Structural Properties of Erasure Codes for Streaming Communication

Ashish Khisti

Joint work with:
Ahmed Badr (Toronto), Wai-Tian Tan (Cisco), John Apostolopoulos (Cisco)

University of Toronto
Department of Electrical and Computer Engineering

Streaming Communication
Structural Properties of Erasure Codes for
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<th>Bit-Rate (Mbps)</th>
<th>Latency (ms)</th>
<th>MSDU (B)</th>
<th>PLR</th>
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<tr>
<td>Video Streaming</td>
<td>4 Mbps</td>
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<td>Video Conferencing</td>
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Applications

Multimedia Streaming
Real-Time Communication System

\[
\begin{align*}
\text{Rate} & \equiv \frac{u}{v} \\
\text{Causal encoder: } x & \equiv [y]s, \ldots, [1]s, [0]s, y = [z]x \\
\text{Stream source} & \equiv [y]s, z \in \mathbb{R}
\end{align*}
\]
Real-Time Communication System
Real-Time Communication System

Rate: \( \frac{u}{v} \)

Causal encoder: \( [x] \)

\( b \in \{ [0] \} \) s \( \cdots \) \( [1] \) \( s \) \( [0] \) \( s \) \( f \) \( = \) \( [i] \) \( x \)

Source

Stream
Real-Time Communication System

Rate

Causal encoder: $x \mapsto \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Output

Input

Time

Source
Real-Time Communication System

Rate = \frac{\log b}{\log n}

\forall \theta \in [\tau] \text{ s.t. } f = [\tau] x

Causal encoder: x

Time

Source stream

Real-Time Communication System
Real-Time Communication System
Real-Time Communication System

Packet Broadcast Channel

\[ \text{Rate} = \frac{u}{v} \]

Causal encoder: \( x \)

\( b \) \( \in \) \([2]s\)

Source

Time
Real-Time Communication System

![Diagram of a real-time communication system with components and equations related to delay constraint decoding and packet erasure.]

- Delay constraint decoding: $([L+\beta], [I], [0])^\beta = [\gamma]$.
- Packet erasure channel: $[\gamma]x = [\gamma]y$ for packet erasure; otherwise $x = [\gamma]y$.
- Causal encoder: $x = [\gamma]y$.
- Rate $\frac{b}{u} \in ([\gamma]s, \ldots, [0]s) f = [\gamma]s$.
Real-Time Communication System

Delay $\mathcal{D}$

$[0]s$

$([L] + \beta) \mathcal{A} \cdots [1] \mathcal{A} [0] \mathcal{A} \beta = [\gamma]s$

Decoder constrained: $[\gamma]s = [\gamma] \mathcal{A}$

Packet erasure channel $\mathcal{X} = [\gamma] \mathcal{A}$

$[\gamma]x = [\gamma] \mathcal{A}$

Rate $\frac{b}{u}$

$[\gamma]s \cdots [1]s [0]s \beta \mathcal{F} = [\gamma] \mathcal{X}$

Causal encoder $\mathcal{X}$

$\mathcal{D} = [\gamma]s$

Source
Streaming Code: Causal Encoder + Delay Constrained Decoder

\[
\]

Delay constrained decoder: \(\delta\) for packet erasure: otherwise \(X = [\delta]X\)

Packet Bresure Channel

\[
\]

Rate

\[
\frac{b}{n} \in ([\delta]s, \ldots, [I]s, [0]s)_{\delta} = [\delta]s 
\]

Causal encoder: \(X = [\delta]X\)

Source

Real-Time Communication System
Proposed Channel Model

Gilbert-Elliott Model

Loss Pattern
Sliding Window Erasure Channel: $C(N, B, W)$

In any sliding window of length $W$, the channel can introduce only one of the following:
- Upto $N$ erasures in arbitrary positions
- An erasure burst of maximum length $B$

