

Fundamental Limits of Spectrum-Sharing in Fading Environments

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Abstract

Traditionally, the frequency spectrum is licensed to users by government agencies in a rigid manner where the licensee has the exclusive right to access the allocated band. Therefore, licensees are protected from *any* interference all the time. From a practical standpoint, however, an unlicensed (secondary) user may share a frequency band with its licensed (primary) owner as long as the interference it incurs is not deemed harmful by the licensee. In a fading environment, a secondary user may take advantage of this fact by opportunistically transmitting with high power when its signal, as received by the licensed receiver, is deeply faded. In this paper we investigate the capacity gains offered by this dynamic spectrum sharing approach when channels vary due to fading. In particular, we quantify the relation between the secondary channel capacity and the interference inflicted on the primary user. We further evaluate and compare the capacity under different fading distributions. Interestingly, our results indicate a significant gain in spectrum access in fading environments compared to the deterministic case.

Index terms: Spectrum sharing, received-power constraint, interference temperature, opportunistic access.

1 Introduction

Recent years have witnessed a dramatic increase in the demand for radio spectrum. This is partly due to the increasing interest of consumers in wireless services which in turn is driving the evolution of wireless networks toward high-speed data networks. With the emergence of new applications and the compelling need for mobile Internet access, the demand for spectrum is expected to grow even more in the coming years.

Spectrum is inherently a limited natural resource access to which is regulated by the government agencies such as the Federal Communications Commission (FCC) in the United States. The traditional approach to spectrum management is very inflexible in the sense that the majority of the frequency bands are exclusively licensed to users and tight limits are set on the permitted transmitted power in order to shield systems from mutual interference all the time. This was the only viable option for protecting systems from interference back in 1920s, when the transmitters could not adapt their parameters and the receivers had little or no robustness toward inter-system interference. However, with many advances in transceiver design, this approach is becoming more of an outdated policy choice rather than a technological necessity.

With most of the spectrum being already allocated, it is becoming exceedingly hard to find vacant bands to either deploy new services or to enhance the existing ones. On the other hand, recent measurements by the Spectrum Policy Task Force (SPTF) within the FCC indicate that many portions of the spectrum are either unused or lightly used for significant periods of time [1]. This finding suggests that currently spectrum scarcity is largely due to the inefficient and rigid regulations rather than the physical shortage of the spectrum.

The FCC's initiative along with several other projects including the Defense Advanced Research Projects Agency (DARPA)'s "Next Generation" (XG) program [2] and the national science foundation's "NeTS-ProWiN" project [3] signal a paradigm shift in the spectrum access policy. This raises several technical and regulatory issues to be addressed by academia as well as policy-makers. The interested reader is referred to [4]- [6] for general overview of the issues associated with the spectrum access policy reform.

In its report to the commission, SPTF proposed secondary access to already-licensed spectrum as a means to mitigate the spectrum shortage. However, such *spectrum-sharing* should be carried out in a controlled fashion so that the primary licensee's operation in the band is

not compromised. Therefore, a secondary user trying to access the licensed spectrum should consider the impact of its transmission on the reception quality of the primary licensee. To this end, the *interference temperature* concept has been introduced in [1] which indicates the tolerable interference level at the primary licensee's receiver. From the licensee's point of view, the secondary access does not affect its operation as long as the total interference power at the licensed receiver remains below a certain threshold. Therefore, the power emitted from a secondary transmitter does not have to be limited as long as the interference inflicted on the primary receiver is below the threshold.

Due to the rigid spectrum access regulations, almost all prior studies of the channel capacity assume constraints on the transmitted power. However, in light of the recent developments in the spectrum access policy reform and building upon the interference temperature concept, a constraint on the received-power at the primary receiver seems much more appropriate within the spectrum-sharing context.

The channel capacity under received-power constraint has been first considered by Gastpar [8] who derived the capacity of different AWGN channels with the average received-power at a third-party receiver (e.g., the licensee's receiver) being constrained. In the absence of fading, it was shown that the transmitted and received power constraints give rise to very similar capacity formulas for the point-to-point AWGN channels [8]. This is due to the fact that for such non-varying channels, the received-power at a third-party receiver is merely a *deterministically* scaled version of the transmitted power. However, this is not the case when channels vary due to fading.

As we shall illustrate later, with the same limit on the received-power level, channel capacity in severe fading (e.g., Rayleigh or Log-normal fading) exceeds that of the non-fading AWGN channel. We recall from [9], however, that with the transmitted-power being constrained instead, capacity is lower under fading (except at low signal-to-noise ratios). This discrepancy may be attributed to the involvement of two (rather than one) fading channel-gains in computing the capacity with a received-power constraint. In a nutshell, when the channel between the secondary user and the third-party receiver is in a deep fade, the user may *opportunistically* transmit at very high power levels. In severe fading there is a higher likelihood for this to be the case, thereby increasing the spectrum access opportunity for the secondary transmitter.

In order to gain more insight into the fundamental limits of spectrum-sharing in fading

environments, we evaluate the channel capacity for a secondary (unlicensed) user operating within a primary (licensed) band subject to a non-interference constraint. This constraint in turn is imposed on either the average or the peak power received at the primary receiver, modeling different situations where the receiver's operation is limited by either the average or the instantaneous interference level.

We study both of the above scenarios by formulating the channel capacity as a maximization problem and solving for the optimum solutions in both settings. In each case, closed-form capacity formulas under different fading channels are provided where possible. Our results suggest that a significant spectrum access gain may be achieved in fading environments. Moreover, simple expressions for this gain under different fading distributions will be derived in Section 5 where we examine the asymptotic capacity in the limit of infinite bandwidth.

The remainder of this paper is organized as follows. In the next section we describe our system model and main assumptions. The capacity under average and peak received-power constraints will be derived in Sections 3 and 4 respectively. The asymptotic capacity analysis will be performed in Section 5. Extensions of the results to the scenarios with correlated fading and multiple primary receivers will be provided in Sections 6 and 7 respectively. Finally, concluding remarks and some directions for further research are discussed in Section 8.

2 System Model

We consider a point-to-point AWGN flat¹ fading channel with perfect channel side information (CSI) available to both the receiver and the transmitter. The capacity of this channel is well-known under both peak and average input power constraints [9], [12]. However, instead of a constraint on the transmitted power, we assume a constraint on the power received at a third-party receiver [8]. As stated earlier, this model is mainly motivated by the interference temperature concept which allows a secondary user to operate within a licensee's spectrum as long as the power received at the licensee's receiver does not exceed a certain threshold, Q .

Let g_0 and g_1 denote the instantaneous channel gains from the secondary transmitter to the primary and secondary receivers respectively as shown in Fig. 1. Both gains are assumed

¹Frequency-selective fading provides an additional dimension for optimization (i.e. by opportunistically transmitting more power at frequencies where the secondary-primary channel has higher attenuation). However, for clarity of exposition, only flat fading is considered here.

to be stationary and ergodic with probability density functions (pdf) $f_0(g_0)$ and $f_1(g_1)$ respectively. Without loss of generality, both channel gains are assumed to be unit-mean². Since the constraint is imposed on the received power, the transmitter should have perfect knowledge of g_0 as well as g_1 (i.e. a two-dimensional CSI). Consequently, the perfect CSI assumption in this context is different from that of [9] which only includes g_1 . Feeding back of g_0 to the secondary transmitter may be carried out directly by the licensee or indirectly through a *band manager* which mediates between the two parties [4]. We note that in practice, for a fast fading secondary-to-primary channel, it may be hard to realize the instantaneous feedback of g_0 . Nonetheless, our results serve to outline the fundamental limits of spectrum sharing by providing capacity upper-bounds in such cases.

