Layered Error-Correction Codes for Real-Time Streaming over Erasure Channels

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Joint Work:
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Causal encoder: $x[i] = f_i(s[0], s[1], \ldots, s[i]) \in F_q^n$

Rate $= \frac{k}{n}$

Source stream
$s[i] \in F_q^k$
Source stream $s[i] \in \mathbb{F}_q^k$ → $s[0]$

Causal encoder: $x[i] = f_i(s[0], s[1], \ldots, s[i]) \in \mathbb{F}_q^n$

Rate = $\frac{k}{n}$

$x[0]$
Real-Time Communication System

Source stream $s[i] \in \mathbb{F}_q^k$

Causal encoder: $x[i] = f_i(s[0], s[1], \ldots, s[i]) \in \mathbb{F}_q^n$

Rate $= \frac{k}{n}$
Causal encoder: $x[i] = f[i](s[0], s[1], \ldots, s[i]) \in F_q^n$
Rate = $\frac{k}{n}$
Real-Time Communication System

Source stream $s[i] \in F_q^k$

Causal encoder: $x[i] = f_i(s[0], s[1], \ldots, s[i]) \in F_q^n$

Rate $= \frac{k}{n}$
Real-Time Communication System

Source stream $s[i] \in \mathbb{F}_q^k$  

Causal encoder: $x[i] = f_i(s[0], s[1], \ldots, s[i]) \in \mathbb{F}_q^n$  
Rate $= \frac{k}{n}$
Real-Time Communication System

Source stream $s[i] \in \mathbb{F}_q^k$  

\[ x[i] = f_i(s[0], s[1], \ldots, s[i]) \in \mathbb{F}_q^n \]

Rate $= \frac{k}{n}$
Real-Time Communication System

Source stream
\[ s[i] \in \mathbb{F}_q^k \]

Causal encoder:
\[ x[i] = f_i(s[0], s[1], \ldots, s[i]) \in \mathbb{F}_q^n \]
Rate: \[ \frac{k}{n} \]

Packet Erasure Channel
\[ y[i] = \ast \text{ for packet erasure; otherwise } y[i] = x[i] \]

\[ y[0] \quad y[1] \quad y[2] \quad \ldots \quad y[i] \quad \ldots \quad y[T-1] \quad y[T] \quad y[T+1] \quad \ldots \]
Real-Time Communication System

Source stream
\( s[i] \in F^k_q \)

\[ \rightarrow \]

\( s[0] \quad s[1] \quad s[2] \quad \ldots \quad s[i] \quad \ldots \quad s[T-1] \quad s[T] \quad s[T+1] \quad \ldots \)

Time

Causal encoder:
\( x[i] = f_i(s[0], s[1], \ldots, s[i]) \in F^n_q \)
Rate = \( \frac{k}{n} \)

Packet Erasure Channel
\( y[i] = * \) for packet erasure; otherwise \( y[i] = x[i] \)

\( y[0] \quad y[1] \quad y[2] \quad \ldots \quad y[i] \quad \ldots \quad y[T-1] \quad y[T] \quad y[T+1] \quad \ldots \)

Delay constrained decoder:
\( s[i] = g_i(y[0], y[1], \ldots, y[i + T]) \)
Real-Time Communication System

Source stream
$s[i] \in \mathbb{F}_q^k$

$\Rightarrow$

\begin{align*}
\mathbf{s}[0] & \quad \mathbf{s}[1] & \quad \mathbf{s}[2] & \quad \cdots & \quad \mathbf{s}[i] & \quad \cdots & \quad \mathbf{s}[T-1] & \quad \mathbf{s}[T] & \quad \mathbf{s}[T+1] & \quad \cdots \\
\text{Causal encoder: } \mathbf{x}[i] = f_i(s[0], s[1], \cdots, s[i]) & \quad \mathbf{x}[T-1] & \quad \mathbf{x}[T] & \quad \mathbf{x}[T+1] & \quad \cdots \\
\text{Rate} = \frac{k}{n} & \\
\end{align*}