Loss Pattern

Gilbert-Elliott Model

Proposed Channel Model
Problem Setup - Sliding Window Erasure Channel Model

Source Model: i.i.d. sequence \( s_t \sim p_s(\cdot) = \text{Unif}\{F_q^b:k\} \)

Streaming Encoder: \( x_t = f_t(s_0, \ldots, s_t) \)

Erasure Channel: (Sliding Window Model)

Delay-Constrained Decoder: \( \hat{s}_t = g_t(y_0, \ldots, y_{t+T}) \)

Rate \( R = \frac{k_n}{q} \) is achievable over the \( C(N, B, W) \) channel, if there is a sequence of encoding and decoding functions, \( f_t(\cdot) \) and \( g_t(\cdot) \), respectively over a sufficiently large field \( F_q \) and rate \( R = \frac{k_n}{q} \).

The supremum of achievable rates is the streaming capacity:

\[
\frac{u}{q} = R
\]
Problem Setup - Sliding Window Erasure Channel Model

Source Model: i.i.d. sequence

\[ s[t] \sim p(s) = \text{Unif}\{F_q^k\} \]

Streaming Encoder:

\[ x[t] = f_t(s[0], \ldots, s[t]) \]

Erasure Channel: (Sliding Window Model)

Delay-Constrained Decoder:

\[ \hat{s}[t] = g_t(y[0], \ldots, y[t+T]) \]

Rate

\[ R = \sup \{ \frac{u}{q} \} \]

Streaming Capacity

Arbitrarily large field size

Worst Case Definition

- Rate \( R \) is achievable over the \( C(N, B, W) \) channel, if there is a sequence of encoding and decoding functions, \( f_t(\cdot) \) and \( g_t(\cdot) \) respectively over a sufficiently large field \( \mathbb{F}_q \) and rate \( (\cdot)^q \)

- The supremum of achievable rates is the streaming capacity.
Theorem

Consider the $C(N, B, W)$ channel, with $W \geq B + 1$, and let the delay be $T$.

**Upper-Bound**

For any rate $R$ code, we have:

$$R \leq N + B \left( \frac{R - 1}{R} \right)$$

**Lower-Bound**

There exists a rate $R$ code that satisfies:

$$R \geq N + B \left( \frac{R - 1}{R} \right)$$

The gap between the upper and lower bound is 1 unit of delay.
Baseline Codes - Full Rank Condition

Baseline Codes - Full Rank Condition

Strong-MDS Codes (Gabidulin'88, Gluesing-Luerssen'06)

Random Linear Codes

Erasure Codes:

By $x - u \cdot \frac{b}{y} \in H$, $H \cdot W \cdot s + \cdots + 1 \cdot H \cdot s + 0 \cdot H \cdot s = d$

$X = \begin{bmatrix}
\begin{array}{cccccccc}
\begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array} & \begin{array}{c}
\text{d} \\
\text{s}
\end{array}
\end{array}
\end{bmatrix}$

$u$
Baseline Codes - Full Rank Condition

- Strongly-MDS Codes (Gabidulin '88, Gluesing-Luerssen '06)
- Random Linear Codes

Erasure Codes:

\[ y - u \times v H \in ?H, \quad \sum_{H} H \cdot H^{-1} s + \cdots + I \cdot I^{-1} s + 0 \cdot H \cdot s = d \]

\[ \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \]

\[ \begin{bmatrix} d \ddots \ddots \ddots \ddots \ddots \ddots \ddots \\ \ddots \ddots \ddots \ddots \ddots \ddots \ddots \\ \ddots \ddots \ddots \ddots \ddots \ddots \ddots \\ \ddots \ddots \ddots \ddots \ddots \ddots \ddots \\ \ddots \ddots \ddots \ddots \ddots \ddots \ddots \\ \ddots \ddots \ddots \ddots \ddots \ddots \ddots \\ \ddots \ddots \ddots \ddots \ddots \ddots \ddots \end{bmatrix} \begin{bmatrix} u \end{bmatrix} \]
Baseline Codes - Full Rank Condition

