Summary of Fourier Transform Properties

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1 Definition and Some Technicalities

When it exists, the *Fourier transform* of a complex-valued signal $g(t)$ is the complex-valued function $G(f)$ defined via

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-\mathrm{j}\, 2\pi f t)\, \mathrm{d}t, \tag{1}$$

where $\mathrm{j} = \sqrt{-1}$ denotes the imaginary unit, and f is any real number. Note that both t and f are real-valued, i.e., $g : \mathbb{R} \to \mathbb{C}$ and also $G : \mathbb{R} \to \mathbb{C}$. Thus G, like g, is a complex-valued signal. Note that, although $g(t)$ may be complex-valued in general, the integral in (1) is not a contour integral as you may have encountered in complex analysis; here, the variable t is real-valued, and the integral can be decomposed into a sum of two ordinary real-valued integrals: one for the real part of the integrand, and one for the imaginary part.

To indicate that $G(f)$ is the Fourier transform of $g(t)$, we sometimes write $G(f) = \mathcal{F}[g(t)]$, or $g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f)$, or $g(t) \rightleftharpoons G(f)$. In the latter notation, the reverse arrow is intended to indicate that $g(t)$ can in fact be recovered, almost everywhere, from $G(f)$ via an inverse Fourier transform operation; see (2).

The integral sign in (1) actually represents quite a complicated mathematical operation. For one thing, the infinite range of integration indicates the need to take limits. To admit a large class of functions $g(t)$ for which the Fourier transform exists, one usually interprets the integral as a Lebesgue integral. The Lebesgue integral strictly generalizes the Riemann integral (the notion of integration taught in calculus): whenever the Riemann integral exists, the Lebesgue integral exists and gives the same value. However, there are functions which are not Riemann integrable, but *are* Lebesgue integrable. Now, the development of the Lebesgue integral is beyond the scope of these notes; for its application in communications, the interested reader is invited to consult the textbook of A. Lapidoth [1]. For our purposes in this course, we will need just one concept from Lebesgue integration theory: the concept of a subset of $\mathbb{R}$ having zero measure, with which we can define the notion of equality "almost everywhere" of two signals.

A *measure* on the real line $\mathbb{R}$ is a function that associates to certain subsets[1] of $\mathbb{R}$ a real, non-negative, number (or ∞). The measurable subsets (those whose measure is defined) must include the empty set, they must be closed under complement, and they must be closed under countable unions and countable intersections; technically, these subsets form a so-called σ-algebra. The empty set has measure zero, and the measure of the union of any countable collection of pairwise disjoint sets is the sum of their measures. Probability measures on $\mathbb{R}$ form one particularly important class of examples with which the reader is undoubtedly familiar.

Now, the *Lebesgue measure* $\mu([a, b])$ associated with a closed finite interval $[a, b]$ is simply its *length*, $b-a$. Henceforth, when we refer to "measure," we will mean Lebesgue measure. The measure of a finite union of disjoint finite intervals is the sum of their measures, i.e., when $a_1 \leq b_1 < a_2 \leq b_2 < a_3 \cdots \leq b_{n-1} < a_n \leq b_n$,

$$\mu([a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n]) = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_n - a_n).$$

As already noted, this notion of additivity is required to extend to *countably* many disjoint finite intervals: if, for $i = 1, 2, 3, \ldots$, I_i is a finite interval, and if, whenever $i \neq j$ we have $I_i \cap I_j = \emptyset$ (i.e., the intervals are pairwise disjoint), then

$$\mu\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} \mu(I_i).$$

If the right-hand side diverges, then the measure of the union is ∞.