\begin{align*}
\mathbf{x}[0] & \quad \mathbf{x}[1] & \quad \mathbf{x}[2] & \quad \cdots & \quad \mathbf{x}[i] & \quad \cdots & \quad \mathbf{x}[T-1] & \quad \mathbf{x}[T] & \quad \mathbf{x}[T+1] & \quad \cdots \\
\text{Packet Erasure Channel} & \\
\mathbf{y}[i] = \ast & \text{for packet erasure; otherwise } \mathbf{y}[i] = \mathbf{x}[i] & \\
\mathbf{y}[0] & \quad \mathbf{y}[1] & \quad \mathbf{y}[2] & \quad \cdots & \quad \mathbf{y}[i] & \quad \cdots & \quad \mathbf{y}[T-1] & \quad \mathbf{y}[T] & \quad \mathbf{y}[T+1] & \quad \cdots \\
\text{Delay constrained decoder: } s[i] = g_i(\mathbf{y}[0], \mathbf{y}[1], \cdots, \mathbf{y}[i + T]) & \\
\end{align*}

\begin{align*}
\text{Delay } T & \\
\end{align*}
Real-Time Communication System

Source stream
\( s[i] \in F_q^k \)

```
\[ s[0] \quad s[1] \quad s[2] \ldots \quad s[i] \ldots \quad s[T-1] \quad s[T] \quad s[T+1] \ldots \]
```

Causal encoder: \( x[i] = f_i(s[0], s[1], \ldots, s[i]) \in F_q^n \)
Rate: \( \frac{k}{n} \)

```
\[ x[0] \quad x[1] \quad x[2] \ldots \quad x[i] \ldots \quad x[T-1] \quad x[T] \quad x[T+1] \ldots \]
```

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\( y[i] = \star \) for packet erasure; otherwise \( y[i] = x[i] \)

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Delay constrained decoder: \( s[i] = g_i(y[0], y[1], \ldots, y[i+T]) \)

```
\[ s[0] \quad s[1] \ldots \]
```

Delay \( T \)

Delay \( T \)
Motivating Example

\[ p_i = s_i \cdot H_0 + s_{i-1} \cdot H_1 + \ldots + s_{i-M} \cdot H_M, \quad H_i \in \mathbb{F}_q^{k \times n-k} \]

Erasure Codes:

- Random Linear Codes
- Strongly-MDS Codes (Gabidulin’88, Gluesing-Luerssen’06 )
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\[ p_i = s_i \cdot H_0 + s_{i-1} \cdot H_1 + \ldots + s_{i-M} \cdot H_M, \quad H_i \in \mathbb{F}_q^{k \times n-k} \]

Recover \( s_0, s_1, s_2, s_3 \)

Erasure Codes:

- Random Linear Codes
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Motivating Example

\[ p_i = s_i \cdot H_0 + s_{i-1} \cdot H_1 + \ldots + s_{i-M} \cdot H_M, \quad H_i \in \mathbb{F}_q^{k \times n-k} \]

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Erasure Codes:

- Random Linear Codes
- Strongly-MDS Codes (Gabidulin’88, Gluesing-Luerssen’06 )

\[
\begin{bmatrix}
p_4 \\
p_5 \\
p_6 \\
p_7 \\
\end{bmatrix} = \begin{bmatrix}
H_4 & H_3 & H_2 & H_1 \\
H_5 & H_4 & H_3 & H_2 \\
0 & H_5 & H_4 & H_3 \\
0 & 0 & H_5 & H_4 \\
\end{bmatrix} \begin{bmatrix}
s_0 \\
s_1 \\
s_2 \\
s_3 \\
\end{bmatrix}
\]
Motivating Example- II

$B = 4, \ T = 8$

Rate 1/2 Baseline Erasure Codes, $T = 7$
Motivating Example II

\[ B = 4, \ T = 8 \]

Rate 1/2 Baseline Erasure Codes, \( T = 7 \)

\[
\begin{align*}
&v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \\
&\quad \uparrow
\\
&v_0, v_1, v_2, v_3
\end{align*}
\]

Rate 1/2 Repetition Code, \( T = 8 \)

\[
\begin{align*}
&u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11} \\
&\quad \uparrow
\\
&u_0, u_1, u_2, u_3
\end{align*}
\]
Motivating Example - II

$B = 4, T = 8$

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Motivating Example - II

\( B = 4, \ T = 8 \)

\[
R = \frac{u + v}{3u + v} = \frac{1}{2}
\]
Motivating Example - II

\( B = 4, \ T = 8 \)

\[
R = \frac{u + v}{2u + v} = \frac{2}{3}
\]
Motivating Example - II

\( B = 4, \ T = 8 \)

\[ R = \frac{u + v}{2u + v} = \frac{2}{3} \]

### Encoding:

1. Split each source symbol into 2 groups \( s_i = (u_i, v_i) \)
2. Apply Erasure code to the \( v_i \) stream generating \( p_i \) parities
3. Repeat the \( u_i \) symbols with a shift of \( T \)
4. Combine the repeated \( u_i \)'s with the \( p_i \)'s
Motivating Example - II

