real-valued exponential, we see that the key to understanding the complex-valued exponential is to understand the function $e^{\mathrm{j}\,b}$ for real b.

For this, we apply the Maclaurin series. We have, for real b,

$$\begin{aligned}
e^{\mathrm{j}\,b} &= \sum_{i=0}^{\infty} \frac{(\mathrm{j}\,b)^i}{i!} \\
&= 1 + \mathrm{j}\,b + \frac{(\mathrm{j}\,b)^2}{2!} + \frac{(\mathrm{j}\,b)^3}{3!} + \frac{(\mathrm{j}\,b)^4}{4!} + \frac{(\mathrm{j}\,b)^5}{5!} + \frac{(\mathrm{j}\,b)^6}{6!} + \frac{(\mathrm{j}\,b)^7}{7!} + \cdots \\
&= 1 + \mathrm{j}\,b - \frac{b^2}{2!} - \frac{\mathrm{j}\,b^3}{3!} + \frac{b^4}{4!} + \frac{\mathrm{j}\,b^5}{5!} - \frac{b^6}{6!} - \frac{\mathrm{j}\,b^7}{7!} + \cdots ,
\end{aligned}$$

where we have used the fact that $\mathrm{j}^2 = -1$, $\mathrm{j}^3 = -\mathrm{j}$, $\mathrm{j}^4 = 1$, etc. Grouping the real and imaginary parts, we find that, for real b,

$$\begin{aligned}
e^{\mathrm{j}\,b} &= \left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \cdots\right) + \mathrm{j}\left(b - \frac{b^3}{3!} + \frac{b^5}{5!} - \frac{b^7}{7!} + \cdots\right) \\
&= \cos b + \mathrm{j} \sin b.
\end{aligned}$$

Thus we find that $e^{\mathrm{j}\,b}$ is essentially a trigonometric function: it has real part $\cos b$ and imaginary part $\sin b$. Applied to $z = a + \mathrm{j}\,b$, the complex exponential returns

$$e^z = e^{a+\mathrm{j}\,b} = e^a(\cos b + \mathrm{j} \sin b).$$

Using the complex exponential allows us to write, in a more compact way, the polar form (3) of a complex number having magnitude r and phase θ: we have

$$r(\cos\theta + \mathrm{j}\sin\theta) = re^{\mathrm{j}\,\theta}.$$

The fact that phases add under complex multiplication now becomes obvious, since

$$r_1 e^{\mathrm{j}\,\theta_1} r_2 e^{\mathrm{j}\,\theta_2} = r_1 r_2 e^{\mathrm{j}(\theta_1+\theta_2)}.$$

Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we have

$$e^{\mathrm{j}\,\theta} = \cos\theta + \mathrm{j}\sin\theta \text{ and } e^{-\mathrm{j}\,\theta} = \cos\theta - \mathrm{j}\sin\theta. \tag{5}$$

Adding the two equations in (5) yields $e^{\mathrm{j}\,\theta} + e^{-\mathrm{j}\,\theta} = 2\cos\theta$, or

$$\cos\theta = \frac{e^{\mathrm{j}\,\theta} + e^{-\mathrm{j}\,\theta}}{2}.$$

Subtracting the two equations in (5) yields $e^{\mathrm{j}\,\theta} - e^{-\mathrm{j}\,\theta} = 2\,\mathrm{j}\sin\theta$, or

$$\sin\theta = \frac{e^{\mathrm{j}\,\theta} - e^{-\mathrm{j}\,\theta}}{2\,\mathrm{j}}.$$

These relationships often allow one to derive various trigonometric identities via the complex exponential function.

Exercises

1. Prove that no rational number x satisfies (1).

2. Work out the rules of arithmetic for elements in $\mathbb{Q}[\sqrt{3}]$.

3. Let $f(x)$ be any polynomial in x with coefficients in $\mathbb{C}$. It can be shown that *every* non-constant polynomial with coefficients in $\mathbb{C}$, i.e., every function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

with $a_0, a_1, \ldots, a_d \in \mathbb{C}$ and $d > 0$ has a zero in $\mathbb{C}$, i.e., $f(x) = 0$ for some $x \in \mathbb{C}$. This means that we cannot find new elements that satisfy simple polynomial relations to adjoin to $\mathbb{C}$. Mathematically speaking, the complex field $\mathbb{C}$ is "algebraically closed." Show that the fact that $\mathbb{C}$ is algebraically closed implies that every polynomial $f(x)$ of degree $d > 0$ *factors* into a product of d degree-one polynomials with complex coefficients. (Hint: show that if $f(x_0) = 0$, then $f(x) = (x - x_0)g(x)$ where $g(x)$ is a polynomial of degree $d - 1$. Then apply the principle of mathematical induction.)

