Multiple Access Channels with Intermittent Feedback and Side Information

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Joint Work with Amos Lapidoth (ETH-Zürich)

ISIT, 2013
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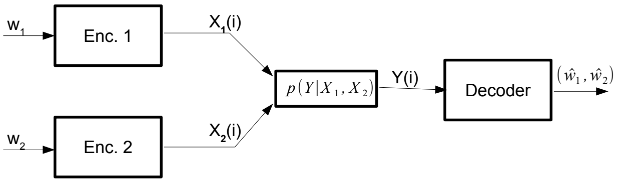
Problem Setup

Multiple Access Channel with Intermittent Feedback



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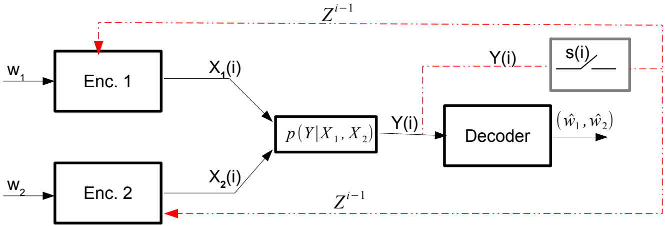
Multiple Access Channel with Intermittent Feedback



- Intermittent Feedback

$$z(i) = \begin{cases} y(i) & s(i) = 1, \\ \star & s(i) = 0. \end{cases}$$

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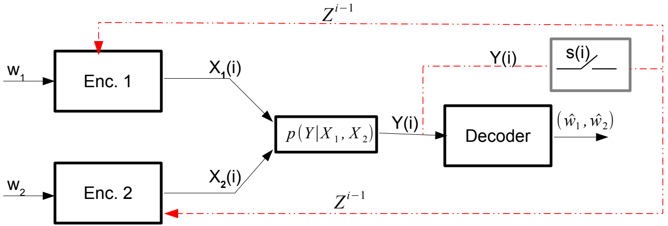

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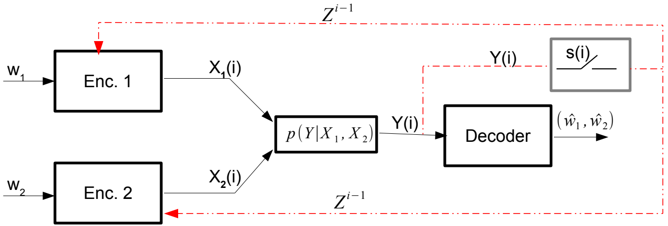

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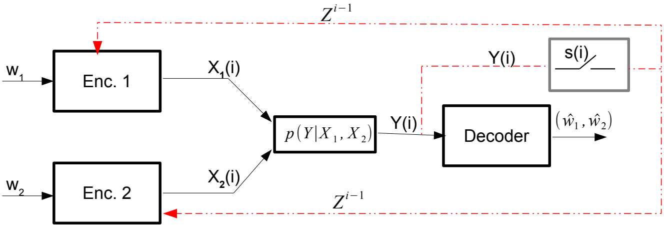
- Encoding Functions: $x_k(i) = f_k(w_k, z^{i-1}, s^{i-1})$, $k = 1, 2$
- Decoding Function: $(\hat{w}_1, \hat{w}_2) = g_n(y^n, s^n)$



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- Decoding Function: $(\hat{w}_1, \hat{w}_2) = g_n(y^n, s^n)$
- Gaussian MAC: $y(i) = x_1(i) + x_2(i) + n(i)$,
  $E[x_k^2] \leq P_k,\ n(i) \sim \mathcal{N}(0, 1)$

Sum-Rate

- No-Feedback: (Ahlswede '71, Liao '72)
- Perfect Feedback: (Ozarow '83)

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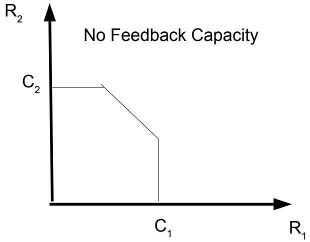