In this paper we consider two different spectrum-sharing constraints, namely the peak and the average received-power constraints. On the other hand, for clarity of exposition no constraints on the transmitted power are assumed. In practice, the transmitted power is limited by the hardware capabilities as well as safety considerations. Thus, the capacities derived in this paper serve as upper-bounds for the capacity of fading channels under received-power constraints.

3 Capacity Under Average Received-Power Constraint

The average received-power constraint has been considered in [8] where the author derives the capacity for different AWGN channels. In particular for a single AWGN channel, the capacity formula was shown to be very similar to the case with the average transmitted-power being constrained. This may be attributed to the fact that in the AWGN case, the received power is merely a *deterministically* scaled version of the transmitted power. As a result, a constraint on the received-power is fundamentally equivalent to one on the transmitted-power in this setting. However, as we shall demonstrate shortly, this result does not generalize to the fading AWGN channel.

In a fading environment, following the same line of development as in [9], it is straightforward (but tedious) to show that the channel capacity is achieved by optimally distributing the transmitted power over time such that the received-power constraint is met. That is, assuming

²Mean of g_0 and g_1 are captured into other parameters as discussed later on.

g_0 and g_1 to be independent of each other³, the channel capacity is the solution to the following optimization problem (cf. [9]),

$$C = \max_{P(g_0, g_1) \geq 0} \int_{g_0} \int_{g_1} B \log \left(1 + \frac{g_1 P(g_0, g_1)}{N_0 B} \right) f_1(g_1) f_0(g_0) dg_1 dg_0 \quad (1)$$

$$\text{s.t.} \quad \int_{g_0} \int_{g_1} g_0 P(g_0, g_1) f_1(g_1) f_0(g_0) dg_1 dg_0 \leq Q \quad (2)$$

where Q is the maximum average power that the primary user can tolerate at its receiver, P and B are the transmitted power and the total available bandwidth respectively and finally N_0 is the power spectral density of the AWGN noise at the secondary receiver. Note that the transmitted power, P , is in general a mapping from the joint fading state (g_0, g_1) to the set of non-negative real numbers, \mathbb{R}_+ . Thus, optimization is over the set of all such mappings having an average received power of at most Q at the primary receiver.

To find the optimal power allocation, $P(g_0, g_1)$, we form the Lagrangian,

$$\begin{aligned} L(P, \lambda) &= \int_{g_0} \int_{g_1} \log \left(1 + \frac{g_1 P(g_0, g_1)}{N_0 B} \right) f_1(g_1) f_0(g_0) dg_1 dg_0 \\ &\quad - \lambda \left(\int_{g_0} \int_{g_1} g_0 P(g_0, g_1) f_1(g_1) f_0(g_0) dg_1 dg_0 - Q \right) \end{aligned} \quad (3)$$

For the optimization problem defined in (1)-(2), the first-order Kuhn-Tucker conditions are necessary and sufficient for optimality [7]. Thus, the optimal power allocation should satisfy the following,

$$\frac{\partial L(P, \lambda)}{\partial P} = \left(\frac{g_1}{g_1 P(g_0, g_1) + N_0 B} - \lambda g_0 \right) f_1(g_1) f_0(g_0) = 0 \quad (4)$$

Solving (4) with the constraint $P(g_0, g_1) \geq 0$ yields,

$$P(g_0, g_1) = \left(\frac{1}{\lambda_0 g_0} - \frac{N_0 B}{g_1} \right)^+ \quad (5)$$

where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$ and λ_0 is determined such that the average received power in (2) is equal to Q i.e.,

$$\int_{g_0} \int_{g_1} \left(\frac{1}{\lambda_0} - N_0 B \frac{g_0}{g_1} \right)^+ f_1(g_1) f_0(g_0) dg_1 dg_0 = Q \quad (6)$$

The optimal power allocation obtained in (5) is reminiscent of the celebrated water-filling-in-

³For the general non-independent case, $f_1(g_1)f_0(g_0)$ should be replaced by the joint pdf of g_0 and g_1 .

time solution corresponding to a constraint on the transmitted-power [9]. We note, however, that the water level in (5) is in general time-varying and is inversely proportional to g_0 . Moreover, the cutoff level for g_1 is $\lambda_0 N_0 B g_0$ i.e., no data will be transmitted if g_1/g_0 is lower than $\lambda_0 N_0 B$. Recall that these observations are in accordance with our intuition. That is, more transmission power is used when either g_1 increases (cf. [9]) or g_0 decreases which is a manifestation of the two-fold opportunistic spectrum access gain in fading channels.

We point out, however, that when g_0 is deterministic (non-fading AWGN) the power allocation of (5) boils down to the regular water-filling-in-time with a transmitted-power constraint of Q/g_0 . In this case, as g_1 increases, $P(g_0, g_1)$ may still be increased without violating (2). This stems from the fact that such high-interference periods will be compensated for by emitting less power when g_1 is small due to fading.

Substituting (5) in (1) results in the following formula for the channel capacity,

$$\begin{aligned} C &= \int_{\frac{g_1}{g_0} \geq \frac{1}{\gamma_0}} \int B \log \left(\gamma_0 \frac{g_1}{g_0} \right) f_1(g_1) f_0(g_0) dg_1 dg_0 \\ &= \int_{\frac{1}{\gamma_0}}^{\infty} B \log (\gamma_0 g_{10}) f_{\frac{g_1}{g_0}}(g_{10}) dg_{10} \end{aligned} \quad (7)$$

where g_{10} and $f_{\frac{g_1}{g_0}}(\cdot)$ denote the random variable g_1/g_0 and its pdf respectively and $\gamma_0 = 1/(\lambda_0 N_0 B)$. It can be seen from (6) that λ_0 (and thus γ_0), depends on the joint channel statistics only through the first-order pdf of g_1/g_0 , $f_{\frac{g_1}{g_0}}(\cdot)$.

In what follows, we shall study the effect of fading characteristics on the gain of opportunistic spectrum access by evaluating the capacity in (7) under different fading statistics.

3.1 AWGN Channel

In the absence of channel fading, g_0 and g_1 are equal to 1 all the time (recall that we have assumed unit-mean channel gains). In this case $P = Q$ and the channel capacity is simply given by,

$$C_{awgn} = B \log (1 + \alpha) \quad (8)$$

where⁴ $\alpha = Q/N_0 B$ is the average signal-to-noise ratio (SNR) for the AWGN channel.

⁴Mean values of g_0 and g_1 may be incorporated into Q and $N_0 B$ respectively

3.2 Log-normal Shadowing

Empirical measurements suggest that medium-scale variations of the received-power, when represented in dB units, follow a normal distribution (see e.g., [18]). In other words, the linear (as opposed to dB) channel gain may be modelled by a log-normal random variable (r.v.) e^X where X is a zero-mean Gaussian r.v. with variance σ^2 . Log-normal shadowing is usually characterized in terms of its dB-spread which is related to σ by $\sigma = 0.1 \log(10) \sigma_{dB}$. The dB-spread typically ranges from 4 to 12 dB as indicated by the empirical measurements [10, 11].