4. Let $f(x)$ be a polynomial with real-valued coefficients. Show that if $f(z) = 0$ for some complex number z, then $f(z^*) = 0$ as well. (Hint: consider $[f(z)]^*$.)

5. A complex-valued function $f(z)$ is said to be periodic with period z_0 if $f(z + z_0) = f(z)$ for all $z \in \mathbb{C}$. Show that e^z is periodic with period $2\pi\,\mathsf{j}$.

6. Show that $(\cos\theta + \mathsf{j}\sin\theta)^n = \cos(n\theta) + \mathsf{j}\sin(n\theta)$, a result known as *De Moivre's formula.*

7. Use De Moivre's formula to show that

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2\theta$$

and

$$\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$$

8. Show that the complex exponential takes on every complex value except 0.

References

[1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering, 3rd Edition*, Prentice-Hall, 2003.

[2] D. A. Wunsch, *Complex Variables with Applications, 3rd Edition*, Addison-Wesley, 2004.

Spoiler Alert

Don't read further unless you want to see the solutions to the exercises.

Solutions to the Exercises

1. Suppose, to the contrary, that $q^2 = 2$ for some rational number q. As a rational number, q can be written as a/b, where a and b are integers, $b \neq 0$, and a and b have no nontrivial common factors. In particular, a and b can be chosen so that at most one of them is even.

   Now if $(a/b)^2 = 2$, then $a^2/b^2 = 2$, so $a^2 = 2b^2$. Thus a^2 is even. Now a cannot be odd, since the square of an odd number is odd; therefore, a must be even, i.e., $a = 2c$ for some integer c. Substituting $2c$ for a, we see that $(2c)^2 = 2b^2$, or $b^2 = 2c^2$. Thus b^2 is even, and so b is even. We find that a and b are both even, in contradiction with our earlier choice of a and b. Thus no such q can possibly exist.

2. Let θ be an element that satisfies $\theta^2 = 3$. It can be shown that θ is irrational. Adding, subtracting and multiplying with rational numbers will always yield numbers of the form $a + b\theta$, with $a, b \in \mathbb{Q}$. Higher powers of θ do not arise. Let a, b, c, d be rational numbers. Then clearly

$$(a + b\theta) \pm (c + d\theta) = (a \pm c) + (b \pm d)\theta,$$

   and

$$\begin{aligned}(a + b\theta)(c + d\theta) &= ac + bd\theta^2 + (bc + ad)\theta \\ &= ac + 3bd + (bc + ad)\theta.\end{aligned}$$

   Finally, provided c and d are not both zero,

$$\begin{aligned}\frac{a + b\theta}{c + d\theta} &= \frac{(a + b\theta)(c - d\theta)}{(c + d\theta)(c - d\theta)} \\ &= \frac{(ac - 3bd) + (bc - ad)\theta}{c^2 - 3d^2}.\end{aligned}$$

   Now the denominator cannot be zero (otherwise θ would be rational), and hence we have a well-defined division operation.

3. Let $f(x) = a_0 + a_1 x + \ldots + a_d x^d$ be a polynomial with complex coefficients. We will write $f(x) \equiv 0$ if $f(x)$ is the zero polynomial (i.e., the polynomial with only zero coefficients). Unless $f(x) \equiv 0$, we will let $\deg f(x)$ denote the *degree* of polynomial $f(x)$, i.e., the largest value of i such that $a_i \neq 0$.

   First, let us show that if $f(x_0) = 0$ (and $f(x) \not\equiv 0$) then $f(x) = q(x)(x - x_0)$ for some polynomial $q(x)$. We will do this via the "division algorithm" for polynomials. Let $f(x)$ and $g(x)$ be two polynomials. Unless $g(x) \equiv 0$, using polynomial long division it is always possible to find polynomials $q(x)$ (the so-called "quotient") and $r(x)$ (the so-called "remainder") such that $f(x) = q(x)g(x) + r(x)$ with $r(x) \equiv 0$ or $\deg r(x) <$

$\deg g(x)$. Note that the remainder is either identically zero, or has a degree *strictly smaller* than that of $g(x)$.