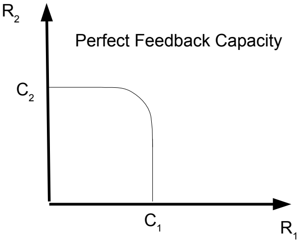

Independent Gaussian Inputs

Jointly Gaussian Inputs

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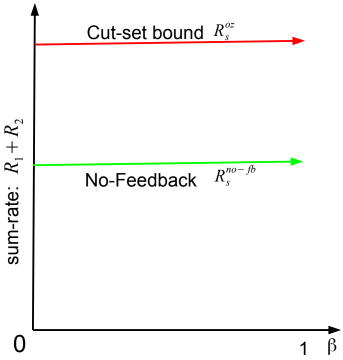

$$\beta \in (0,1): \qquad R_1 + R_2 \overset{?}{\geq} (1-\beta)\cdot R_{\text{sum}}^{\text{cut}-\text{set}} + \beta \cdot R_{\text{sum}}^{\text{no}-\text{fb}}$$

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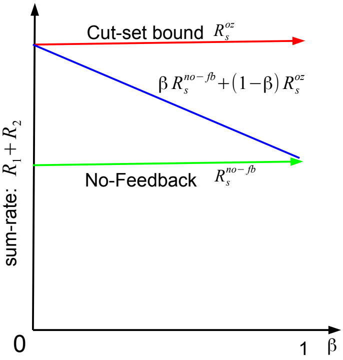

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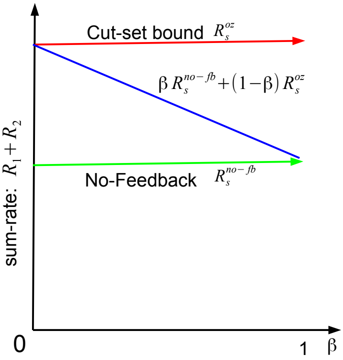

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Intermittent Feedback: Outer Bound

Theorem

Any achievable rate pair (R_1, R_2) satisfies:

$$(R_1, R_2) \in \bigcap_{\substack{\sigma_1^2 \geq 0, \sigma_2^2 \geq 0 \\ \sigma_1^2 + \sigma_2^2 = 1}} \mathcal{C}(\sigma_1^2, \sigma_2^2)$$

where $\mathcal{C}(\sigma_1^2, \sigma_2^2)$ is defined by:

$$R_1 \leq \frac{\beta}{2} \log(1 + P_1) + \frac{\bar{\beta}}{2} \log\left(1 + \frac{P_1}{\sigma_1^2}\right)$$

$$R_2 \leq \frac{\beta}{2} \log(1 + P_2) + \frac{\bar{\beta}}{2} \log\left(1 + \frac{P_2}{\sigma_2^2}\right)$$

$$R_1 + R_2 \leq \frac{\beta}{2} \log(1 + P_1 + P_2) + \frac{\bar{\beta}}{2} \left[\log\left(1 + \frac{P_1}{\sigma_1^2}\right) + \log\left(1 + \frac{P_2}{\sigma_2^2}\right)\right]$$

for every $\sigma_1^2 \geq 0, \sigma_2^2 \geq 0$ with $\sigma_1^2 + \sigma_2^2 = 1$ and $\bar{\beta} \triangleq (1 - \beta)$.