\( B = 4, T = 8 \)

\[
R = \frac{u + v}{2u + v} = \frac{2}{3}
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Encoding:

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$B = 4$, $T = 8$

$R = \frac{u + v}{2u + v}$

Encoding:

1. Split each source symbol into 2 groups $s_i = (u_i, v_i)$
2. Apply Erasure code to the $v_i$ stream generating $p_i$ parities
3. Repeat the $u_i$ symbols with a shift of $T$
4. Combine the repeated $u_i$'s with the $p_i$'s
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\( B = 4, \ T = 8 \)

Encoding:

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**Encoding:**

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5. Maximally Short Codes (Martinian and Sundberg '04, Martinian and Trott '07)
Key Questions

- Can we construct streaming codes for realistic channels?

![Gilbert-Elliott Model diagram]

Gilbert-Elliott Model

- How much performance gains can we obtain?
- What are the fundamental metrics for low-delay error correction codes?
Problem Setup

System Model

- **Source Model**: i.i.d. sequence \( s[t] \sim p_s(\cdot) = \text{Unif}(\mathbb{F}_q^k) \)
- **Streaming Encoder**: \( x[t] = f_t(s[1], \ldots, s[t]), x[t] \in (\mathbb{F}_q)^n \)
- **Erasure Channel**
- **Delay-Constrained Decoder**: \( \hat{s}[t] = g_t(y[1], \ldots, y[t + T]) \)
- **Rate** \( R = \frac{k}{n} \)
Problem Setup

System Model

- **Source Model**: i.i.d. sequence \(s[t] \sim p_s(\cdot) = \text{Unif}\{(\mathbb{F}_q)^k\}
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- **Rate** \(R = \frac{k}{n}\)

\((N, B, W) = (2, 3, 6)\)

- \(W = 6\)
- \(N = 2\)
Problem Setup

System Model

- Source Model: i.i.d. sequence \( s[t] \sim p_s(\cdot) = \text{Unif}\{\mathbb{F}_q^k\} \)
- Streaming Encoder: \( x[t] = f_t(s[1], \ldots, s[t]), x[t] \in (\mathbb{F}_q)^n \)
- Erasure Channel
- Delay-Constrained Decoder: \( \hat{s}[t] = g_t(y[1], \ldots, y[t + T]) \)
- Rate \( R = \frac{k}{n} \)

\[(N, B, W) = (2, 3, 6)\]

\[\begin{array}{ccccccccccccc}
8 & \times & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\end{array}\]

- \( W = 6 \)
- \( N = 2 \)
Problem Setup

System Model

- **Source Model**: i.i.d. sequence \( s[t] \sim p_s(\cdot) = \text{Unif}\{(\mathbb{F}_q)^k\} \)
- **Streaming Encoder**: \( x[t] = f_t(s[1], \ldots, s[t]), \ x[t] \in (\mathbb{F}_q)^n \)
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\[(N,B,W) = (2,3,6)\]

- \( W = 6 \)
- \( B = 3 \)
Problem Setup

System Model

- Source Model: i.i.d. sequence $s[t] \sim p_s(\cdot) = \text{Unif}\{\mathbb{F}_q^k\}$
- Streaming Encoder: $x[t] = f_t(s[1], \ldots, s[t]), x[t] \in (\mathbb{F}_q)^n$
- Erasure Channel
- Delay-Constrained Decoder: $\hat{s}[t] = g_t(y[1], \ldots, y[t + T])$
- Rate $R = \frac{k}{n}$

$(N, B, W) = (2, 3, 6)$

W = 6  
B = 3

Capacity: $C(N, B, W, T)$
Theorem

Consider the $C(N, B, W)$ channel, with $W \geq B + 1$, and let the delay be $T$.