If E_1 and E_2 are measurable sets with $E_1 \subseteq E_2$, then we must have $\mu(E_1) \leq \mu(E_2)$. If $E_1, E_2, \ldots$ is any countable collection of measurable sets (not necessarily disjoint), we must have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

A set $S \subseteq \mathbb{R}$ is said to be a *set of measure zero* if (a) μ is defined on S and (b) $\mu(S) = 0$. For example, $\mu(\{a\}) = \mu([a, a]) = a - a = 0$, i.e., every singleton set is a set of measure zero. From the additivity property for measures, it follows that every finite or countably infinite subset of $\mathbb{R}$ must also be a set of measure zero. For example, the rational numbers $\mathbb{Q}$ are a

[1] but not necessarily to *all* subsets

set of measure zero. Strangely enough, it is also possible to construct uncountable subsets of $\mathbb{R}$ that have zero measure[2], but for the purposes of intuition, it often suffices to think of a set of measure zero as being finite or countably infinite. Now, if S is a set of measure zero, we will require that *every* subset of S is measurable, necessarily with a measure of zero. Of course, in particular, the empty set has measure zero.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be equal to zero *almost everywhere* (abbreviated "ae") if $f(x) = 0$ for every real number x, except possibly for x contained in a set of measure zero. When this is the case, we will write $f(x) = 0$ ae or $f(x) \overset{\mathsf{ae}}{=} 0$. For example, the function

$$\mathbb{I}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & \text{otherwise} \end{cases}$$

is equal to zero almost everywhere, since it is nonzero only on rational values of x, and the rational numbers are a set of measure zero. It is easy to show that the product of any function with a function that is equal to zero almost everywhere is another function that is equal to zero almost everywhere, i.e., if $f(x)$ is any function, and if $z(x) \overset{\mathsf{ae}}{=} 0$, then $f(x)z(x) \overset{\mathsf{ae}}{=} 0$.

Two functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are said to be *equal almost everywhere* if $f_1(x) - f_2(x) \overset{\mathsf{ae}}{=} 0$. In other words, two real-valued functions $f_1(x)$ and $f_2(x)$ are equal almost everywhere if $\{x : f_1(x) \neq f_2(x)\}$ is a set of measure zero. When this is the case, we will write $f_1(x) = f_2(x)$ ae or $f_1(x) \overset{\mathsf{ae}}{=} f_2(x)$.

The relation $\overset{\mathsf{ae}}{=}$ ("equal almost everywhere") between two functions is in fact an equivalence relation, i.e., a relation that is reflexive, symmetric, and transitive. Indeed, this relation is obviously reflexive ($f(x) \overset{\mathsf{ae}}{=} f(x)$ for any $f(x)$) and symmetric (if $f(x) \overset{\mathsf{ae}}{=} g(x)$ then $g(x) \overset{\mathsf{ae}}{=} f(x)$). To show transitivity, suppose that $f(x) \overset{\mathsf{ae}}{=} g(x)$ and $g(x) \overset{\mathsf{ae}}{=} h(x)$. Let $A = \{x : f(x) \neq g(x)\}$, let $B = \{x : g(x) \neq h(x)\}$, and let $C = \{x : f(x) \neq h(x)\}$. By definition, A and B are sets of measure zero; furthermore, $A \cup B$ is measurable, and since $\mu(A \cup B) \leq \mu(A) + \mu(B) = 0$, we see that $A \cup B$ is itself a set of measure zero. Finally, observe that $C \subseteq A \cup B$. Since C is a subset of a set (namely, $A \cup B$) of measure zero, we have that C itself is a set of measure zero, which implies that $f(x) \overset{\mathsf{ae}}{=} h(x)$. This shows that $\overset{\mathsf{ae}}{=}$ is indeed transitive.

A significant result of Lebesgue integration theory states that, if f_1 and f_2 are Lebesgue integrable functions, then

$$\int_a^b f_1(x)\,\mathrm{d}x = \int_a^b f_2(x)\,\mathrm{d}x \text{ for all } a < b$$

if and only if $f_1(x) \overset{\mathsf{ae}}{=} f_2(x)$. In other words, the integrals of the two functions agree (on all possible intervals) if and only if the two functions are equal almost everywhere. Stated

[2]One such example is the so-called "Cantor set," containing numbers in the interval $[0, 1]$ whose ternary expansion does not contain the digit "1".

another way: the Lebesgue integral does not (in fact, it cannot) distinguish between two integrands in the same equivalence class. For example, for any $a < b$, we have

$$\int_a^b \mathbb{I}_{\mathbb{Q}}(x)\,\mathrm{d}x = \int_a^b 0\,\mathrm{d}x = 0.$$

Interestingly, though the function $\mathbb{I}_{\mathbb{Q}}(x)$ is not Riemann integrable, it *is* Lebesgue integrable (its integral over any finite interval is zero).

