Source Broadcasting over Erasure Channels: Distortion Bounds and Code Design

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Motivation

Setup: Broadcast to Heterogenous Users

One Source and Multiple Receivers
- Receiver $i$: Channel loss rate $\varepsilon_i$
- Receiver $i$: Required Fraction $D_i$
Joint Source-Channel Coding

- Binary Source Sequence: \( s^k \in \{0, 1\}^k \)
- Erasure Broadcast Channel: \( (\varepsilon_1 < \varepsilon_2) \)
- Bandwidth Expansion Factor: \( b = \frac{n}{k} \)
- Erasure Distortion:

\[
d(s_i, \hat{s}_i) = \begin{cases} 
0, & s_i = \hat{s}_i, \\
1, & \hat{s}_i = \star, \\
\infty, & \text{else.}
\end{cases}
\]

\[
d(s^k, \hat{s}^k) = \frac{1}{k} \sum_{i=1}^{k} d(s_i, \hat{s}_i) = D, \text{ then } (1 - D) \text{ fraction of source symbols available to the destination.}
\]
Given a degree distribution: \( P(u) = p_1 u + p_2 u^2 + \ldots + p_L u^L \)

Sample each symbol \( x_i \) as follows:

- Sample \( d \in [1, L] \) from the distribution \([p_1, p_2, \ldots, p_L]\)
- Sample \( d \) out of \( k \) symbols, \( s_{i_1}, \ldots, s_{i_d} \) and let
  \[ x_i = s_{i_1} \oplus \ldots \oplus s_{i_d} \]
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Sample each symbol \( x_i \) as follows:

1. Sample \( d \in [1, L] \) from the distribution \([p_1, p_2, \ldots, p_L]\)
2. Sample \( d \) out of \( k \) symbols, \( s_{i_1}, \ldots, s_{i_d} \) and let

\[
 x_i = s_{i_1} \oplus \ldots \oplus s_{i_d}
\]
Given a degree distribution: $P(u) = p_1 u + p_2 u^2 + \ldots + p_L u^L$

Sample each symbol $x_i$ as follows:

- Sample $d \in [1, L]$ from the distribution $[p_1, p_2, \ldots, p_L]$
- Sample $d$ out of $k$ symbols, $s_{i_1}, \ldots, s_{i_d}$ and let
  $$x_i = s_{i_1} \oplus \ldots \oplus s_{i_d}$$
Robust Soliton Distribution: Near Optimal for Lossless Recovery over all channels

Partial Recovery: Only a fraction of source symbols need to be recovered by all receivers

Optimized Degree Distribution
Gaussian Source: $s^k_i \overset{i.i.d.}{\sim} \mathcal{CN}(0, \sigma^2)$

AWGN Channel: $z^n_i \overset{i.i.d.}{\sim} \mathcal{CN}(0, N_i)$

Degradation Order: $N_2 > N_1$

Power Constraint: $E[x(i)^2] \leq P$

Quadratic Distortion Measure $d(s, \hat{s}) = (s - \hat{s})^2$

Characterize achievable pairs $(b, D_1, D_2)$. 

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For $b = 1$, uncoded transmission is optimal.

Problem Remains Open in General

Significant Prior Work:


Joint Source Channel Coding

Classical Coding Schemes

- Systematic Lossy Coding Scheme
- Mittal-Phamdo Coding Scheme
Analog Phase: \( x^k = \sqrt{\frac{P}{\sigma^2}} s^k \)

Digital Phase: \( R^{wz} = (b - 1)C_2(P) \)

\[
D_2 = \frac{\sigma^2}{2^{2bC_2(P)}}, \quad D_1 = \frac{\sigma^2}{2^{2bC_2(P)} + \left( \frac{P}{N_1} - \frac{P}{N_2} \right)}
\]
Mittal and Phamdo (2002)

- Digital Phase: \( R_q = \frac{1}{2} \log_2 \frac{\sigma^2}{D_q} = (b - 1)C_2(P) \)
- Analog Phase: \( e^k = s^k - c^k \)
- \( D_2 = \frac{\sigma^2}{2^{2bC_2(P)}} \), \( D_1 = \frac{\sigma^2}{2^{2bC_2(P)}} \left(1+\frac{P}{N_1} \right) \left(1+\frac{P}{N_2} \right) \leq D_1^{\text{systematic}} \)
Binary Source, Erasure Channel

Erasure Distortion

Point-to-Point Channel

- $s^k \in \{0,1\}^k$
- $s^k$ i.i.d. $\sim Ber(1/2)$
- Erasure Distortion: $R(D) = 1 - D$
- i.i.d. Erasure Channel: $C = 1 - \varepsilon$