**Upper-Bound** For any rate $R$ code, we have:

$$\left( \frac{R}{1 - R} \right) B + N \leq \min(W, T + 1)$$

**Lower-Bound:** There exists a rate $R$ code that satisfies:

$$\left( \frac{R}{1 - R} \right) B + N \geq \min(W, T + 1) - 1.$$  

The gap between the upper and lower bound is 1 unit of delay.
Streaming Codes - Burst Erasure Channels

\[ C(N = 1, B, W \geq T + 1) = \frac{T}{T+B} \]

- Assume \( s_i \in \mathbb{F}_q^T \). Let \( v_i \in \mathbb{F}_q^{T-B} \), \( u_i, p_i \in \mathbb{F}_q^B \)
- Recovery of \( \{v_i\} \) by time \( T - 1 \)
  - Number of Unknowns: \( B \) symbols over \( \mathbb{F}_q^{T-B} \)
  - Number of Equations: \( T - B \) symbols over \( \mathbb{F}_q^{B} \)
- Recovery of \( u_i \) at time \( i + T \)
Streaming Codes - Burst Erasure Channels

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$C(N = 1, B, W \geq T + 1) = \frac{T}{T+B}$

- Assume $s_i \in \mathbb{F}^T_q$. Let $v_i \in \mathbb{F}^{T-B}_q$, $u_i, p_i \in \mathbb{F}^B_q$
- Recovery of $\{v_i\}$ by time $T - 1$
  - Number of Unknowns: $B$ symbols over $\mathbb{F}^{T-B}_q$
  - Number of Equations: $T - B$ symbols over $\mathbb{F}^B_q$
- Recovery of $u_i$ at time $i + T$
Streaming Codes - Isolated Erasures

$C(N \geq 2, B, W)$

$T = 8$

- Erasures at time $t = 0$ and $t = 8$
- $u_0$ cannot be recovered due to a repetition code
Proposed Approach: Layering

$C(N \geq 2, B, W)$

Layered Code Design

- **Burst-Erasure Streaming Code** $C_1 : (u_i, v_i, p_i + u_{i-T})$
- **Erasure Code:** $q_i = \sum_{t=1}^{M} u_{i-t} \cdot H^u_t$, $q_i \in \mathbb{F}_q^k$
- **$C_2$:** $(u_i, v_i, p_i + u_{i-T}, q_i)$

\[
R = \frac{T}{T + B + k}, \quad k = \frac{N}{T - N + 1} B
\]
Proposed Approach: Layering

$C(N \geq 2, B, W)$

Layered Code Design

- Burst-Erasure Streaming Code $C_1: (u_i, v_i, p_i + u_{i-T})$
- Erasure Code: $q_i = \sum_{t=1}^{M} u_{i-t} \cdot H_i^u$, $q_i \in \mathbb{F}_q^k$
- $C_2: (u_i, v_i, p_i + u_{i-T}, q_i)$

$$R = \frac{T}{T + B + k}, \quad k = \frac{N}{T - N + 1} B$$
**Proposed Approach: Layering**

\( C(N \geq 2, B, W) \)

**Layered Code Design**

- **Burst-Erasure Streaming Code** \( C_1 : (u_i, v_i, p_i + u_{i-T}) \)
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- **\( C_2 \)**: \( (u_i, v_i, p_i + u_{i-T}, q_i) \)

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\( C(N \geq 2, B, W) \)

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\[
R = \frac{T}{T + B + k}, \quad k = \frac{N}{T - N + 1} B
\]
Upper Bound

\[ W \geq T + 1 \]

- Periodic Erasure Channel with \( P = T + 1 - N + B \).
- \( B \) erasures in each burst.
- Guard Interval = \( T + 1 - N \).

Show that any low-delay code recovers every symbol on this erasure channel.

\[
R \leq \frac{T + 1 - N}{T + 1 - N + B}
\]
Simulation Results

Gilbert-Elliott Channel \((\alpha, \beta) = (5 \times 10^{-4}, 0.5), T = 12\) and \(R = 12/23\)

**Gilbert Elliott Channel**

- **Good State:** \(\text{Pr}(\text{loss}) = \varepsilon\)
- **Bad State:** \(\text{Pr}(\text{loss}) = 1\)
Simulation Results
Gilbert-Elliott Channel \((\alpha, \beta) = (5 \times 10^{-4}, 0.5), \ T = 12\) and \(R = \frac{12}{23}\)

<table>
<thead>
<tr>
<th>Code</th>
<th>N</th>
<th>B</th>
<th>Code</th>
<th>N</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strongly MDS</td>
<td>6</td>
<td>6</td>
<td>MiDAS</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>Burst-Erasure</td>
<td>1</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Simulation Results-II

Fritchman Channel \((\alpha, \beta) = (1e-5, 0.5)\) and \(T = 40\) and \(R = 40/79\), 9 states

- \(\alpha = 1e - 5\)
- \(\beta = 0.5\)