Now, let us return to Fourier transforms. Suppose that $g(t)$ has Fourier transform $G(f)$, and that $h(t) = g(t)$ a.e. Then we have that $h(t) = g(t) + z(t)$ where $z(t) \overset{\mathsf{ae}}{=} 0$. Since the Fourier transform of $z(t)$ is $Z(f) = 0$, it follows that $H(f) = G(f) + Z(f) = G(f)$. In other words, if two functions are equal almost everywhere, then they have the same Fourier transform.

It is possible to recover a signal $g(t)$ whose Fourier transform is $G(f)$ via the *inverse Fourier transform*:

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(\mathrm{j}\, 2\pi f t)\,\mathrm{d}f. \tag{2}$$

We sometimes write $g(t) = \mathcal{F}^{-1}[G(f)]$. Note that it is not necessary for $\mathcal{F}^{-1}[\mathcal{F}[g(t)]]$ to return $g(t)$; however, the inverse Fourier transform of the Fourier transform of $g(t)$ will return a function that is equal to $g(t)$ almost everywhere. In other words,

$$\mathcal{F}^{-1}[\mathcal{F}[g(t)]] \overset{\mathsf{ae}}{=} g(t).$$

For all engineering applications, two signals that are equal almost everywhere are indeed indistinguishable by any physical system. Thus, in defining, say, the unit rectangle function, we can define

$$\mathrm{rect}_1(t) = \begin{cases} 1 & -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \mathrm{rect}_2(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \mathrm{rect}_3(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ \frac{1}{2} & t \in \{\pm 1/2\} \\ 0 & \text{otherwise} \end{cases}$$

(or any one of many other possible definitions) as is convenient, since these functions are indeed equal almost everywhere.

The reader should note that many textbook authors (including the authors of the recommended text for this course) do not carefully distinguish between '$=$' and '$\overset{\mathsf{ae}}{=}$' when comparing two functions. The reader must therefore take care to understand which sense of "equality" is intended.

2 Properties of the Fourier Transform

Of course the Fourier transform enjoys many interesting and useful properties. We summarize the main ones here. Proofs are given in the textbook. As mentioned already, $\mathrm{j} = \sqrt{-1}$ denotes the purely imaginary unit. If z is a complex number, then z^* denotes its complex conjugate.

2.1 Symmetries

The following are equivalent:

$$\begin{array}{rclcrcl} g(t) & \rightleftharpoons & G(f) & \qquad & g(-t) & \rightleftharpoons & G(-f) \\ g^*(t) & \rightleftharpoons & G^*(-f) & & g^*(-t) & \rightleftharpoons & G^*(f) \\ G(t) & \rightleftharpoons & g(-f)\dagger & & G(-t) & \rightleftharpoons & g(f) \\ G^*(t) & \rightleftharpoons & g^*(f) & & G^*(-t) & \rightleftharpoons & g^*(-f) \end{array}$$

† this relation is known as time-frequency duality.

Thus $g(t) = g^*(t)$ if and only if $G(f) = G^*(-f)$, i.e., a signal is real-valued if and only if its Fourier transform exhibits Hermitian symmetry.

Furthermore $g(t) = g^*(t) = g(-t)$ (i.e., $g(t)$ is real and even) if and only if $G(f) = G^*(-f) = G(-f)$ (i.e., $G(f)$ is purely real and even).

Finally $g(t) = g^*(t) = -g(-t)$ (i.e., $g(t)$ is real and odd) if and only if $G(f) = G^*(-f) = -G(-f)$ (i.e., $G(f)$ is purely imaginary and odd).

2.2 Time Dilation

Suppose $g(t) \rightleftharpoons G(f)$. If a is real and nonzero, then

$$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

In particular, $g(-t) \rightleftharpoons G(-f)$.

2.3 Linearity

Suppose $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then, for all scalars $a \in \mathbb{C}$,

$$a g_1(t) + g_2(t) \rightleftharpoons a G_1(f) + G_2(f).$$

2.4 Time-shifting

If $g(t) \rightleftharpoons G(f)$, then for all $t_0 \in \mathbb{R}$, $g(t - t_0) \rightleftharpoons \exp(-\mathrm{j}\, 2\pi f t_0) G(f)$.