Erasure Codes:

\[
\begin{align*}
\begin{bmatrix}
\mathbf{b} \\
\mathbf{u} \times \mathbf{H}
\end{bmatrix} \subseteq \mathbf{H}, \\
\mathbf{H} \cdot \mathbf{W} \cdot \mathbf{s} + \cdots + \mathbf{H} \cdot \mathbf{I} \cdot \mathbf{s} + \mathbf{0} \cdot \mathbf{H} \cdot \mathbf{s} = \mathbf{d}
\end{align*}
\]
Baseline Codes - Full Rank Condition

- Strongly-MDS Codes (Gabidulin’88, Gluesing-Luerssen’06)
- Random Linear Codes

Erasure Codes:

\[
x - u \times b \in \mathcal{H} \quad 'W' \quad H \cdot W^{-1} s + \cdots + I \cdot I^{-1} s + 0 \cdot H \cdot s = d
\]
Baseline Codes - Full Rank Condition

\[ \begin{pmatrix} x \\ \vdots \end{pmatrix} = \begin{pmatrix} s_0 \\ \vdots \end{pmatrix} \cdot H_0 + \begin{pmatrix} s_1 \\ \vdots \end{pmatrix} \cdot H_1 + \cdots + \begin{pmatrix} s_M \\ \vdots \end{pmatrix} \cdot H_M, \]

\( H_i \in \mathbb{F}_q^{n \times k} \)

\( H \cdot \mathbf{s} + \cdots + I \cdot \mathbf{s} + 0 \cdot \mathbf{H} = \mathbf{d} \)

Erasur Codes:

- Strongly-MDS Codes (Gabidulin '88, Glueising-Lurerssen '06)
- Random Linear Codes
Baseline Codes - Full Rank Condition

Erasure Codes:
- Random Linear Codes
- Strongly-MDS Codes (Gabidulin’88, Gluesing-Luerssen’06)

\[ p_i = s_i \cdot H_0 + s_{i-1} \cdot H_1 + \ldots + s_{i-M} \cdot H_M, \]
\[ H_i \in \mathbb{F}_q^{k \times n-k} \]

\[ p_0, p_1, \ldots, p_n \]

\[ s_0, s_1, \ldots, s_n \]
Baseline Codes - Full Rank Condition

Strongly-MDS Codes (Gabidulin'88, Gluesing-Luerssen'06)

Random Linear Codes

Erasure Codes:

\[ \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \in \mathbb{F}_q^n \]

\[ H \cdot W^{-1} \mathbf{s} + \cdots + I \cdot I^{-1} \mathbf{s} + 0 \cdot H \cdot \mathbf{s} = \mathbf{d} \]

\[ \mathbf{x} \]

\[ \begin{bmatrix} d_1 & d_2 & \cdots & d_k \end{bmatrix} \]
Baseline Codes - Full Rank Condition

- Strongly-MDS Codes (Gabidulin'88, Gluesing-Luerssen'06)
- Random Linear Codes

Erasure Codes:

\[
\begin{align*}
\begin{bmatrix}
V \\
H
\end{bmatrix} 
& \in \mathbb{F}_q^{b \times (k+u)} \\
\mathbf{H} 
& = \begin{bmatrix}
\mathbf{H}_1 \\
\mathbf{H}_2 \\
\vdots \\
\mathbf{H}_M
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x} - \mathbf{y} 
& = \begin{bmatrix}
\mathbf{d}_1 \\
\mathbf{d}_2 \\
\vdots \\
\mathbf{d}_M
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x} 
& = \begin{bmatrix}
\mathbf{s}_1 \\
\mathbf{s}_2 \\
\vdots \\
\mathbf{s}_M
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x} 
& = \begin{bmatrix}
\mathbf{d}_1 \\
\mathbf{d}_2 \\
\vdots \\
\mathbf{d}_M
\end{bmatrix}
\end{align*}
\]
Baseline Codes - Full Rank Condition