Now let us assume $g_0 = e^{X_0}$ and $g_1 = e^{X_1}$, where X_0 and X_1 are two independent zero-mean Gaussian random variables with equal variance of σ^2 . Then $g_1/g_0 = e^Y$ is a log-normal r.v. with $Y = X_1 - X_0$ being a zero-mean Gaussian r.v. with a variance of $2\sigma^2$. Now rewriting (6) using $g_1/g_0 = e^Y$,

$$\int_{-\infty}^{\log \gamma_0} \frac{(\gamma_0 - e^y)}{2\sqrt{\pi}\sigma} \exp\left\{-\frac{y^2}{4\sigma^2}\right\} dy = \frac{\gamma_0}{2} \left[\operatorname{erf}\left(\frac{\log \gamma_0}{2\sigma}\right) + 1 \right] - \frac{e^{\sigma^2}}{2} \left[\operatorname{erf}\left(\frac{\log \gamma_0}{2\sigma} - \sigma\right) + 1 \right] = \alpha \quad (9)$$

where $\alpha = Q/N_0B$ as defined before and $\operatorname{erf}(x)$ is the error function defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Thus, for a given α , $\gamma_0(\alpha)$ may be obtained from (9) and the capacity is given by,

$$\begin{aligned} C_{\lognormal} &= \int_{-\log \gamma_0(\alpha)}^{\infty} B \log(\gamma_0(\alpha) e^y) \frac{1}{2\sqrt{\pi}\sigma} \exp\left\{-\frac{y^2}{4\sigma^2}\right\} dy \\ &= B \left[\frac{\log \gamma_0(\alpha)}{2} \operatorname{erf}\left(\frac{y}{2\sigma}\right) - \frac{\sigma}{\sqrt{\pi}} \exp\left\{-\frac{y^2}{4\sigma^2}\right\} \right]_{-\log \gamma_0(\alpha)}^{\infty} \\ &= B \left[\frac{\log \gamma_0(\alpha)}{2} \left(1 + \operatorname{erf}\left(\frac{\log \gamma_0(\alpha)}{2\sigma}\right) \right) + \frac{\sigma}{\sqrt{\pi}} \exp\left\{-\frac{\log^2 \gamma_0(\alpha)}{4\sigma^2}\right\} \right] \end{aligned} \quad (10)$$

The capacity under log-normal fading for different dB-spreads is shown in Fig. 2. Results indicate that the capacity grows with increasing variance of the log-normal fading. This is mainly due to the higher probability of g_0 being in deep fades when σ^2 becomes larger, thereby allowing very high transmit power levels. Moreover, with a dB-spread of 12dB (e.g., in heavily built-up urban areas), the capacity under log-normal fading is consistently higher than that of the AWGN channel (cf. [9]).

3.3 Rayleigh Fading

With channel fading following the Rayleigh pdf, both g_0 and g_1 would be exponentially distributed with unit-mean. In this case g_1/g_0 may be shown to have a log-logistic distribution defined as (see Appendix A),

$$f_{\frac{g_1}{g_0}}(x) = \frac{1}{(1+x)^2}, \quad x \geq 0 \quad (11)$$

Now γ_0 may be obtained by combining (6) and (11) as follows. Recall that g_0 and g_1 are independent and identically distributed. Therefore, g_0/g_1 and g_1/g_0 will have the same pdf. Now, rewriting (6) using the pdf of g_0/g_1 given in (11) yields,

$$\begin{aligned} \int_0^{\gamma_0} (\gamma_0 - x) \frac{1}{(1+x)^2} dx &= \left[-\frac{1+\gamma_0}{1+x} - \log(1+x) \right]_0^{\gamma_0} \\ &= \gamma_0 - \log(1+\gamma_0) \\ &= \frac{Q}{N_0 B} \end{aligned}$$

Thus γ_0 satisfies the following,

$$\gamma_0 - \log(1+\gamma_0) = \alpha \quad (12)$$

It is worth noting that under the average received-power constraint and Rayleigh fading, γ_0 is determined from (12) which obviates the need for numerical integration (cf. [9]). Now from (7) the channel capacity may be written as follows,

$$\begin{aligned} C_{rayleigh} &= \int_{\frac{1}{\gamma_0(\alpha)}}^{\infty} B \log(\gamma_0(\alpha)x) \frac{1}{(1+x)^2} dx \\ &= B \left[\log x - \log(1+x) - \frac{\log(\gamma_0 x)}{1+x} \right]_{\frac{1}{\gamma_0(\alpha)}}^{\infty} \\ &= B \log(1+\gamma_0(\alpha)) \end{aligned} \quad (13)$$

where $\gamma_0(\alpha)$ is the solution of (12) for a given α . It can be seen from (12) that $\gamma_0(\alpha) > \alpha$ for all $\alpha > 0$. Therefore,

$$C_{rayleigh} = B \log(1+\gamma_0(\alpha)) > B \log(1+\alpha) = C_{awgn}$$

Thus, under the same average received-power constraint, the capacity of a Rayleigh fading channel is greater than that of the pure AWGN channel for all values of the average SNR, α . We note, however, that with the average transmitted power being constrained, this statement holds only at low SNR levels [9].

It should be noted that the statement above holds for the case where both g_0 and g_1 are fading. Indeed, with a deterministic g_0 , fading of g_1 will reduce the capacity compared to the case where both g_0 and g_1 are non-fading (i.e. pure AWGN). Nonetheless, we point out that in either case, dynamic spectrum sharing provides a capacity gain over the fixed spectrum access regime where $Q = 0$.

3.4 Nakagami Fading

A more versatile fading model is the Nakagami- m distribution which better fits a wide range of empirical data by adjusting a single parameter. This parameter, m , measures the ratio of the line-of-sight (LOS) signal power to that of the multi-path component. For a unit-mean channel gain, the distribution is given by,

$$f_{\sqrt{g}}(y) = \frac{2m^m y^{2m-1}}{\Gamma(m)} e^{-my^2}, \quad m \geq 0.5 \quad (14)$$

where $\Gamma(\cdot)$ is the gamma function defined as, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. It can be seen that Rayleigh distribution is a special case of the Nakagami distribution with $m = 1$.

For a Nakagami fading channel, channel power gain g is distributed according to the following Gamma distribution (see e.g., [10]),

$$f_g(x) = \frac{m^m x^{m-1}}{\Gamma(m)} e^{-mx}, \quad x \geq 0 \quad (15)$$

With g_0 and g_1 distributed according to (15) with their m parameter being m_0 and m_1 respectively, g_1/g_0 may be shown to have the following probability distribution (see Appendix A),

$$f_{\frac{g_1}{g_0}}(x) = \left(\frac{m_0}{m_1}\right)^{m_0} \frac{x^{m_1-1}}{\mathcal{B}(m_0, m_1) \left(x + \frac{m_0}{m_1}\right)^{m_0+m_1}}, \quad x \geq 0 \quad (16)$$

where $\mathcal{B}(a, b)$ is the beta function defined as,

$$\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

This distribution is known as the beta-prime distribution or beta distribution of the second kind (see p. 248 of [13]). When $m_0 = m_1 = m$, (16) reduces to,

$$f_{\frac{g_1}{g_0}}(x) = \frac{x^{m-1}}{\mathcal{B}(m, m) (x+1)^{2m}}, \quad x \geq 0 \quad (17)$$