2.5 Modulation

If $g(t) \rightleftharpoons G(f)$, then for all $f_0 \in \mathbb{R}$, $\exp(\mathrm{j}\,2\pi f_0 t) g(t) \rightleftharpoons G(f - f_0)$.

2.6 Area Property

If $g(t) \rightleftharpoons G(f)$, then

$$\int_{-\infty}^{\infty} g(t)\,\mathrm{d}t = G(0) \quad \text{and} \quad g(0) = \int_{-\infty}^{\infty} G(f)\,\mathrm{d}f.$$

2.7 Derivatives

If $g(t) \rightleftharpoons G(f)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \rightleftharpoons \mathrm{j}\,2\pi f \cdot G(f)$$

and

$$-\mathrm{j}\,2\pi t \cdot g(t) \rightleftharpoons \frac{\mathrm{d}}{\mathrm{d}f} G(f).$$

2.8 Convolution

The *convolution* of functions g and h is the function $f = g \star h$ defined by

$$f(x) = \int_{-\infty}^{\infty} g(y) h(x - y)\,\mathrm{d}y.$$

We will write $(g \star h)(x)$ to denote the particular value that the function f takes at value x.

If $g(t) \rightleftharpoons G(f)$ and $h(t) \rightleftharpoons H(f)$, then

$$(g * h)(t) \rightleftharpoons G(f) \cdot H(f)$$

and

$$g(t) \cdot h(t) \rightleftharpoons (G \star H)(f).$$

2.9 Time Correlation

If $x(t) \rightleftharpoons X(f)$ and $y(t) \rightleftharpoons Y(f)$, then

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t - \tau)\,\mathrm{d}\tau \rightleftharpoons X(f) Y^*(f).$$

2.10 Rayleigh's Energy Theorem (Parseval's Theorem)

$$\int_{-\infty}^{\infty} |g(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} |G(f)|^2 \, \mathrm{d}f.$$

3 The Dirac Delta

It is useful to introduce a "function" $\delta(f)$, the *Dirac delta*, that plays the role of the Fourier transform of the signal $g(t) = 1$, i.e.,

$$1 \rightleftharpoons \delta(f).$$

Then, since $g(t) \cdot 1 = g(t)$, we must have, for all f,

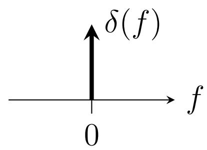
$$G(f) = (G \star \delta)(f) = \int_{-\infty}^{\infty} G(\lambda)\delta(f-\lambda) \, \mathrm{d}\lambda = \int_{-\infty}^{\infty} \delta(\lambda) G(f-\lambda) \, \mathrm{d}\lambda.$$

Now let $G(f) = \mathrm{rect}(f/a)$. Then

$$\int_{\infty}^{\infty} \delta(\lambda) \, \mathrm{rect}((f-\lambda)/a) \, \mathrm{d}\lambda = \int_{f-a/2}^{f+a/2} \delta(\lambda) \, \mathrm{d}\lambda = \mathrm{rect}(f/a).$$

As long as $|f| > a/2$ this integral is zero, which implies, for any pair of numbers b and c, $b < c$, with $c < 0$ or $b > 0$, $\int_b^c \delta(\lambda) \, \mathrm{d}\lambda = 0$, whereas if $b < 0 < c$, $\int_b^c \delta(\lambda) \, \mathrm{d}\lambda = 1$. Thus we may think of a Dirac delta $\delta(f)$ as a unit area "function" that is zero (almost) everywhere, except at $f = 0$, as depicted below.

It is important *not* to confuse the Dirac delta with a function that is equal to zero almost everywhere. In fact, the Dirac delta is not a "function;" as no function exists that is equal to zero almost everywhere, yet has unit integral. A Dirac delta is really just a convenient book-keeping device, which is nicely behaved (and has a well-defined meaning) only under convolution (with functions). A mathematically rigorous approach to the Dirac delta is given by the theory of distributions; see e.g., [2].