\[
\begin{bmatrix}
  s_0 \\
  s_1 \\
  s_2 \\
  s_3 \\
  s_4 \\
  s_5 \\
  s_6 \\
  s_7 \\
\end{bmatrix} \cdot \begin{bmatrix}
  H_0 & H_1 & 0 & 0 \\
  H_1 & H_0 & H_1 & 0 \\
  H_0 & H_1 & H_0 & H_1 \\
  H_1 & H_0 & H_1 & H_0 \\
\end{bmatrix} = \begin{bmatrix}
  d_0 \\
  d_1 \\
  d_2 \\
  d_3 \\
\end{bmatrix}
\]

Erasure Codes:

\[
\begin{align*}
\frac{b}{u} \times \frac{b}{u} \in \mathbb{H} \\
H_0 & \cdot H_1 \cdot \cdots \cdot H_{b-1} \cdot s + \cdots + H_0 \cdot s + 0 \cdot s = d
\end{align*}
\]

Random Linear Codes

Strongly-MDS Codes (Gabidulin, 88, Gluesing-Luerssen, 06)

Recover
Rate \( \frac{1}{2} \) Baseline Erasure Codes, \( T = 7 \)

\[
\frac{B}{T} = 8, \quad R = \frac{B}{T} = 8
\]
Rate 1/2 Repetition Code, $\mathcal{L} = 8$

Rate 1/2 Baseline Erasure Codes, $\mathcal{L} = 7$

$$\frac{3}{\varphi} = \frac{B + \mathcal{L}}{\mathcal{L}} = R, \quad \mathcal{L} = 8, \quad R = 4, \quad B = ?$$
Streaming Code - Burst Erasure Channel

Encoding:

1. Source Splitting:
   \[ s_i = (u_i, v_i) \]

2. Erasure Code on \( v_i \):
   \[ \text{Generate} \ v_i ! (v_i, p_i) \]
   where \( p_i \) is obtained from a Strongly-MDS code.

3. Repetition Code on \( u_i \):
   \[ \text{Repeat the} \ u_i \ 's \text{with a shift of} \ T \]

Merging:

1. Combine the repeated \( u_i \)’s with the \( p_i \)’s.

Rate:

\[ R = \frac{T + B}{9} \]
Streaming Code - Burst Erasure Channel

\[ R = \frac{u+v}{3u+v} = \frac{1}{2} \]

\[ B = 4, T = 8, R = \frac{T}{T+B} = \frac{2}{3} \]
Streaming Code - Burst Erasure Channel

\[ R = \frac{\lambda + n \cdot 2}{\lambda + n} = \frac{e}{z} \]

Encoding:

1. Source Splitting:
   \[ s_i = (u_i, v_i) \]
   where \( u_i \in \mathbb{F}_B \) and \( v_i \in \mathbb{F}_T \).

2. Erasure Code on \( v_i \):
   Generate \( v_i! \) from a Strongly-MDS code.

3. Repetition Code on \( u_i \):
   Repeat the \( u_i \) symbols with a shift of \( T \).

4. Merging:
   Combine the repeated \( u_i \)'s with the \( p_i \)'s.

Rate:

\[ R = \frac{T + B}{2T} = \frac{e}{z} \]

Burst Erasure Channel
Streaming Code - Burst Erasure Channel

Encoding:

\[ \frac{\lambda + n \geq}{\lambda + n} = R \]

Merging: Combine the repeated \( u_i \)'s with the \( p_i \)'s

Repetition Code on \( u_i \): Repeat the \( u_i \)'s symbols with a shift of \( \mathcal{L} \)

Strongly-MDS code:

- Erasure Code on \( v_i \): Generate \( v_i \) from a \( B \) is obtained from a

Source Splitting:

- \( s_i = (u_i, v_i) \)
- \( u_i^2 \in \mathbb{F}_b \)
- \( v_i^2 \in \mathbb{F}_b \)

- Generate \( v_i \) from \( u_i \)
- Repeat the \( u_i \)'s symbols with a shift of \( \mathcal{L} \)
- Combine the repeated \( u_i \)'s with the \( p_i \)'s

Rate:

\[ \frac{B + \mathcal{L}}{\mathcal{L}} = R \]

\( B \), \( T \), \( R \)
Encoding:

1. **Source Splitting:**
   - Split the source into two parts: $s_i = (u_i, v_i)$, where $u_i^b \in F_B q$ and $v_i^b \in F_T B_q$.

2. **Erase Code on $v_i$:** Generate $(v_i, p_i)$ using a strongly-MDS code, where $p_i^b \in F_B q$ is obtained from $v_i^b$.

3. **Repetition Code on $u_i$:** Repeat the $u_i$ symbols with a shift of $T$.

4. **Merging:** Combine the repeated $u_i$'s with the $p_i$'s.

5. **Rate:**
   \[ \frac{B + T}{B} = R \]
Streaming Code - Burst Erasure Channel

Encoding:

1. Source Splitting: $s_i = (u_i, v_i)$, $u_i \in F_{b_q^n}$, $v_i \in F_{T_{b_q^n}}$

2. Erasure Code on $v_i$: Generate $\lambda \leftarrow \forall \beta \in F_{b_q^n}$ where $p_i \in \beta$

3. Repetition Code on $u_i$: Repeat the $u_i$ symbols with a shift of $\lambda$

4. $s_i = (u_i, v_i)$

5. Merging: Combine the repeated $u_i$'s with the $p_i$'s

Rate: $R = \frac{b+\lambda}{\lambda} = \frac{\lambda+n}{\lambda+n}$

B = 4, T = 8, R = 8

Strongly-MDS code:

$\forall \lambda \in F_{b_q^n}$ is obtained from a $b_q^n$-MDS code.

$\forall \beta \in F_{b_q^n}$ where $p_i \in \beta$

$\forall \beta \in F_{b_q^n}$ where $p_i \in \beta$
Encoding:

\[ \frac{B + \frac{L}{T}}{L} = R \]

\[ \frac{\lambda + n \gamma}{\lambda + n} = R \]

Streaming Code - Burst Erasure Channel

Source Splitting: 

- Erasure Code on \( u_i \): Repeat the \( u_i \) symbols with a shift of \( L \)
- Repetition Code on \( u_i \): Repeat the \( u_i \) symbols with a shift of \( L \)
- Strongly-MDS code: is obtained from a

Merging: Combine the repeated \( u_i \)s with the \( p_i \)s

Rate: 

\[ \frac{B + \frac{L}{T}}{L} = R \]
S. Rate: \( R = \frac{B + \frac{L}{T}}{L} \)

4. Merge: Combine the repeated \( u_i \)'s with the \( p_i \)'s

3. Repetition Code on \( u_i \): Repeat the \( u_i \)'s symbols with a shift of \( L \)

2. Strongly-MDS code: \( b_i \in \mathbb{F}^L \) is obtained from a

1. Erasure code on \( \lambda \): Generate \( \lambda \) where \( P \in \mathbb{F}^L \) is a

Source Splitting: \( s_i \)’s

Encoding:

\[
\begin{align*}
\xi & = \frac{\lambda + n \zeta}{\zeta} = R \\
\end{align*}
\]

Streaming Code - Burst Erasure Channel

\[
\begin{align*}
\frac{\xi}{\zeta} &= \frac{B + \frac{L}{T}}{L} = R \\
A \cdot \frac{L}{T} &= B \\
\end{align*}
\]
Source Splitting: $s_i = (u_i, v_i)$

1. Erasure Code on $v_i$: Generate $v_i = (v_i, p_i)$ where $p_i \in \mathbb{F}_{B_q}$ is obtained from a Strongly-MDS code.