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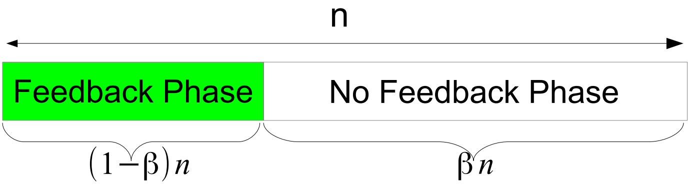
Two-Phase MAC:



Achievable Sum-Rate: $R_1 + R_2 = \bar{\beta} \cdot R_{\text{sum}}^{\text{cut}-\text{set}} + \beta \cdot R_{\text{sum}}^{\text{No}-\text{Feedback}}$

Upper Bound: $R_1 + R_2 \leq \bar{\beta} \cdot R_{\text{sum}}^{\text{parallel}} + \beta \cdot R_{\text{sum}}^{\text{No}-\text{Feedback}}$

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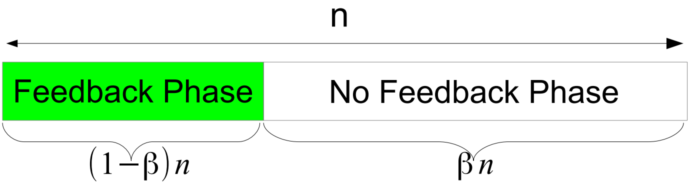

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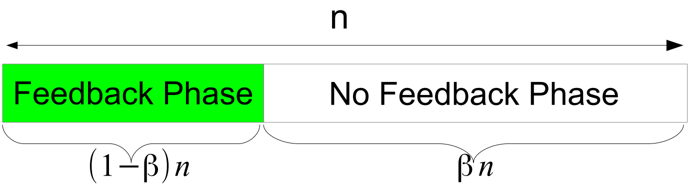
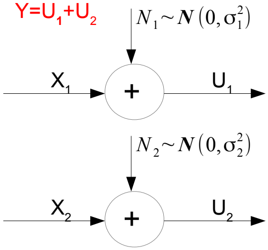
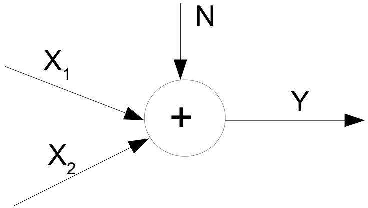


Feedback Phase



No-Feedback Phase



Intermittent Feedback: Outer Bound

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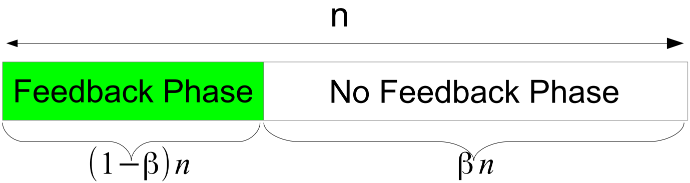

Lemma (Conditional Independence Lemma - Two Phase Channel)

Let $m = (1-\beta)n$. For any coding scheme

- $x_k(i) = f_{k,i}(w_k, u_{1,1}^{i-1}, u_{2,1}^{i-1})$, *for* $i = 1, 2, \ldots, m$
- $x_k(i) = f_{k,i}(w_k, u_{1,1}^{m}, u_{2,1}^{m})$, *for* $i = m+1, \ldots, n$

Then $x_1(i) \leftrightarrow (u_1^m, u_2^m) \leftrightarrow x_2(i)$, *for* $i \in \{m+1 \ldots, n\}$ *holds.*

See also: Willems: ('82), ('83), ('85), Bracher et. al. ('12)

Intermittent Feedback: Outer Bound

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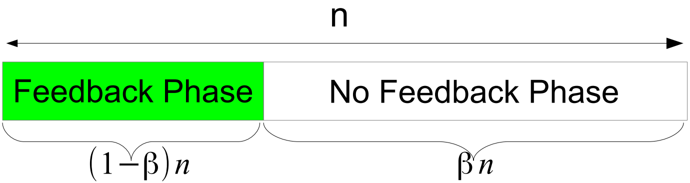

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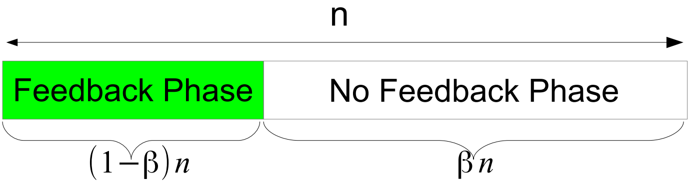