![Diagram of Fritchman Channel with states and transition probabilities](image)
Simulation Results-II
Fritchman Channel \((\alpha, \beta) = (1e-5, 0.5)\) and \(T = 40\) and \(R = 40/79\), 9 states

\[
\begin{array}{c|c|c}
\text{Code} & \text{N} & \text{B} \\
\hline
\text{Strongly MDS} & 20 & 20 \\
\text{Burst Erasure} & 1 & 39 \\
\end{array}
\]
Simulation Results - III

Gilbert-Eliott Channel \((\alpha, \beta) = (5 \times 10^{-5}, 0.2), T = 50\) and \(R \approx 0.6\)

<table>
<thead>
<tr>
<th>Code</th>
<th>N</th>
<th>B</th>
<th>Code</th>
<th>N</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strongly MDS</td>
<td>20</td>
<td>20</td>
<td>MiDAS</td>
<td>4</td>
<td>30</td>
</tr>
<tr>
<td>Burst-Erasure</td>
<td>1</td>
<td>33</td>
<td>PRC</td>
<td>4</td>
<td>25</td>
</tr>
</tbody>
</table>
Streaming Codes — Burst plus Isolated Erasures
Badr-Khisti-Tan-Apostolopoulos (2013)

When do Earlier Constructions Fail?

- Burst spans $t \in [0, 2]$  
- Isolated Erasure $i \in [7, 10]$  
- Interference from $v_i$ during the recovery of $u_0, \ldots, u_2$
Streaming Codes — Burst plus Isolated Erasures
Badr-Khisti-Tan-Apostolopoulos (2013)

Layered Construction for Partial Recovery

\[
\begin{align*}
\mathbf{r}_i &= \sum_{t=1}^{M} \mathbf{v}_{i-t} \cdot \mathbf{H}_t^v \\
\text{Shift in Repetition Code: } &\mathbf{p}_i + \mathbf{u}_{i-(T-\Delta)} \\
\text{Isolated Loss: } &\mathbf{v} \leq (\Delta + 1)\mathbf{r} \\
\text{Burst Loss: } &\mathbf{v}(B+1) \leq (T - \Delta - B)(\mathbf{u} + \mathbf{r})
\end{align*}
\]
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Layered Construction for Partial Recovery

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Layered Construction for Partial Recovery

\[
\mathbf{r}_i = \sum_{t=1}^{M} \mathbf{v}_{i-t} \cdot \mathbf{H}_t^\nu \mathbf{v}_0 \mathbf{v}_1 \mathbf{v}_2
\]

- **Shift in Repetition Code:** \( \mathbf{p}_i + \mathbf{u}_{i-(T-\Delta)} \)
- **Isolated Loss:** \( \nu \leq (\Delta + 1)\mathbf{r} \)
- **Burst Loss:** \( \nu(B + 1) \leq (T - \Delta - B)(\mathbf{u} + \mathbf{r}) \)
Streaming Codes — Burst plus Isolated Erasures
Badr-Khisti-Tan-Apostolopoulos (2013)

Code-Params

\[
\begin{align*}
v &= (T - \Delta + 1)(\Delta - B - 1) \\
u &= (B + 1)(T - \Delta + 1) \\
r &= (\Delta - B - 1)
\end{align*}
\]

Rate

\[
R = \frac{u + v}{2u + v + r} = \frac{\Delta(T - \Delta) + (B + 1)}{\Delta(T - \Delta) + (B + 1)(T - \Delta + 2)}
\]

\[
\Delta^* = T + 1 - \sqrt{T - B}
\]
Layering Principle

- Start with a code for $C(N, B, W)$
- Add additional parity check symbols for more sophisticated patterns.
Unequal Source/Channel Rates

Source \[s[0]\] \rightarrow \text{Time}

- Source model: i.i.d. process with
  \[s[i] \sim \text{uniform over } F\]

- Streaming encoder:
  \[x[i,j] = f_{i,j}(s[0], s[1], \ldots, s[i]) \in F^n_q\]

- Macro-packet:
  \[X[i, : ] = [x[i,1] \mid \ldots \mid x[i,M]]\]

- Rate:
  \[R = H(s) n \times M = k n \times M\]

- Packet erasure channel: erasure burst of maximum
  \[B\] channel packets

- Delay-constrained decoder:
  \[s[i] \text{ recovered by macro-packet } i + T\]
Source model: i.i.d. process with $s[i] \sim \text{uniform over } \mathbb{F}_q^k$
Unequal Source/Channel Rates

Source model: i.i.d. process with \( s[i] \sim \text{uniform over } F_q^k \)

- Source
- Encoder
- Time
- Macro-packet \( X[0,:] \)
- Channel packet \( x[0,1] \)