2. Repetition Code on $u_i$: Repeat the $u_i$ symbols with a shift of $T$.

3. Repeat the $u_i$ symbols with a shift of $T$.

4. Combine the repeated $u_i$'s with the $p_i$'s.

5. Combine the repeated $u_i$'s with the $p_i$'s.

$\frac{B+T}{L} = R$
Robust Extension: $CN, B, W$ Channel

Layered Code Design

Burst-Erasure Streaming Code:

Erasure Code: $q_i = PM_{t=1} u_i \cdot H u_t$

Concatenation: $b \vdash \gamma \vdash n + b \vdash d \vdash n \vdash k$ $\vdash n$ $\vdash n$ $\vdash n$

Attains the lower bound

$\frac{\gamma + B + L}{L} = R$

$\vdash n + b \vdash d \vdash n \vdash k$
Distance and Span Properties

Consider \((n, k, m)\) Convolutional code: \(x_i = \sum_{j=0}^{m} s_{i-j} G_{ij}\)
Consider \((n, k, m)\) convolutional code: 

\[ x_i = \sum_{j=0}^{m} s_{i-j} G_j \]
Distance and Span Properties

Consider \((n, k, m)\) Convolutional code:

\[
x_i = P_{m j=0} s_{ij} G_j
\]

Column Distance:

\[
d_{C} = \min \left[ s_0, \ldots, s_T \right] \quad \text{wt} = \min \left[ 0 \neq 0s \right] [s_s, \ldots, 0s] = \nu p
\]

Consider \((n, k', m)\) Convolutional code: \(x\).
Consider \((n, k, m)\) convolutional code:

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
1-\ell & \cdots & 0 \\
\ell & \cdots & 1 \\
\end{pmatrix}
\begin{bmatrix}
J \mathbf{s} & \cdots & 0 \mathbf{s}
\end{bmatrix}
\text{wt}\min_{0 \neq \theta_0} = p
\]

Column Distance: \(d_T\)

Consider \((n, k, m)\) convolutional code: \(x\). Distance and Span Properties
Consider $(n, k, m)$ convolutional code: $x = \sum_{i=0}^{T} s_i G_i$. \[ d_T = \min_{0 \neq s} \left[ s_0, \ldots, s_T \right] \]

Column Distance: 
\[
\begin{bmatrix}
G_0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
J^{-1}G & \cdots & 0 \\
JG & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
J s & \cdots & 0 \\
\end{bmatrix}
\]

Column Span in $[0,3]$
Consider \((n, k, m)\) convolutional code:

\[
x_i = P_{m}^j = 0 \\
\text{Column Distance: } \quad d_{\text{C}} = \min \left[ s_0, \ldots, s_T \right] = 0 \\
\text{Column Span: } \quad c_{\text{C}} = \min \left[ \sum_{i=0}^{T} s_i \right] = c_T \\
\text{Column Distance: } \quad d_T = \min \left[ \sum_{i=0}^{T} s_i \right] = T_p \\
\text{Distance and Span Properties}
\]
Consider a $C(N, B, W)$ channel with delay $T$ and $W^{-1} + 1$. A streaming code is feasible over this channel if and only if it satisfies:

$$1 + B \geq T \text{ and } 1 + N \geq T \text{ and } W^{-1} + 1.$$
Theorem

Consider a \( C(N,B,W) \) channel with delay \( T \) and \( W_T + 1 \). A streaming code is feasible over this channel if and only if it satisfies:

\[
\frac{H - 1}{1} + L \geq Tp + Tc \left( \frac{H - 1}{H} \right)
\]

Field that satisfies:

There exists a rate \( R \) code (MIDAS Code) over a sufficiently large field that satisfies:

\[
\frac{H - 1}{1} + 1 + L \geq Tp + Tc \left( \frac{H - 1}{H} \right)
\]