Outer Bound for the Two-Phase Channel.

$$R_1 \leq \frac{\bar{\beta}}{2}\log\left(1+\frac{P_1^F}{\sigma_1^2}\right)+\frac{\beta}{2}\log\left(1+P_1^{NF}\right)$$

$$R_2 \leq \frac{\bar{\beta}}{2}\log\left(1+\frac{P_2^F}{\sigma_1^2}\right)+\frac{\beta}{2}\log\left(1+P_2^{NF}\right)$$

$$R_1+R_2 \leq \frac{\bar{\beta}}{2}\left\{\log\left(1+\frac{P_1^F}{\sigma_1^2}\right)+\log\left(1+\frac{P_2^F}{\sigma_2^2}\right)\right\}+\frac{\beta}{2}\log\left(1+P_1^{NF}+P_2^{NF}\right)$$

for some $P_k^F \geq 0, P_k^{NF} \geq 0$ with $\bar{\beta}P_k^F+\beta P_k^{NF} \leq P_k$.

Outer Bound — Memoryless State

$\mathsf{s}^n \sim \prod_{i=1}^{n} p_{\mathsf{s}}(s_i)$

- Strictly Causal Encoder CSI
- Preclude Power Adaptation i.e., $P_k^F = P_k^{NF} = P_k$

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State Dependent Channel Enhancement

Feedback: $s(i) = 1$

$$y(i) = z(i) = \{u_1(i), u_2(i)\}$$
$$u_1(i) = x_1(i) + n_1(i)$$
$$u_2(i) = x_2(i) + n_2(i)$$
$$n_1(i) \perp n_2(i), n_1(i) + n_2(i) = n(i)$$

Erasure: $s(i) = 0$

$$y(i) = x_1(i) + x_2(i) + n(i)$$
$$z(i) = \star$$

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Lemma (Conditional Independence Lemma - Memoryless State)

For the enhanced MAC Channel, let $\Omega(i) = (\mathsf{z}(i), \mathsf{s}(i))$. For any coding scheme $\mathsf{x}_k(i) = f_{k,i}(\mathsf{w}_k, \Omega^{i-1})$, for $i = 1, 2, \ldots, n$ we have that $\mathsf{x}_1(i) \leftrightarrow \Omega^{i-1} \leftrightarrow \mathsf{x}_2(i)$

Outer Bound — Memoryless State

$s^n \sim \prod_{i=1}^{n} p_s(s_i)$

- Strictly Causal Encoder CSI
- Preclude Power Adaptation i.e., $P_k^F = P_k^{NF} = P_k$

$$\begin{aligned} n(R_1 + R_2) &\leq I(w_1, w_2; y^n, s^n) \\ &\leq \sum_{i=1}^{n} I(x_1(i), x_2(i); y(i) | z^{i-1}, s^{i-1}, s(i)) \end{aligned} \qquad (1)$$

- $x_1(i) \leftrightarrow (z^{i-1}, s^{i-1}) \leftrightarrow x_2(i)$
- $x_1(i)$ and $x_2(i)$ are independent of $s(i)$.