Delay-constrained decoder: \( s[i] \) recovered by macro-packet \( i + T \)
Unequal Source/Channel Rates

- **Source model**: i.i.d. process with $s[i] \sim \text{uniform over } \mathbb{F}_q^k$
- **Streaming encoder**: $x[i, j] = f_{i,j}(s[0], s[1], \ldots, s[i]) \in \mathbb{F}_q^n$
- **Macro-packet**: $X[i,:] = [x[i, 1] | \ldots | x[i, M]]$
- **Rate**: $R = \frac{H(s)}{n \times M} = \frac{k}{n \times M}$

[Diagram showing the process with source, encoder, macro-packet, and channel packet.]
Unequal Source/Channel Rates

- **Source model**: i.i.d. process with \( s[i] \sim \text{uniform over } F_q^n \)
- **Streaming encoder**: \( x[i,j] = f_{i,j}(s[0], s[1], \ldots, s[i]) \in F_q^n \)
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Macro-packet: \( X[i, :] = [x[i, 1] | \ldots | x[i, M]] \)

Rate: \( R = \frac{H(s)}{n \times M} = \frac{k}{n \times M} \)

Packet erasure channel: erasure burst of maximum \( B \) channel packets
Unequal Source/Channel Rates

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- **Macro-packet:** $X[i,:] = [x[i, 1] \mid \ldots \mid x[i, M]]$
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Source model: i.i.d. process with $s[i] \sim \text{uniform over } \mathbb{F}_q^k$

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Delay-constrained decoder: $s[i]$ recovered by macro-packet $i + T$
Unequal Source/Channel Rates

Source-splitting based scheme

- Split \( s[i] \) into sub-symbols \((s[i, 1], s[i, 2], \ldots, s[i, M])\).
- Apply a streaming code for \( M = 1 \).
- Decoding Delay \( T' = M \cdot T \).
- \( R = \frac{MT}{MT+B} \)
Unequal Source/Channel Rates
Patil-Badr-Khisti-Tan 2013

- **Source Splitting:**
  \[ R = \frac{MT}{MT + B} \]

- **Strongly MDS:**
  \[ R = 1 - \frac{B}{M(T+1)} \]
Source Splitting:

\[ R = \frac{MT}{MT+B} \]

Strongly MDS:

\[ R = 1 - \frac{B}{M(T+1)} \]

Capacity: Let \( B = bM + B' \), where \( B' \in \{0, \ldots, M - 1\} \)

\[
C = \begin{cases} 
\frac{T}{T+b}, & 0 \leq B' \leq \frac{b}{T+b} M \\
\frac{M(T+b+1)-B}{M(T+b+1)}, & \frac{b}{T+b} M \leq B' \leq M - 1
\end{cases}
\]
Key Idea:

1. Apply code for $M = 1$ on complete source-packet to generate the entire macro-packet.
2. Map/reshape the generated macro-packet into $M$ individual channel-packets.
Source splitting

- split $s[i]$ into two groups

$$s[i] = (s_1[i], \ldots, s_k[i])$$
$$= (u_1[i], \ldots, u_{k_u}[i], v_1[i], \ldots, v_{k_v}[i])$$

$u_{vec}[i]$  $v_{vec}[i]$
2 Parity generation

- layer 1: \((k_u + k_v, k_v, T)\) Strongly-MDS code applied to \(v_{vec}[\cdot]\) generating \(p_{vec}[\cdot]\)
- layer 2: repetition code on \(u_{vec}\) with a shift of \(T\)
Overall combined parity: $q_{vec}[i] = p_{vec}[i] + u_{vec}[i - T]$
Mapping of macro-packet to individual channel-packets

- Map the generated macro-packet into $M$ individual channel-packets

Rate of the code $= \frac{k_u + k_v}{2k_u + k_v}$
Overall macro-packet structure:

\[ i \quad i + 1 \quad \cdots \quad i + b \quad \cdots \quad i + T - 1 \quad i + T \]

Overall macro-packet arrangement
Key Fact: The worst case burst pattern starts at the beginning of the macro-packet.