Now, let $f(x)$ be a polynomial with coefficients in $\mathbb{C}$, and suppose $f(x_0) = 0$ for some $x_0 \in \mathbb{C}$. Let $g(x) = x - x_0$. Note that $\deg g(x) = 1$. By the division property described above, we can find $q(x)$ and $r(x)$ such that

$$f(x) = q(x)(x - x_0) + r(x),$$

with $r(x) \equiv 0$ or $\deg r(x) < 1$. In either case $r(x) = a_0$ for some complex number a_0. However, evaluating $f(x)$ at $x = x_0$ yields $f(x_0) = 0 = q(x_0)(x_0 - x_0) + r(x_0) = r(x_0)$, so $a_0 = 0$. This shows that $f(x) = q(x)(x - x_0)$. If $f(x)$ has degree $d > 0$, then $q(x)$ must have degree $d - 1$ (since degrees add under polynomial multiplication).

Now if $f(x)$ is a polynomial over $\mathbb{C}$ of degree one, then the claimed property is obviously true, since $f(x) = f(x)$ is a "factorization" of $f(x)$ into a "product" of degree-one polynomials. Now assume, for $d \geq 1$, that every polynomial of degree d over $\mathbb{C}$ factors as a product of d degree-one polynomials. Let $f(x)$ be a polynomial of degree $d + 1$ over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, $f(x)$ has a zero, i.e., $f(x_0) = 0$ for some $x_0 \in \mathbb{C}$. By our previous result, this implies that

$$f(x) = q(x)(x - x_0) \tag{6}$$

for some polynomial $q(x)$ of degree d. By our hypothesis, however, $q(x)$ itself factors as a product of d degree-one polynomials. Substituting this factorization for $q(x)$ in (6), we obtain a factorization of $f(x)$ as a product of $d + 1$ degree-one polynomials.

Since the claimed property holds for all degree-one polynomials, and we have just shown that whenever the property holds for all polynomials of degree $d \geq 1$ it also holds for all polynomials of degree $d + 1$, it follows from the Principle of Mathematical Induction that the property holds for all $d \geq 1$.

4. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$ be a polynomial with $a_0, a_1, \ldots, a_d \in \mathbb{R}$. Note that $[f(x)]^*$, the complex conjugate of $f(x)$ can be written as

$$\begin{aligned} [f(x)]^* &= (a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d)^* \\ &= a_0^* + a_1^* x^* + a_2^* (x^2)^* + \cdots + a_d^* (x^d)^* \\ &= a_0 + a_1 x^* + a_2 (x^*)^2 + \cdots + a_d (x^*)^d \\ &= f(x^*), \end{aligned}$$

where, in the second last equality we have used the property that $a_i^* = a_i$ and $(x^i)^* = (x^*)^i$. Thus if $f(z) = 0$ for some z, then $f(z^*) = [f(z)]^* = 0$ also.

5. For all $z \in \mathbb{C}$ we have $e^{z+2\pi \mathsf{j}} = e^z e^{2\pi \mathsf{j}} = e^z$, since $e^{2\pi \mathsf{j}} = \cos(2\pi) + \mathsf{j}\sin(2\pi) = 1$.

6. We have $(\cos\theta + \mathsf{j}\sin\theta)^n = (e^{\mathsf{j}\theta})^n = e^{\mathsf{j}n\theta} = \cos(n\theta) + \mathsf{j}\sin(n\theta)$.

7. From De Moivre's formula, we have

$$\cos(3\theta) + \mathsf{j}\sin(3\theta) = (\cos\theta + \mathsf{j}\sin\theta)^3$$
$$\cos^3(\theta) + 3\cos^2(\theta)\,\mathsf{j}\sin(\theta) + 3\cos(\theta)\,\mathsf{j}^2\sin^2(\theta) + \mathsf{j}^3\sin^3(\theta)$$
$$= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + \mathsf{j}[3\cos^2(\theta)\sin($$

   Equating real and imaginary components yields the desired identities.

8. Let $z = re^{\mathsf{j}\theta}$ with $r > 0$. Then $z = e^{\ln r}e^{\mathsf{j}\theta} = e^{\ln r + \mathsf{j}\theta}$, which shows that z is in the range of the complex exponential. On the other hand, if $a + b\mathsf{j}$ is a complex value with $a, b \in \mathbb{R}$, then $e^{a+b\mathsf{j}} = e^a e^{\mathsf{j}b}$ has magnitude e^a, which is nonzero. Thus 0, which has magnitude 0, is *not* in the range of the complex exponential.