Theorem

For any rate \( R \) convolutional code and any \( T \) the Column-Distance \( d_T \) and Column-Span \( c_T \) satisfy the following:

\[
\frac{H - 1}{1} + \frac{1}{T} \geq T p \Rightarrow c_T + d_T \geq T + 1 \cdot \frac{1}{R}
\]

There exists a rate \( R \) code (MiDAS Code) over a sufficiently large field that satisfies:

\[
\frac{H - 1}{1} + \frac{1}{T} \geq T p \Rightarrow c_T + d_T \geq T + 1 \cdot \frac{1}{R}
\]

Consider a \( C(N,B,W) \) channel with delay \( T \) and \( W \) and any \( T \) the following:

\[
V + B \geq c_T \text{ and } N \geq T p \text{ satisfies: streaming code is feasible over this channel if and only if it}
\]
Simulation Result

Gilbert Elliott Channel

Good State: \( \Pr(\text{loss}) = e \)

Bad State: \( \Pr(\text{loss}) = 1 \)

Simulation Length = 10^7

Gilbert Channel \( (\alpha, \beta) = (5 \times 10^{-4}, 0.5) \), \( T = 12 \) and \( R \approx 0.5 \)

Burst Length vs. Probability of Occurrence
Simulation Results

Gilbert-Elliott Channel

$\left( 5 \times 10^{-4} \right), T = 12$ and $R = 0.5$

Strongly MDS

$\left( 5 \times 10^{-4} \right), T = 12$

Burst-Erasurce

$\left( 5 \times 10^{-4} \right), T = 12$

Table:

<table>
<thead>
<tr>
<th>Code</th>
<th>Strongly MDS</th>
<th>Burst-Erasurce</th>
<th>I II</th>
</tr>
</thead>
<tbody>
<tr>
<td>N B</td>
<td>N B</td>
<td>I II</td>
<td></td>
</tr>
</tbody>
</table>
Simulation Results - II

Fritchman Channel

\( \beta = 0.5\)

\[ \beta = \text{Probability of Occurrence} \]

\[ \text{Histogram of Burst Lengths for 9-State Fritchman Channel} \]

Analytical vs. Actual

16/23
Fritchman Channel

Simulation Results

<table>
<thead>
<tr>
<th>Code</th>
<th>NB</th>
<th>Code</th>
<th>NB</th>
</tr>
</thead>
<tbody>
<tr>
<td>MiDAS-II 4</td>
<td>35</td>
<td>MiDAS-I 8</td>
<td>31</td>
</tr>
</tbody>
</table>

Burst Erasure:

- Strongly MDS: 20
- Code: 20

Uncoded

- RLC
- SC0
- FLC
- Unencoded

Loss Probability

9 States - Fritchman Channel $(\alpha^t) = (1e-005, 0.5) \ T = 40$

9 States - Fritchman Channel $(\alpha^t) = (1e-05, 0.5) \ T = 40$

$R = 40/79$, $9$ states

Simulation Results
Multicast Streaming Codes

Motivation

Delay adapts to Channel State

Receiver 2: Weaker Channel State
Receiver 1: Good Channel State

$B_1 > B_2$

$\text{Src. Stream}$

Encoder

Decoder 1

Decoder 2

$B_1$  

$B_2$

$\text{Burst Erasure}$

$\text{Broadcast Channel}$

Receiver 1: Good Channel State
Receiver 2: Weaker Channel State

Delay adapts to Channel State

Delay $= T_1$

Delay $= T_2$
Multicast Streaming Codes

- Concatenation Lower Bound: $C \geq \frac{\frac{T_2 T_1 B_2}{T_1 T_2} + \frac{1}{B_1}}{1}$
- Single User Upper Bound: $C \geq \min \left( \frac{T_1 T_2}{B_1}, B_2 \right)$
- Capacity function $C(T_1, T_2, B_1, B_2)$

```
\begin{align*}
\text{Capacity Function} & \geq \min \left( \frac{T_1 T_2}{B_1}, B_2 \right) \\
\text{Encoder} & \\
\text{Decoder 1} & \text{Delay} = T_1 \\
\text{Decoder 2} & \text{Delay} = T_2 \\
\text{Src. Stream} & \\
\end{align*}
```