Let $B = bM + B'$. Two cases depending on whether $B' \leq (1 - R)M$:

- **Burst only erases symbols from** $\mathbf{u}_{\text{vec}}[i + b]$
  - Recovery of $\mathbf{v}$ symbols: $(k_u + k_v)b = k_u T$
  - Optimal spitting ratio: $\frac{k_u}{k_v} = \frac{b}{T-b}$
  - $R = \frac{T}{T+b}$

- **Burst erases symbols from** $\mathbf{v}_{\text{vec}}[i + b]$
  - Recovery of $\mathbf{v}$ symbols: $(k_u + k_v)b + (B'n - k_u) = k_u T$
  - Optimal spitting ratio: $\frac{k_u}{k_v} = \frac{B}{M(T+b+1) - 2B}$
  - $R = \frac{M(T+b+1) - B}{M(T+b+1)}$
Theorem

For the streaming setup considered, with any $M$, $T$ and $B$, the streaming capacity $C$ is given by the following expression:

$$C = \begin{cases} \frac{T}{T+b}, & B' \leq \frac{b}{T+b} M, \ T \geq b, \\ \frac{M(T+b+1)-B}{M(T+b+1)}, & B' > \frac{b}{T+b} M, \ T > b, \\ \frac{M-B'}{M}, & B' > \frac{M}{2}, \ T = b, \\ 0, & T < b. \end{cases}$$

where the constants $b$ and $B'$ are defined via

$$B = bM + B', \quad B' \in \{0, 1, \ldots, M - 1\}, \quad b \in \mathbb{N}^0.$$
Robust Extension for $M = 1$

- Append an additional layer parity checks containing Strongly-MDS code on $u$

\[
\begin{array}{cccccccccccc}
  & u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} \\
  v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \\
  p_0 +u_{-8} & p_1 +u_{-7} & p_2 +u_{-6} & p_3 +u_{-5} & p_4 +u_{-4} & p_5 +u_{-3} & p_6 +u_{-2} & p_7 +u_{-1} & p_8 +u_0 & p_9 +u_1 & p_{10} +u_2 \\
  q_0 & q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 & q_8 & q_9 & q_{10} \\
\end{array}
\]

- $R = \frac{u+v}{2u+v+k}$.

- Very close to being optimal for $k = \frac{N}{T-N+1} B$.
Robust Extension for any M

Many possibilities for the placement of Strongly-MDS parities for u!

Approach I:
- Repetition code replaced by Strongly-MDS code for u
- Rate of the code unchanged.
- \( N = r \) is achievable.

Approach II
- Append additional layer of Strongly-MDS code for u
- \( R = \frac{k_u + k_v}{2k_u + k_v + k_s} \).
- \( k_s \) chosen according to given \( N \).
Related Works

- Burst Erasure Channel: Maximally Short Codes (Block Code + Interleaving), Martinian and Sundberg (IT-2004), Martinian and Trott (ISIT-2007)
Related Works

- **Burst Erasure Channel**: Maximally Short Codes (Block Code + Interleaving), Martinian and Sundberg (IT-2004), Martinian and Trott (ISIT-2007)

- **Other Variations of Streaming Codes**
  - Burst and isolated loss patterns (Badr-Khisti-Tan-Apostolopoulos, ISIT 2013)
  - Unequal Source Channel Rates (Patil-Badr-Khisti-Tan Asilomar 2013)
  - Multicast Extension (Khisti-Singh 2009, Badr-Lui-Khisti Allerton 2010)
  - Parallel Channels (Lui-Badr-Khisti CWIT 2011)
  - Multi-Source Streaming Codes (Lui Thesis, 2011)
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- Tree Codes: Schulman (IT 1996), Sahai (2001), Martinian and Wornell (Allerton 2004), Sukhavasi and Hassibi (2011)
Append an additional layer parity checks containing Strongly-MDS code on $u$

$$R = \frac{u+v}{2u+v+k}.$$ 

Very close to being optimal for $k = \frac{N}{T-N+1}B$.

MiDAS $\rightarrow$ (Near) Maximum Distance And Span tradeoff
Consider \((n, k, m)\) Convolutional code: \(x_i = \sum_{j=0}^{m} s_{i-j} G_j\)
Consider \((n, k, m)\) Convolutional code: \(x_i = \sum_{j=0}^{m} s_{i-j} G_{j}\)
Distance and Span Properties

Consider \((n, k, m)\) Convolutional code: \(x_i = \sum_{j=0}^{m} s_{i-j} G_j\)

Column Distance: \(d_T\)

\[
d_T = \min_{\left[ s_0, \ldots, s_T \right]} \text{wt} \left[ \begin{array}{c} s_0 \\ \vdots \\ 0 \\ \end{array} \right] \begin{bmatrix} G_0 & G_1 & \cdots & G_T \\ 0 & G_0 & \cdots & G_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & G_0 \end{bmatrix}
\]
Distance and Span Properties