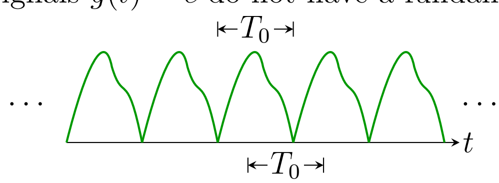
4 Fourier Transform Pairs

$$
\begin{aligned}
1 &\rightleftharpoons \delta(f) \\
\delta(t) &\rightleftharpoons 1 \\
\delta(t - t_0) &\rightleftharpoons \exp(-\mathrm{j}\,2\pi f t_0) && t_0 \in \mathbb{R} \\
\exp(\mathrm{j}\,2\pi f_0 t) &\rightleftharpoons \delta(f - f_0) && f_0 \in \mathbb{R} \\
\exp(-at)u(t) &\rightleftharpoons \frac{1}{a + \mathrm{j}\,2\pi f} && 0 < a \in \mathbb{R} \\
\operatorname{rect}(t) &\rightleftharpoons \operatorname{sinc}(f) \\
\cos(2\pi f_0 t) &\rightleftharpoons \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) && f_0 \in \mathbb{R} \\
\sin(2\pi f_0 t) &\rightleftharpoons \frac{1}{2\mathrm{j}}\delta(f - f_0) - \frac{1}{2\mathrm{j}}\delta(f + f_0) && f_0 \in \mathbb{R} \\
\operatorname{sgn}(t) &\rightleftharpoons \frac{1}{\mathrm{j}\,\pi f} \\
\frac{1}{\pi t} &\rightleftharpoons -\mathrm{j}\operatorname{sgn}(f) \\
u(t) &\rightleftharpoons \frac{1}{\mathrm{j}\,2\pi f} + \frac{1}{2}\delta(f)
\end{aligned}
$$

5 Periodic Signals and Fourier Series

A signal $g(t)$ is *periodic* with period $T > 0$ if, for all $t \in \mathbb{R}$, $g(t + T) = g(t)$. The smallest period of a periodic signal is called the *fundamental period*, T_0. The reciprocal $f_0 = 1/T_0$ is called the *fundamental frequency*.

Every period is an integer multiple of the fundamental period (if it exists). Among periodic signals, only the constant signals $g(t) = c$ do not have a fundamental period.



If $g(t)$ is periodic with period T, then $g(t)$ has complex-exponential Fourier series

$$g(t) = \sum_{k=-\infty}^{\infty} c_k \exp(\mathrm{j}\,2\pi k t / T)$$

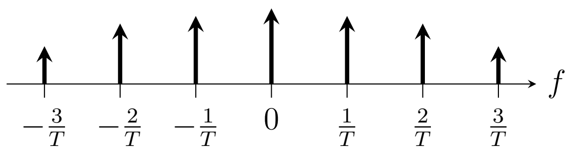
where

$$c_k = \frac{1}{T}\int_T g(t)\exp(-\mathrm{j}\,2\pi kt/T)\,\mathrm{d}t,$$

where $\int_T$ is the integral over any interval of length T. Then

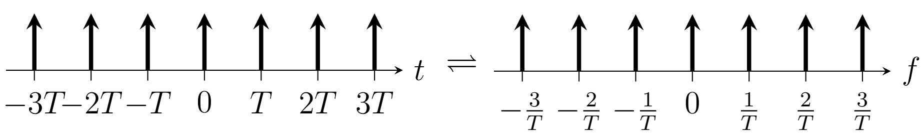
$$G(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T}\right),$$

a *line spectrum.*



The following Fourier transform pair is sometimes called "the picket-fence miracle;" the Fourier transform of a delta-function train (a "picket fence") is another such function (with the reciprocal spacing between pickets). More precisely, for any $T > 0$,

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \rightleftharpoons \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$



This Fourier transform pair is useful for understanding the *sampling theorem.*

6 Energy, Correlation, Orthogonality

The *energy* of a signal $g(t)$ is

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2\,\mathrm{d}t,$$

when this integral exists. A signal of finite energy is called an *energy signal.*

The *correlation* between two signals $g_1(t)$ and $g_2(t)$ is

$$\langle g_1(t), g_2(t)\rangle = \int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,\mathrm{d}t. \quad \text{(Note the complex conjugate.)}$$

Evidently,

$$E_g = \langle g(t), g(t)\rangle.$$

Two signals $g_1(t)$ and $g_2(t)$ are said to be *orthogonal*, written $g_1(t) \perp g_2(t)$, if $\langle g_1(t), g_2(t)\rangle = 0$.