Outer Bound — Memoryless State

$\mathsf{s}^n \sim \prod_{i=1}^n p_{\mathsf{s}}(s_i)$

- Strictly Causal Encoder CSI
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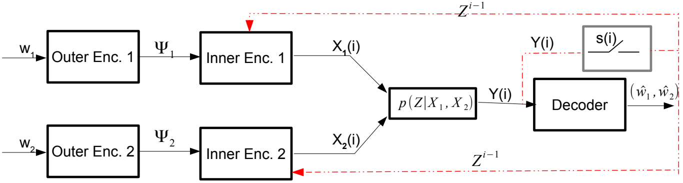
$$
\begin{aligned}
& I(\mathsf{x}_1(i), \mathsf{x}_2(i); \mathsf{y}(i) | \mathsf{z}^{i-1}, \mathsf{s}^{i-1}, \mathsf{s}(i)) \\
& = \beta I(\mathsf{x}_1(i), \mathsf{x}_2(i); \mathsf{y}_0(i) | \mathsf{z}^{i-1}, \mathsf{s}^{i-1}, \mathsf{s}(i) = 0) \\
& \qquad + \bar{\beta} I(\mathsf{x}_1(i), \mathsf{x}_2(i); \mathsf{y}_1(i) | \mathsf{z}^{i-1}, \mathsf{s}^{i-1}, \mathsf{s}(i) = 1)
\end{aligned}
$$

- $\mathsf{x}_1(i) \leftrightarrow (\mathsf{z}^{i-1}, \mathsf{s}^{i-1}) \leftrightarrow \mathsf{x}_2(i)$
- $\mathsf{x}_1(i)$ and $\mathsf{x}_2(i)$ are independent of $\mathsf{s}(i)$.
- Optimality of Gaussian Inputs

Lower Bound

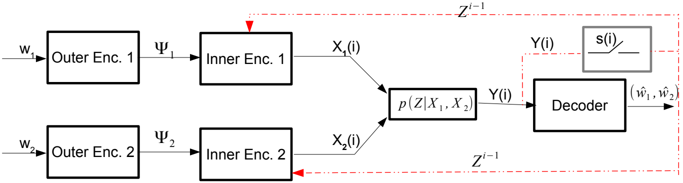
Concatenated Coding Scheme



- Outer Encoder: Length N block code

$$w_1 \to (\psi_1(1), \ldots, \psi_1(N)), w_2 \to (\psi_2(1), \ldots, \psi_2(N))$$

- Inner Encoder: Iteratively Refine $(\psi_1(j), \psi_2(j))$ until feedback symbol is erased.



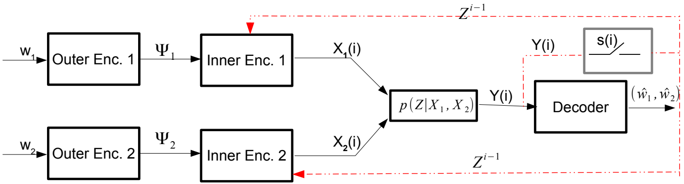
Iterative Refinement Scheme:

- Initialize: $x_1(1) = \psi_1(1)$ and $x_2(1) = \psi_2(1)$
- Until $z_i = \star$, we let

$$x_1(i) = \frac{1}{\beta_{1,i}} \left(\psi_1(1) - E\left[\psi_1(1)|z^{i-1}\right]\right), x_2(i) = \frac{(-1)^{i-1}}{\beta_{2,i}} \left(\psi_2(1) - E\left[\psi_2(1)|z^{i-1}\right]\right)$$

- If $z_i = \star$ reset to $x_1(1) = \psi_1(2)$ and $x_2(1) = \psi_2(2)$, proceed.

Lower Bound
Concatenated Coding Scheme



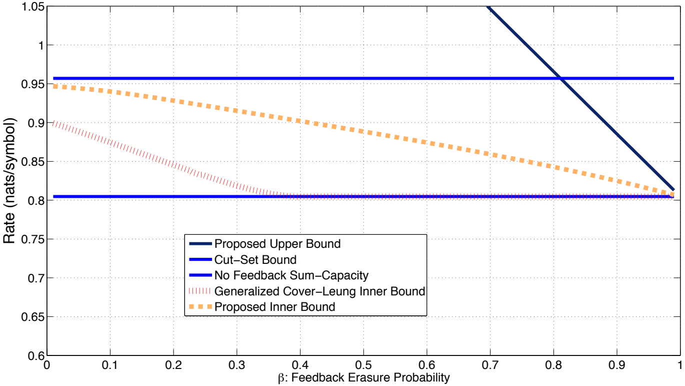
Induced Channel: $(\psi_1, \psi_2) \to (y_1, \ldots, y_T, T)$
$\Pr(T = t) = \beta(1-\beta)^{t-1}$.