Separation Theorem: $R(D) \leq b \cdot C$

$$D \geq \Delta(b, \varepsilon) = \max\{0, 1 - b(1 - \varepsilon)\}$$

$$b \geq \beta(D, \varepsilon) = \frac{1 - D}{1 - \varepsilon}, \quad 0 \leq D \leq 1$$
Split $s^k$ into two subsequences
- Transmit first $q = k \frac{D^*}{\varepsilon}$ bits uncoded
- Transmit remaining $k \left(1 - \frac{D^*}{\varepsilon}\right)$ bits at rate $1 - \varepsilon$

\[
\frac{D^*}{\varepsilon} + \frac{1 - D^*}{1 - \varepsilon} = \beta(D^*, \varepsilon)
\]
Mittal-Phamdo Coding Scheme

Achievable \((D_1, D_2)\):

\[ D_2 = \Delta(b, \varepsilon_2) = 1 - b(1 - \varepsilon_2), \quad D_1 = \frac{\varepsilon_1}{\varepsilon_2}D_2. \]
Generalized Mittal-Phamdo Scheme

Erasure Setup

- Split $s^k$ into three groups
- First $\alpha \cdot k$ symbols: transmit uncoded
- Next $\beta \cdot k$ symbols: apply rate $C_2 = (1 - \varepsilon_2)$ code
- Last $\gamma \cdot k$ symbols: apply rate $C_1 = (1 - \varepsilon_1)$ code
Generalized Mittal-Phamdo Scheme

Erasure Setup

Source: $S^k$

- Source: $s^k$
- Channel Input: $x^n$

Bandwidth expansion: $b = \alpha + \frac{\beta}{1 - \varepsilon_2} + \frac{\gamma}{1 - \varepsilon_1}$

User 1 recovery: $\alpha(1 - \varepsilon_1) + \beta + \gamma$

User 2 recovery: $\alpha(1 - \varepsilon_2) + \beta + \gamma(1 - \varepsilon_2)$
Generalized Mittal-Phamdo Scheme

Erasure Setup

Source: $s^k$

Source: $s^k$

Channel Input: $x^n$

$$\min_{\alpha, \beta, \gamma} \left\{ \alpha + \frac{\beta}{1 - \varepsilon_2} + \frac{\gamma}{1 - \varepsilon_1} \right\}$$

s.t. $\alpha + \beta + \gamma \leq 1$, $\alpha, \beta, \gamma \geq 0$,

$\alpha(1 - \varepsilon_1) + \beta + \gamma \geq 1 - D_1$,

$\alpha(1 - \varepsilon_2) + \beta + \gamma(1 - \varepsilon_2) \geq 1 - D_2$. 
Solution to LP Program

- **Case 1:** $D_2 \in [0, D_1 \frac{\varepsilon_2}{\varepsilon_1}]$
  - $b^* = \beta(D_2, \varepsilon_2)$
  - $\alpha = \frac{1-D_2-\beta}{1-\varepsilon_2}$, $\beta \in \left[1 - \frac{D_2}{\varepsilon_2}, \left(\frac{1-D_2}{1-\varepsilon_2} - \frac{1-D_1}{1-\varepsilon_1}\right) \frac{(1-\varepsilon_1)(1-\varepsilon_2)}{\varepsilon_2-\varepsilon_1}\right]$
  - $\gamma = 0$

- **Case 2:** $D_2 \in [D_1 \frac{\varepsilon_2}{\varepsilon_1}, \varepsilon_2]
  - $b^* = b^{\text{Inner}}$
  - $\alpha = \frac{D_1}{\varepsilon_1}$, $\beta = 1 - \frac{D_2}{\varepsilon_2}$, $\gamma = \frac{D_2}{\varepsilon_2} - \frac{D_1}{\varepsilon_1}$

- **Case 3:** $D_2 \in [\varepsilon_2, 1]$
  - $b = \beta(D_1, \varepsilon_1)$
  - $\alpha = \frac{1-D_1-\gamma}{1-\varepsilon_1}$, $\beta = 0$, $\gamma = \left[1 - \frac{D_1}{\varepsilon_1}, \left(\frac{1-D_1}{1-\varepsilon_1} - \frac{1-D_2}{1-\varepsilon_2}\right) \frac{(1-\varepsilon_1)}{\varepsilon_1}\right]$
Bandwidth-Distortion Tradeoff

Fix $D_1$, Find $b$ vs $D_2$

For Hamming Distortion, Improved Outer Bound: Tan-K-Soljanin ('13)
Optimization Problem for Systematic Rateless Codes

subject to:  \( \min_{b, p_1, \ldots, p_L} b \)

\[- \ln(\varepsilon_i) + \ln(1 - u) + (1 - \varepsilon_i)(b - 1)P'(u) \geq 0, \]

\( \forall u \in [0, 1 - D_i], i = 1, 2 \)
Conclusions

Summary:

- Lossy Broadcasting to Heterogenous Receivers
- JSCC Perspective involving Erasure Channels
- Generalization of Mittal-Phamdo Scheme
- Practical Code Designs

Future Work:

- Extension to more than two receivers.
- Robust Extensions
- Unequal Bandwidth
Distortion in Analog Phase: \( \varepsilon \)
Distortion in W-Z codeword: \( d(s^k, c^k) \approx \frac{D^*}{\varepsilon} \)
Overall Reconstruction Distortion: \( D^* = \Delta(b, \varepsilon) \)