Thus, applying (6), $\gamma_0(\alpha)$ is the solution to the following equation,

$$\frac{1}{\mathcal{B}(m, m)} \int_0^{\gamma_0} (\gamma_0 - x) \frac{x^{m-1}}{(1+x)^{2m}} dx = \alpha \quad (18)$$

consequently the channel capacity is given by,

$$C_{nakagami} = \int_{\frac{1}{\gamma_0}}^{\infty} B \log(\gamma_0(\alpha)x) \frac{x^{m-1}}{\mathcal{B}(m, m)(1+x)^{2m}} dx, \quad m \geq 0.5 \quad (19)$$

For general m , (18) and (19) do not admit simple closed-form formulas and the capacity should be evaluated numerically. However, for $m = 2$, some manipulation of (6) yields the following relationship between α and γ_0 ,

$$\frac{\gamma_0^3}{(1+\gamma_0^2)} = \alpha \quad (20)$$

By substituting the γ_0 satisfying (20), denoted by $\gamma_0(\alpha)$, into (19), capacity may be derived as below,

$$C_{nakagami} = B \left[\log(1 + \gamma_0(\alpha)) - \frac{\gamma_0(\alpha)}{(1 + \gamma_0(\alpha))^2} \right] \quad (21)$$

Fig. 3 compares the channel capacity under average received-power constraint for different fading distributions as obtained in (8), (13) and (21). Evidently the capacity under Rayleigh fading is higher than that of the AWGN channel for all values of α (cf. [9]). For less severe fading channels, capacity reduces and may fall below the AWGN channel's capacity at relatively higher values of α .

4 Capacity Under Peak Received-Power Constraint

From the spectrum-sharing point of view, an average received-power constraint as defined in (2) is reasonable when the primary licensee's QoS is determined by the average SNR (e.g., for a delay-insensitive service). We note, however, that in many situations the licensee's QoS would be limited by the instantaneous SNR at the receiver which renders a peak received-power constraint more appropriate. Thus, in this section we study the fading channel capacity defined in (1), with (2) being replaced by the following constraint,

$$g_0 P(g_0, g_1) \leq Q \quad (22)$$

where, with a slight abuse of notation, Q represents the *peak* received-power this time. Assuming no other limitation on the power of secondary transmitter, the capacity is maximized by transmitting at maximum instantaneous power allowed by the licensee, Q/g_0 . Thus,

$$\begin{aligned} C &= \int_{g_0} \int_{g_1} B \log \left(1 + \frac{g_1}{g_0} \frac{Q}{N_0 B} \right) f_1(g_1) f_0(g_0) dg_1 dg_0 \\ &= \int_{\frac{g_1}{g_0}} B \log(1 + \alpha x) f_{\frac{g_1}{g_0}}(x) dx \end{aligned} \quad (23)$$

where $\alpha = Q/N_0 B$ as defined before. Clearly for the AWGN case, peak and average received-power constraints give rise to same capacity as given by (8).

In what follows, we derive the capacity for Rayleigh fading channels. The capacity under log-normal and general Nakagami fading should be obtained numerically.

Under Rayleigh fading, g_1/g_0 has a log-logistic distribution as defined in (11). Thus, the channel capacity is given by,

$$\begin{aligned} C_{rayleigh} &= B \int_0^\infty \log(1 + \alpha x) \frac{1}{(1+x)^2} dx \\ &= B \left[-\log(1 + \alpha x) \left(\frac{1}{1+x} + \frac{\alpha}{1-\alpha} \right) + \frac{\alpha}{1-\alpha} \log(1+x) \right]_0^\infty \\ &= B \frac{\alpha \log \alpha}{\alpha - 1} \end{aligned} \quad (24)$$

The capacity under peak received-power constraint is plotted in Fig. 4 for different fading distributions. It is interesting to note that the capacity under both Rayleigh and Nakagami

($m = 2$) fading is consistently higher than that of the AWGN case. Moreover, comparing figures 3 and 4 for the same α , capacities in the former case are generally higher than those of the latter which is due to the more restrictive nature of the peak (as opposed to average) received-power constraint.

5 Asymptotic Capacity Analysis

It is well-known that the capacity of an AWGN channel approaches a constant value as the channel bandwidth goes to infinity [14]. This result has been extended to fading channels by Kennedy [15] who showed that the capacity of an infinite-bandwidth Rayleigh fading channel approaches the same limit as that of an infinite-bandwidth AWGN channel with the same average received power.

In this section we investigate the asymptotic capacity when the power received at a third-party receiver is constrained. In particular, we derive a simple formula for the asymptotic capacity under general fading distributions, thereby facilitating capacity comparisons in the large-bandwidth regime. Moreover, applying the formula to different settings provides more insight into the impact of fading distribution parameters (e.g., m for Nakagami fading) on the capacity.

For the AWGN channel, asymptotic capacity, \mathcal{C}_{awgn} is simply given by,

$$\mathcal{C}_{awgn} = \lim_{B \rightarrow \infty} B \log \left(1 + \frac{Q}{N_0 B} \right) = \frac{Q}{N_0} \quad (25)$$

Now let us consider a fading AWGN channel with peak received power constraint, Q . Taking the limit of (23) as B goes to infinity yields the asymptotic capacity, \mathcal{C}_{fading}

$$\begin{aligned} \mathcal{C}_{fading} &= \lim_{B \rightarrow \infty} \int_{g_0} \int_{g_1} B \log \left(1 + \frac{g_1 Q}{g_0 N_0 B} \right) f_1(g_1) f_0(g_0) dg_1 dg_0 \\ &= \int_{g_0} \int_{g_1} \lim_{B \rightarrow \infty} \left[B \log \left(1 + \frac{g_1 Q}{g_0 N_0 B} \right) \right] f_1(g_1) f_0(g_0) dg_1 dg_0 \\ &= \int_{g_0} \int_{g_1} \frac{g_1 Q}{g_0 N_0} f_1(g_1) f_0(g_0) dg_1 dg_0 \\ &= \frac{Q}{N_0} E \left(\frac{g_1}{g_0} \right) \end{aligned} \quad (26)$$

where $E(g_1/g_0)$ is the expected value of g_1/g_0 . Since g_1/g_0 is convex in g_0 , by Jensen's inequality (e.g., [17, p. 77]),

$$\mathcal{C}_{fading} \geq \frac{Q}{N_0} \frac{E(g_1)}{E(g_0)} = \frac{Q}{N_0} = \mathcal{C}_{awgn} \quad (27)$$

where we have used the fact that both g_0 and g_1 are unit-mean. Thus, the asymptotic capacity of a fading channel under a peak received-power constraint is at least equal to that of the corresponding non-fading AWGN channel. Note that when g_0 is not fading, the inequality in (27) turns into equality no matter g_1 is fading or not. The equality also holds for the pathological case $g_0 = g_1$ (e.g., collocated primary and secondary receivers).

Let $\mathcal{P}_{avg}(Q)$ and $\mathcal{P}_{peak}(Q)$ denote the two sets of power allocations satisfying (2) and (22) respectively, i.e.,

$$\begin{aligned} \mathcal{P}_{avg}(Q) &= \left\{ P(g_0, g_1) : E_{g_0, g_1} [g_0 P(g_0, g_1)] \leq Q, \forall g_0, g_1 : P(g_0, g_1) \geq 0 \right\} \\ \mathcal{P}_{peak}(Q) &= \left\{ P(g_0, g_1) : g_0 P(g_0, g_1) \leq Q, \forall g_0, g_1 : P(g_0, g_1) \geq 0 \right\} \end{aligned}$$

It is readily observed that,

$$\mathcal{P}_{peak}(Q) \subseteq \mathcal{P}_{avg}(Q)$$

Therefore, if the spectrum-sharing constraint is relaxed to limit the average (rather than the peak) received-power to Q , the asymptotic capacity will generally increase further.