Consider \((n, k, m)\) Convolutional code: 
\[ x_i = \sum_{j=0}^{m} s_{i-j} G_j \]

![Trellis Diagram](image)

**States**

**Column Distance** in \([0, 3]\)

**Column Distance:** 
\[ d_T = \min_{[s_0, \ldots, s_T]} \text{wt} \left( \begin{bmatrix} s_0 & \ldots & s_T \end{bmatrix} \begin{bmatrix} G_0 & G_1 & \ldots & G_T \\ 0 & G_0 & \ldots & G_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & G_0 \end{bmatrix} \right) \]
Consider \((n, k, m)\) Convolutional code: \(x_i = \sum_{j=0}^{m} s_{i-j}G_j\)

\[c_T = \min_{[s_0, \ldots, s_T]} \text{span} \begin{bmatrix} s_0 & \ldots & s_T \end{bmatrix} \begin{bmatrix} G_0 & G_1 & \cdots & G_T \\ 0 & G_0 & \cdots & G_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & G_0 \end{bmatrix} \]
Theorem

Consider a $C(N, B, W)$ channel with delay $T$ and $W \geq T + 1$. A streaming code is feasible over this channel if and only if it satisfies: $d_T \geq N + 1$ and $c_T \geq B + 1$
Theorem

Consider a $C(N, B, W)$ channel with delay $T$ and $W \geq T + 1$. A streaming code is feasible over this channel if and only if it satisfies: $d_T \geq N + 1$ and $c_T \geq B + 1$

Theorem

For any rate $R$ convolutional code and any $T \geq 0$ the Column-Distance $d_T$ and Column-Span $c_T$ satisfy the following:

$$\left(\frac{R}{1 - R}\right) c_T + d_T \leq T + 1 + \frac{1}{1 - R}$$

There exists a rate $R$ code (MiDAS Code) over a sufficiently large field that satisfies:

$$\left(\frac{R}{1 - R}\right) c_T + d_T \geq T + \frac{1}{1 - R}$$
Motivation

- $B_1 < B_2$
- Receiver 1: Good Channel State
- Receiver 2: Weaker Channel State
- Delay adapts to Channel State
Example: $B_1 = 2$, $T_1 = 4$, $B_2 = 3$ and $T_2 = 5$.

- A burst of length $B_1$ results in a delay of $T_1$.
- A burst of length $B_2$ results in a delay of $T_2$.
- Stretch-Factor: $s = \frac{T_2 - B_1}{T_1 - B_1}$

Tradeoff between $s$ and $\text{Pr}(\text{loss})$. 
Theorem

There exists a \( \{(B_1, T_1), (B_2, T_2)\} \) DE-SCo construction of rate
\[
R = \frac{T_1}{T_1 + B_1}
\]
provided

\[
T_2 \geq T_2^* \triangleq \frac{B_2}{B_1} T_1 + B_1.
\]

This code has polynomial-time encoding and decoding complexity.
Multicast (Low Delay) Capacity

Capacity Function

- Capacity function \( C(T_1, T_2, B_1, B_2) \)
- Single User Upper Bound: \( C \leq \min \left( \frac{T_1}{T_1+B_1}, \frac{T_2}{T_2+B_2} \right) \)
- Concatenation Lower Bound: \( C \geq \frac{1}{1+\frac{B_1}{T_1}+\frac{B_2}{T_2}} \)
Assume w.l.o.g. $B_2 \geq B_1$

<table>
<thead>
<tr>
<th>Region</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\frac{T_2}{T_2 + B_2}$</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{T_1}{T_1 + B_1}$</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{T_2 - B_1}{T_2 - B_1 + B_2}$</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{T_1}{T_1 + B_2}$</td>
</tr>
<tr>
<td>E</td>
<td>??</td>
</tr>
</tbody>
</table>
Other Extensions

- Multiple Erasure Bursts (Li-Khisti-Girod 2011) - Interleaved Low-Delay Codes
- Multiple Links (Lui-Badr-Khisti 2011) - Layered coding for burst erasure channels
- Multiple Source Streams with Different Decoding Delays (Lui 2011) - Embedded Codes
Conclusions

- Error Correction Codes for Real-Time Streaming
- Deterministic Channel Models $C(N, B, W)$
- Tradeoff between achievable $N$ and $B$
- MiDAS Constructions
- Column-Distance and Column-Span Tradeoff
- Partial Recovery Codes for Burst + Isolated Erasures
- Unequal Source-Channel Rates