$$R_1 + R_2 \leq \frac{1}{E[T]} I(\psi_1, \psi_2; y_1, \ldots, y_T | T),$$
$$R_k \leq \frac{1}{E[T]} I(\psi_k; y_1, \ldots, y_T | \psi_{\bar{k}}, T)$$

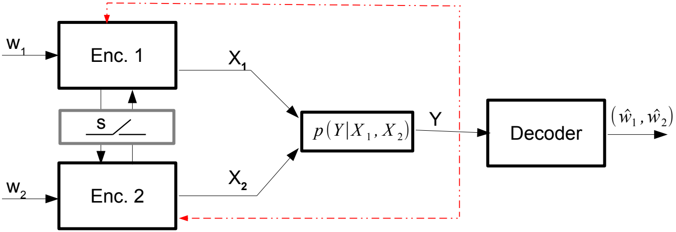
Computation based on Ozarow ('83), Lapidoth-Wigger ('10)

Numerical Comparisons

$$P_1 = P_2 = 2$$



MAC With Intermittent Cribbing + Perfect Feedback



Encoder Side Information

$$z(i) = \begin{cases} y(i), & s(i) = 0 \\ (y(i), x_1(i), x_2(i)), & s(i) = 1 \end{cases}$$

- Encoding Function: $x_k(i) = f_{k,i}(w_k, z^{i-1}, s^{i-1})$
- Decoding Function: $(\hat{w}_1, \hat{w}_2) = g_n(y^n)$

MAC With Intermittent Cribbing + Perfect Feedback

Semi-deterministic MAC (SD-MAC): $x_1 = f(y, x_2)$

Theorem

The capacity region of SD-MAC consists of all (R_1, R_2) *that satisfy:*

$$\begin{aligned} R_1 &\leq \beta I(x_1; y|u, x_2) + \bar{\beta} H(x_1|u) \\ R_2 &\leq \beta I(x_2; y|u, x_1) + \bar{\beta} H(x_2|u) \\ R_1 + R_2 &\leq I(x_1, x_2; y) \end{aligned}$$

for some u *that satisfies* $x_1 \to u \to x_2$ *and* $u \to (x_1, x_2) \to y$.

- Achievability: Superposition Block-Markov Coding
- Converse: Independence Lemma

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Extreme Cases:

- No Cribbing, Only Feedback ($\beta = 1$), Willems ('82)
- Perfect Cribbing + Feedback ($\beta = 0$), Bracher et. al ('12)

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Conclusions

MAC with Intermittent Side Information

- Gaussian MAC with Intermittent Feedback
- SD-MAC with Intermittent Cribbing + Perfect Feedback

Conclusions

MAC with Intermittent Side Information

- Gaussian MAC with Intermittent Feedback
  - Inner Bound based on Concatenated Coding + Iterative Refinement
  - Outer Bound that reduces to No-Feedback Capacity when $\beta \to 1$.
- SD-MAC with Intermittent Cribbing + Perfect Feedback
  - Capacity Region
  - Achievability based on Superposition Block Markov Coding

Conclusions

MAC with Intermittent Side Information

- Gaussian MAC with Intermittent Feedback
  - Inner Bound based on Concatenated Coding + Iterative Refinement
  - Outer Bound that reduces to No-Feedback Capacity when $\beta \to 1$.
- SD-MAC with Intermittent Cribbing + Perfect Feedback
  - Capacity Region
  - Achievability based on Superposition Block Markov Coding

Future Work

- Noisy Feedback (Gastpar-Kramer '06, Wigger-Lapidoth '10, Tandon-Ulukus '11)
- Independent Erasures
- Tighter Outer Bounds