The inequality in (27) holds irrespective of the fading statistics. However, each fading distribution gives rise to a different asymptotic capacity. We define the asymptotic capacity gain for a fading channel, \mathcal{G} , as follows,

$$\mathcal{G} = \frac{\mathcal{C}_{fading}}{\mathcal{C}_{awgn}} = E\left(\frac{g_1}{g_0}\right) \quad (28)$$

Under log-normal fading $g_1/g_0 = e^Y$, where Y is a zero-mean Gaussian r.v. with a variance of $2\sigma^2$. It may be easily shown that the expected value of g_1/g_0 is related to the mean and variance of Y through,

$$\mathcal{G} = E\left(\frac{g_1}{g_0}\right) = e^{\mu_Y + \sigma_Y^2/2} = e^{\sigma^2} \quad (\text{Log-normal fading}) \quad (29)$$

where σ^2 is the variance of normally distributed $\log g_0$ and $\log g_1$. Therefore, the asymptotic

capacity gain is an exponentially increasing function of σ^2 .

Under Rayleigh fading, g_1/g_0 has a log-logistic distribution with infinite mean. Thus, in the absence of any other constraints, \mathcal{G} is infinite. For the general Nakagami- m fading it may be shown that (see Appendix B),

$$\mathcal{G} = \frac{m}{m-1}, \quad m > 1 \quad (\text{Nakagami-}m \text{ fading}) \quad (30)$$

Therefore, under Nakagami fading, as fading becomes less severe (i.e. as m increases) the asymptotic capacity gain reduces. In the limit as m goes to infinity, the Nakagami fading channel reduces to an AWGN channel and \mathcal{G} becomes unity.

6 Correlated Fading

Up to this point, the capacity derivations were based on the assumption that g_0 and g_1 vary independently. As observed earlier, when g_0 is small compared to g_1 , significant capacity gains may be achieved. We note, however, that in practice there may be a degree of correlation between g_0 and g_1 which in turn would reduce the probability of such events. In this section we examine the impact of fading correlation on the channel capacity under a peak received-power constraint. Extension of the results to an average received-power constrained scenario is straightforward.

6.1 Correlated Log-normal Shadowing

The correlation between log-normally distributed g_0 and g_1 may be considered via a joint Gaussian pdf for $X_0 = \log g_0$ and $X_1 = \log g_1$. Equivalently, one may model X_0 and X_1 as follows [19],

$$X_i = \sqrt{\rho}X + \sqrt{1-\rho}Y_i, \quad i = 0, 1 \quad (31)$$

where X, Y_0 and Y_1 are independent zero-mean Gaussian random variables with equal variance of σ^2 and ρ is the correlation coefficient between X_0 and X_1 . It can be seen that g_0 and g_1 become equal if ρ is set to 1 and they would be independent log-normal variables if $\rho = 0$.

Under the model defined by (31) g_1/g_0 is given by,

$$\frac{g_1}{g_0} = e^{\sqrt{1-\rho}(Y_1-Y_0)} \quad (32)$$

It can be seen that as in the i.i.d. case, g_1/g_0 is log-normally distributed. However, $\log g_1/g_0$ has a variance of $2(1-\rho)\sigma^2$ instead of $2\sigma^2$. Thus, the previous capacity results apply here too if the dB spread of the log-normal shadowing is scaled by $\sqrt{1-\rho}$.

6.2 Correlated Nakagami Fading

When $\sqrt{g_0}$ and $\sqrt{g_1}$ are correlated Nakagami- m variables with a correlation coefficient of ρ , g_1/g_0 would be distributed according to the following pdf [20],

$$f_{\frac{g_1}{g_0}}(x) = \frac{(1-\rho)^m x^{m-1}}{\mathcal{B}(m, m)(1+x)^{2m}} \left(1 - \frac{4\rho x}{(1+x)^2}\right)^{-(m+\frac{1}{2})}, \quad x \geq 0 \quad (33)$$

Plugging the pdf above into (23) and integrating numerically, one is able to obtain the capacity for different degrees of correlation between the two channels. Capacity under a peak received-power constraint of $\alpha = 0$ dB is plotted versus the correlation coefficient in Fig. 5 for different fading distributions. Capacity of the non-fading AWGN case is also plotted for comparison.

7 Multiple Primary Users

The previous results were derived under the assumption that spectrum is being shared between the secondary user and one primary receiver. When more primary receivers are present, received (and hence transmitted) power of the secondary user would be subject to additional constraints resulting in a capacity reduction. To quantify this effect, in this section we derive the capacity subject to peak received-power constraint when n primary receivers are present. For mathematical tractability only Rayleigh fading case is considered however, following the same procedure, capacity for other fading distributions subject to peak/average received-power constraint may be evaluated numerically.

Let g_{0i} denote the channel gain of the secondary transmitter to the i th primary receiver.

Consequently, the received-power constraint in (22) is replaced by the following n constraints,

$$g_{0i}P(g_{01}, \dots, g_{0n}, g_1) \leq Q_i, \quad i = 1, \dots, n$$

or equivalently,

$$P(g_{01}, \dots, g_{0n}, g_1) \leq \min_i \frac{Q_i}{g_{0i}}, \quad i = 1, \dots, n \quad (34)$$

where $P(g_{01}, \dots, g_{0n}, g_1)$ is the optimal power allocation corresponding to the joint channel state $(g_{01}, \dots, g_{0n}, g_1)$. In the absence of any other constraints, capacity is achieved when the maximum allowed power according to (34) is transmitted. For ease of exposition let us assume $Q_i = Q, i = 1, \dots, n$. Thus, the capacity is given by,

$$C = \int_X B \log \left(1 + x \frac{Q}{N_0 B} \right) f_X(x) dx \quad (35)$$

where $X = g_1 / \max_i g_{0i}$. When $g_1, g_{01}, \dots, g_{0n}$ are i.i.d. unit-mean exponential random variables, the pdf of X is given by (see Appendix C),

$$f_X(x) = n \sum_{k=0}^{n-1} (-1)^k \frac{\binom{n-1}{k}}{(1+x+k)^2} \quad (36)$$

Substituting $f_X(x)$ from (36) into (35) yields the capacity under peak received-power constraint and Rayleigh fading when n primary users are present,

$$\begin{aligned} C &= B \int_X \log(1 + \alpha x) f_x(x) dx \\ &= B \int_0^\infty \log(1 + \alpha x) n \sum_{k=0}^{n-1} (-1)^k \frac{\binom{n-1}{k}}{(1+x+k)^2} dx \\ &= nB \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^\infty \frac{\log(1 + \alpha x)}{(1+x+k)^2} dx \\ &= nB \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\alpha \log[(1+k)\alpha]}{(1+k)\alpha - 1} \end{aligned} \quad (37)$$

where $\alpha = Q/N_0B$ as before. Simple inspection verifies that (37) boils down to (24) when $n = 1$. Fig. 6 shows the plot of capacity in (37) for $n = 1, 2, 3$. The capacity of the non-fading AWGN channel is also plotted for comparison. As expected, the capacity degrades with

increasing number of primary receivers. In particular, for a wide range of α , capacity with $n = 3$ is smaller compared to the non-fading AWGN case. That is, fading has a detrimental effect on the capacity as n gets larger since there is a higher probability of having a large channel gain toward at least one primary receiver.