If $g_1(t) \perp g_2(t)$ then

$$\begin{aligned} E_{g_1+g_2} &= \langle g_1(t) + g_2(t), g_1(t) + g_2(t)\rangle \\ &= \underbrace{\langle g_1(t), g_1(t)\rangle}_{E_{g_1}} + \underbrace{\langle g_1(t), g_2(t)\rangle}_{0} + \underbrace{\langle g_2(t), g_1(t)\rangle}_{0} + \underbrace{\langle g_2(t), g_2(t)\rangle}_{E_{g_2}} \\ &= E_{g_1} + E_{g_2} \end{aligned}$$

(and conversely).

The *Cauchy–Schwarz inequality* states that every pair $g_1(t)$ and $g_2(t)$ of energy signals satisfies

$$|\langle g_1(t), g_2(t)\rangle|^2 \leq \langle g_1(t), g_1(t)\rangle \cdot \langle g_2(t), g_2(t)\rangle,$$

where equality holds if and only if $g_2(t) = a g_1(t)$ for some $a \in \mathbb{C}$.

Thus, for any two energy signals $g_1(t)$ and $g_2(t)$,

$$\left| \int_{-\infty}^{\infty} g_1(t) g_2^*(t)\,\mathrm{d}t \right|^2 \leq \int_{-\infty}^{\infty} |g_1(t)|^2\,\mathrm{d}t \cdot \int_{-\infty}^{\infty} |g_2(t)|^2\,\mathrm{d}t.$$

Let $g(t) \rightleftharpoons G(f)$. Since $E_g = \int_{-\infty}^{\infty} |g(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |G(f)|^2\,\mathrm{d}f$, we define the *energy spectral density* of $g(t)$ as

$$\Psi_g(f) = |G(f)|^2 = G(f)G^*(f).$$

The inverse Fourier transform of $\Psi_g(f)$ is the (deterministic) *autocorrelation function*

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t) g^*(t - \tau)\,\mathrm{d}t = g(\tau) * g^*(-\tau).$$

Note, this is a function of the lag variable τ. Note that $E_g = R_g(0)$.

If $x(t)$ is applied to a linear time-invariant system with impulse response $h(t)$, the output signal is $y(t) = x(t) \star h(t)$.

Note that

$$\begin{aligned} R_y(\tau) &= y(\tau) \star y^*(-\tau) \\ &= x(\tau) \star h(\tau) \star (x(-\tau) * h(-\tau))^* \\ &= (x(\tau) \star x^*(-\tau)) \star (h(\tau) \star h^*(-\tau)) \\ &= R_x(\tau) \star R_h(\tau) \end{aligned}$$

If follows that

$$\Psi_y(f) = \Psi_x(f)|H(f)|^2.$$

The *power* of a signal $g(t)$ is defined as

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \,\mathrm{d}t,$$

when this integral exists. The power of an energy signal is zero. A signal of finite nonzero power is called a *power signal*.

Let $g_T(t) = g(t) \cdot rect(t/T)$ and suppose $g_T(t) \rightleftharpoons G_T(f)$. Then

$$\begin{aligned} P_g &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |g_T(t)|^2 \,\mathrm{d}t \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |G_T(f)|^2 \,\mathrm{d}f. \end{aligned}$$

The (deterministic) *power spectral density* of a power signal $g(t)$ is defined as

$$S_g(f) = \lim_{T\to\infty} \frac{1}{T} |G_T(f)|^2.$$

Acknowledgements

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References

[1] Amos Lapidoth, *A Foundation in Digital Communication*. Cambridge University Press, 2009.

[2] Robert S. Strichartz, *A Guide to Distribution Theory and Fourier Transforms*. World Scientific, 2003.