Burst Erasure Broadcast Channel
Multicast Streaming Setup

Encoder

Decoders 1, 2

Capacity Function

\[ C \left( \frac{T_1 + B_1}{T_2 + B_2}, \frac{T_2 + B_2}{T_1 + B_1} \right) \]

Single User Upper Bound:

\[ C \geq \min \left( \frac{T_1 + B_1}{T_2 + B_2}, \frac{T_2 + B_2}{T_1 + B_1} \right) \]

Concatenation Lower Bound:

\[ C \leq \frac{T_1 + B_1}{T_2 + B_2} \]

Delay = \( T_1 \) for Decoder 1

Delay = \( T_2 \) for Decoder 2

Burst Erasure Channel

Src. Stream
Assume w.l.o.g. $B_2 \geq B_1$.  

**Multicast Streaming Capacity**

Badr-Khisti-Lui (IT Trans. To Appear 2015)
Other Extensions

Mismatched Streaming Codes (Patil-Badr-Khisti-Tan Asilomar 2013)

Partial Recovery Streaming Codes (Patil-Badr-Khisti-Tan Asilomar 2013)

Other Recent Results: Leong-Ho (ISIT 2012)

Burst Erasure Channels: Martinian and Sundberg (IT-2004)

Other Results

Source Streaming with Different Decoding Delays (Lui (unpublished) 2011)

- Embedded Codes

Multiple Links (Lui-Badr-Khisti CWIT 2011) - Layered coding

Interleaved Low-Delay Codes

Multiple Erasure Bursts (Li-Khisti-Cirio Asilomar 2011) - Layered coding

Leong-Qureshi-Ho (ISIT 2013)

Other Recent Results: Leong-Ho (ISIT 2012)

Multiple Source Streams with Different Decoding Delays (Lui)

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Burst Erasure Channels: Martinian and Sundberg (IT-2004)
Conclusions

Real-Time Communication over Channels with Burst and Isolated Erasures

- Analyses of Probabilistic Channels
- MiDAS Codes
  - Systems Theoretic Approach (e.g., Dual Codes for MiDAS)
- Improved constructions for short-inter-burst gaps

Future Work

- Distance and Span Metrics
- Layering Approach
  - MiDAS Codes: Near Optimal Distance/Span Tradeoff
  - Sliding Window Erasure Channel Model
- Improved constructions for short-inter-burst gaps
References


Sliding Window Erasure Channel: Remarks

(N, B, W) = (2, 3, 6)

W = 6
N = 2

C(N, B, W) = Bursect Channel

(2, 3, 6)
Sliding Window Erasure Channel: Remarks

$$(N, B, W) = (2, 3, 6)$$

$W = 6$

$N = 2$
Sliding Window Erasure Channel: Remarks

\((N, B, W) = (2, 3, 6)\)

\(W = 6\)

\(N = 2\)
Sliding Window Erasure Channel: Remarks

$\mathbf{0} \quad (N, B, W) = (2, 3, 6)$

$W = 6$

$B = 3$

$(N, B, W) = (2, 3, 6)$
Sliding Window Erasure Channel: Remarks

\((N, B, W) = (2, 3, 6)\):

- \(N = 1\) (Burst-Erase Channel)
- \(W = 6\)
- \(B = 3\)

\((N, B, W) = (2, 3, 6)\):

- \(B = 3\)
- \(W = 6\)
C(N) \subseteq M \text{ Burst-Erase Channel}

(N, B, W) = (2, 3, 6)

W = 6
B = 3

(N, B, W) = (2, 3, 6)