8 Concluding Remarks

Dynamic spectrum sharing is emerging as a new paradigm for efficient spectrum access. Reforming the conventional policy to enable the adoption of this approach would enhance the spectrum utilization across the licensed spectrum, thereby paving the way for introduction of new wireless applications and services. In this context, with the conventional regulatory constraints on the transmitted power being either relaxed or removed, a constraint on the received-interference seems more appropriate. Building upon this argument we investigated the capacity of fading channels subject to constraints on the power received at a third-party (primary) receiver. Our results indicate that in many cases significant capacity gains may be achieved if the channels are varying due to fading.

This paper serves as a first step toward quantifying the fundamental limits of spectrum sharing in fading environments. We intend to extend and generalize our results in several directions. Throughout this paper we assumed perfect knowledge of the joint channel state. The effect of imperfect feedback (e.g., feedback quantization or outdated CSI) on the opportunistic spectrum access gain remains to be analyzed.

Another essential extension would be to consider multiple secondary users. This would raise the issues of fairness and multiple-access control. In particular, under an average *transmitted-power* constraint for individual users, the sum throughput of a multiple-access channel is maximized when at each instant only the user having the best channel gain is allowed to transmit [21]. Furthermore, the channel capacity under Rayleigh fading with such one-by-one scheduling is higher than the corresponding AWGN case. This is due to the fact that at each moment, with high probability, there will be a user with a very good channel condition (the so-called "multiuser diversity" gain). We conjecture that similar gains would be achievable under the *received-power* constraint if at each instant, the user with the best joint channel state (i.e. highest g_1/g_0) is scheduled for transmission.

Appendix A

We derive the distribution of g_1/g_0 for the general case where g_0 and g_1 have Gamma distributions with parameters m_0 and m_1 respectively (i.e. $\sqrt{g_0}$ and $\sqrt{g_1}$ have Nakagami distribution with parameters m_0 and m_1). When $m_0 = m_1 = 1$, the pdf reduces to that corresponding to the Rayleigh fading case.

Define two new random variables X and Y as,

$$X = \frac{g_1}{g_0} \quad Y = g_0 + g_1$$

Then the Jacobian is given by,

$$J = \begin{vmatrix} -g_1/g_0^2 & 1/g_0 \\ 1 & 1 \end{vmatrix} = -\frac{g_0 + g_1}{g_0^2} = -\frac{(1+x)^2}{y}$$

Thus,

$$dg_0 dg_1 = \frac{y}{(1+x)^2} dx dy$$

Consequently, using (15) the joint pdf of X and Y may be written as,

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{m_0^{m_0} m_1^{m_1}}{\Gamma(m_0)\Gamma(m_1)} \exp\left\{-\frac{m_0 y + m_1 x y}{1+x}\right\} \left(\frac{y}{1+x}\right)^{m_0-1} \left(\frac{x y}{1+x}\right)^{m_1-1} \frac{y}{(1+x)^2} \\ &= \frac{m_0^{m_0} m_1^{m_1}}{\Gamma(m_0)\Gamma(m_1)} \exp\left\{-\frac{m_0 y + m_1 x y}{1+x}\right\} y^{m_0+m_1-1} x^{m_1-1} (1+x)^{-(m_0+m_1)} \quad (\text{A.1}) \end{aligned}$$

The marginal distribution of $X = g_1/g_0$ may be obtained by integrating $f_{X,Y}(x, y)$ with respect to y ,

$$\begin{aligned} f_{\frac{g_1}{g_0}}(x) &= \int_0^\infty f_{X,Y}(x, y) dy \\ &= \frac{m_0^{m_0} m_1^{m_1}}{\Gamma(m_0)\Gamma(m_1)} x^{m_1-1} (1+x)^{-(m_0+m_1)} \int_0^\infty e^{-a y} y^b dy \end{aligned}$$

where $a = (m_0 + m_1 x)/(1+x)$ and $b = m_0 + m_1 - 1$. Through a change of variable, the latter

integral may be shown to be equal to $\Gamma(b+1)/a^{b+1}$. Thus,

$$f_{\frac{g_1}{g_0}}(x) = \left(\frac{m_0}{m_1}\right)^{m_0} \frac{x^{m_1-1}}{\mathcal{B}(m_0, m_1) \left(x + \frac{m_0}{m_1}\right)^{m_0+m_1}}, \quad x \geq 0 \quad (\text{A.2})$$

Note that for $m_0 = m_1 = 1$, the distribution in (A.2) coincides with the log-logistic distribution as claimed earlier.

Appendix B

If X is a random variable with beta-prime distribution as defined in (17), its mean is given by,

$$E(X) = \frac{1}{\mathcal{B}(m, m)} \int_0^\infty \frac{x^m}{(1+x)^{2m}} dx \quad (\text{B.1})$$

From (3.194-3) in [16], the integral above is equal to $\mathcal{B}(m+1, m-1)$. Thus,

$$\begin{aligned} E(X) &= \frac{\mathcal{B}(m+1, m-1)}{\mathcal{B}(m, m)} \\ &= \frac{\Gamma(m+1)\Gamma(m-1)}{\Gamma(m)\Gamma(m)} \\ &= \frac{m}{m-1} \end{aligned} \quad (\text{B.2})$$

where the last equality follows from the identity $\Gamma(m+1) = m\Gamma(m)$.

Appendix C

Let g_{0i} ($i = 1, \dots, n$) be i.i.d. unit-mean exponential random variables. Also assume that g_1 is independent of all g_{0i} and has the same distribution. Now define g_0 as,

$$g_0 = \max_i g_{0i}, \quad i = 1, \dots, n$$

Then, the cumulative distribution function (CDF) of g_0 is given by,

$$F_{g_0}(z) = \prod_{i=1}^n F_{g_{0i}}(z) = (1 - e^{-z})^n$$

After differentiation, the pdf of g_0 may be written as follows,

$$f_{g_0}(z) = ne^{-z}(1 - e^{-z})^{n-1}, \quad z \geq 0$$

Defining $X = g_1/g_0$ and $Y = g_0 + g_1$, and following the procedure of Appendix A, the pdf of g_1/g_0 is given by the following integral,

$$\begin{aligned} f_X(x) &= \int_0^\infty e^{-\frac{xy}{1+x}} n e^{-\frac{y}{1+x}} (1 - e^{-\frac{y}{1+x}})^{n-1} \frac{y}{(1+x)^2} dy \\ &= n \int_0^\infty e^{-y} (1 - e^{-\frac{y}{1+x}})^{n-1} \frac{y}{(1+x)^2} dy \\ &= \frac{n}{(1+x)^2} \int_0^\infty e^{-y} y \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \exp\left\{-\frac{ky}{1+x}\right\} dy \\ &= n \sum_{k=0}^{n-1} (-1)^k \frac{\binom{n-1}{k}}{(1+x)^2} \int_0^\infty y \exp\left\{-\left(1 + \frac{k}{1+x}\right)y\right\} dy \\ &= n \sum_{k=0}^{n-1} (-1)^k \frac{\binom{n-1}{k}}{(1+x+k)^2} \end{aligned}$$

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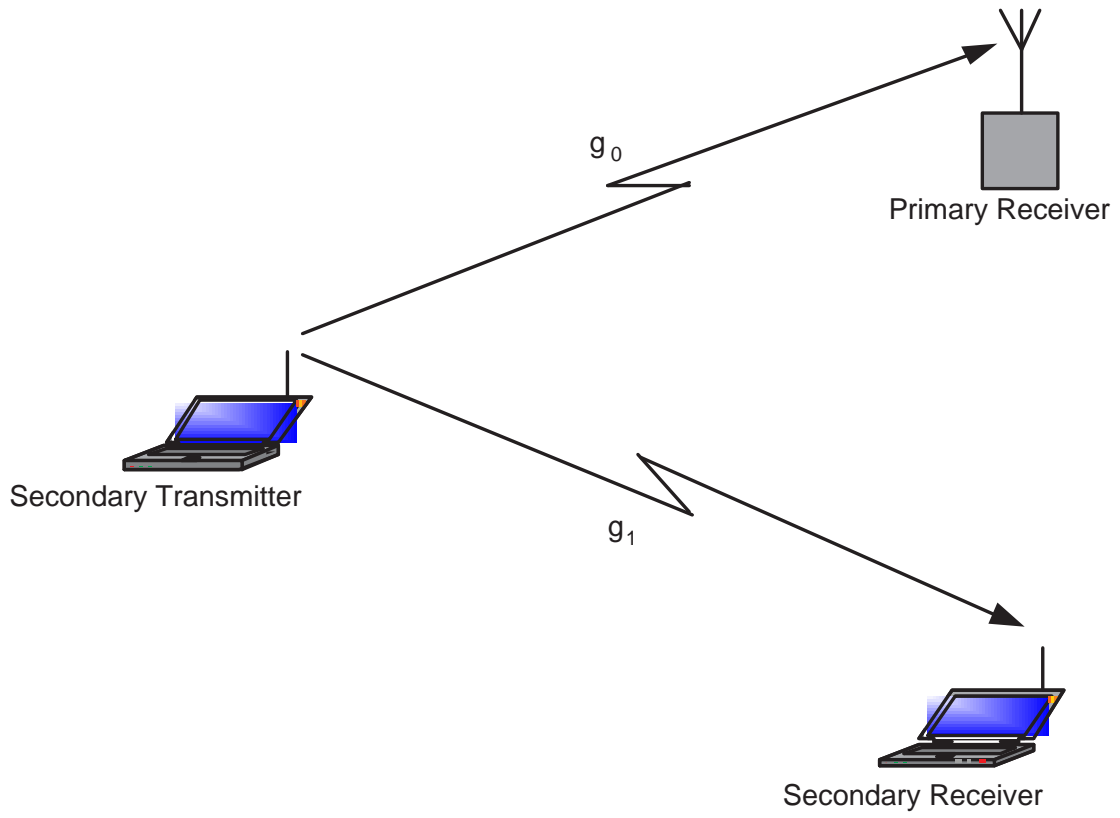


Figure 1: A secondary user sharing the spectrum with the primary user

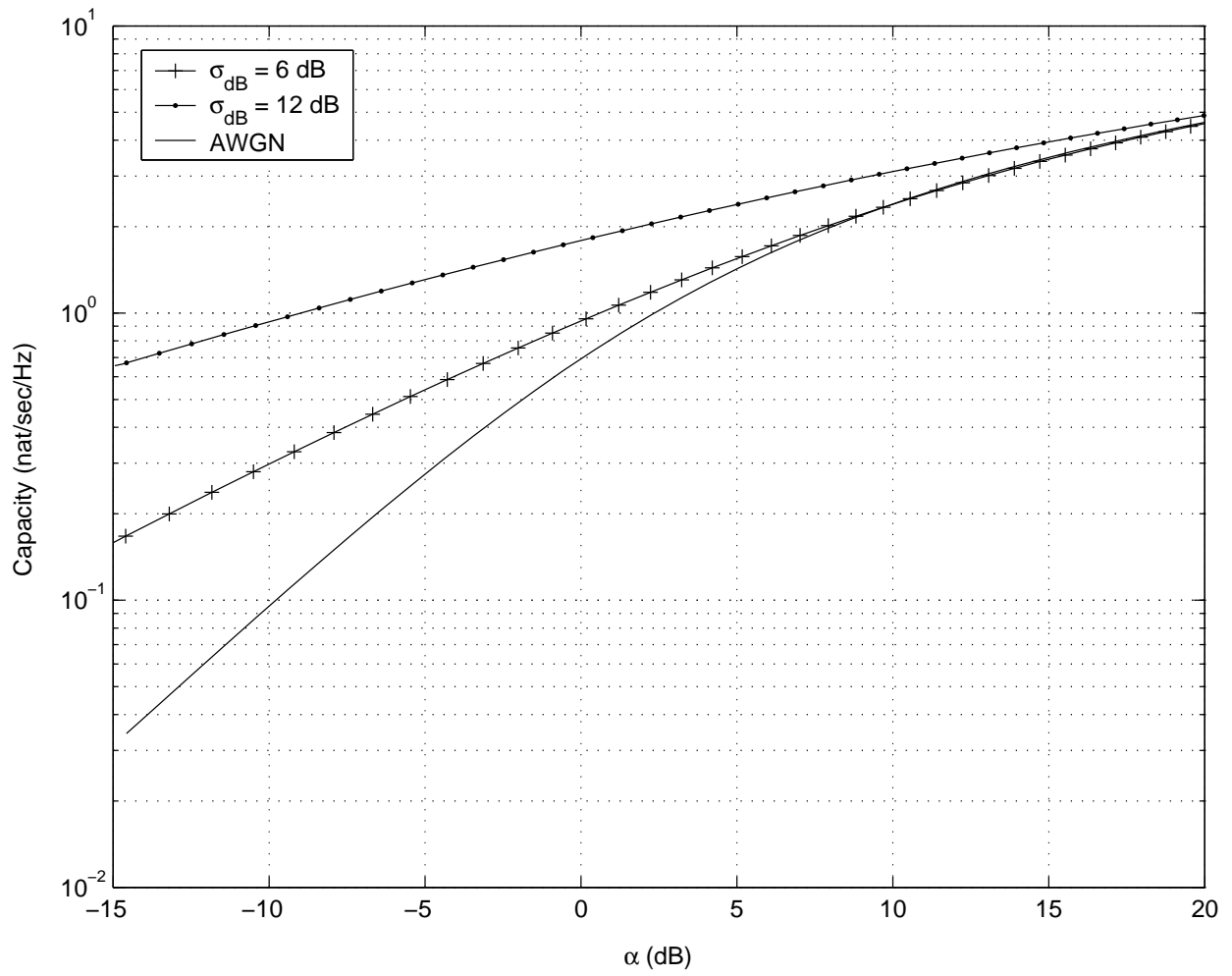


Figure 2: Capacity under log-normal fading and average received-power constraint vs. α

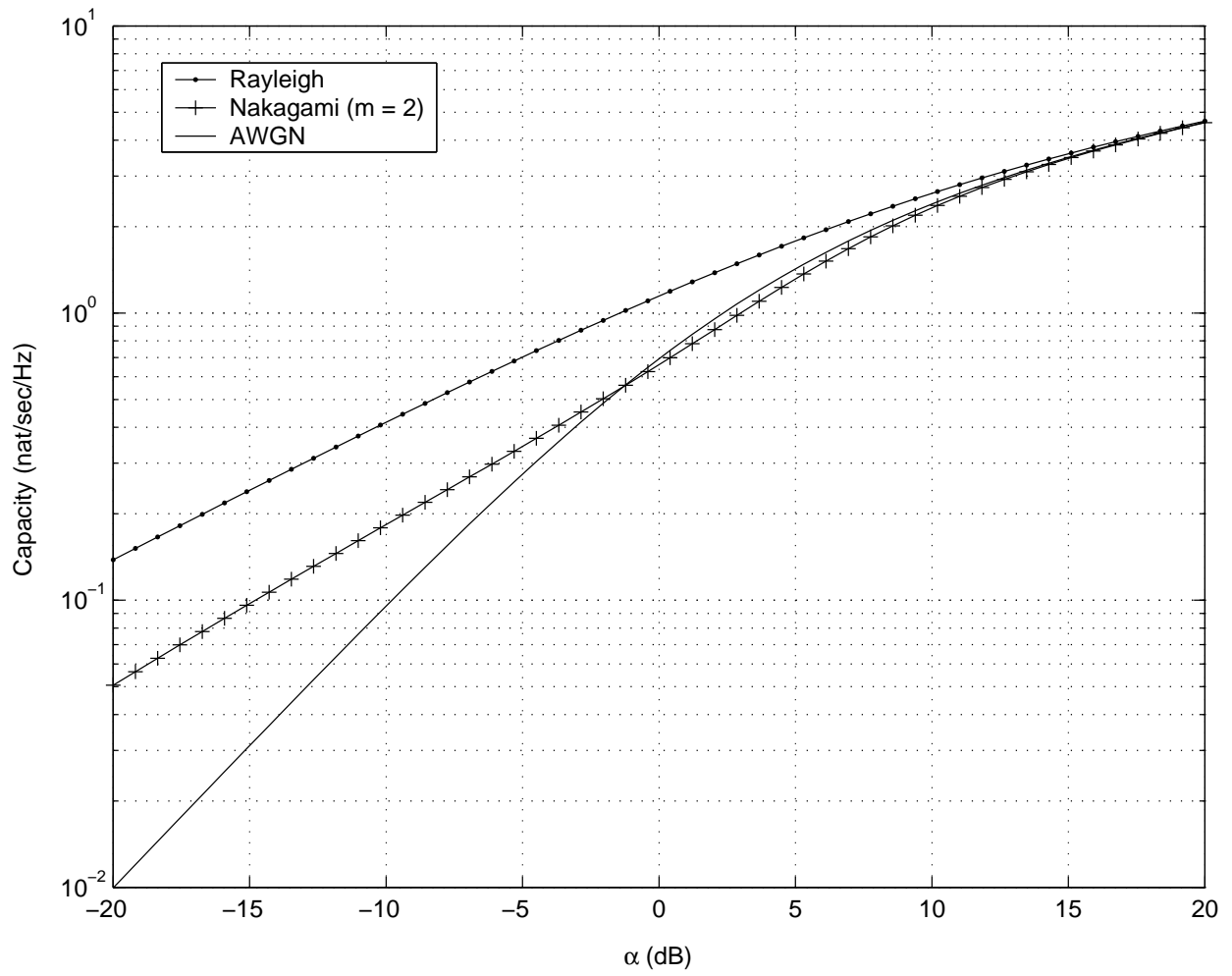


Figure 3: Capacity under average received-power constraint vs. α

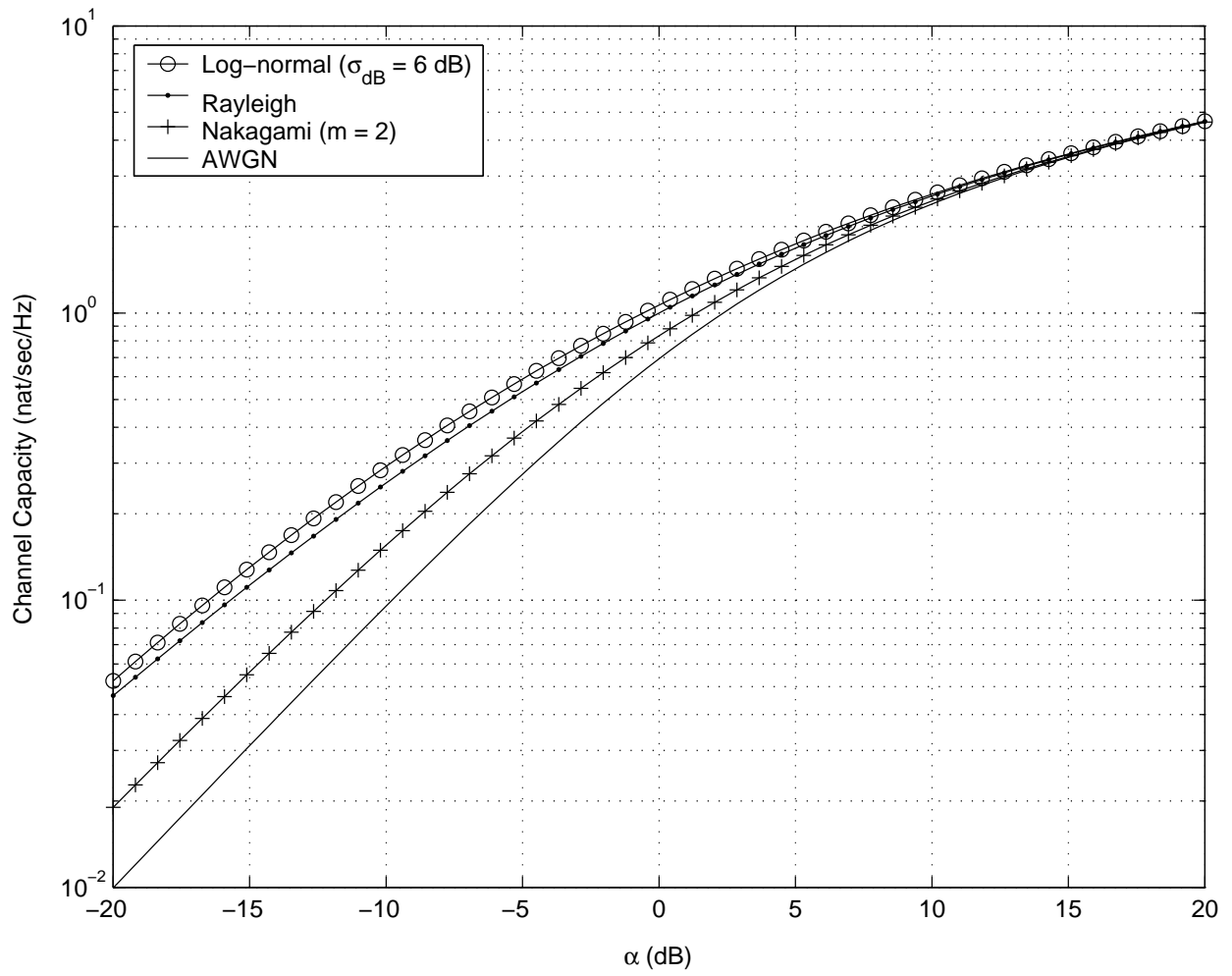


Figure 4: Capacity under peak received-power constraint vs. α

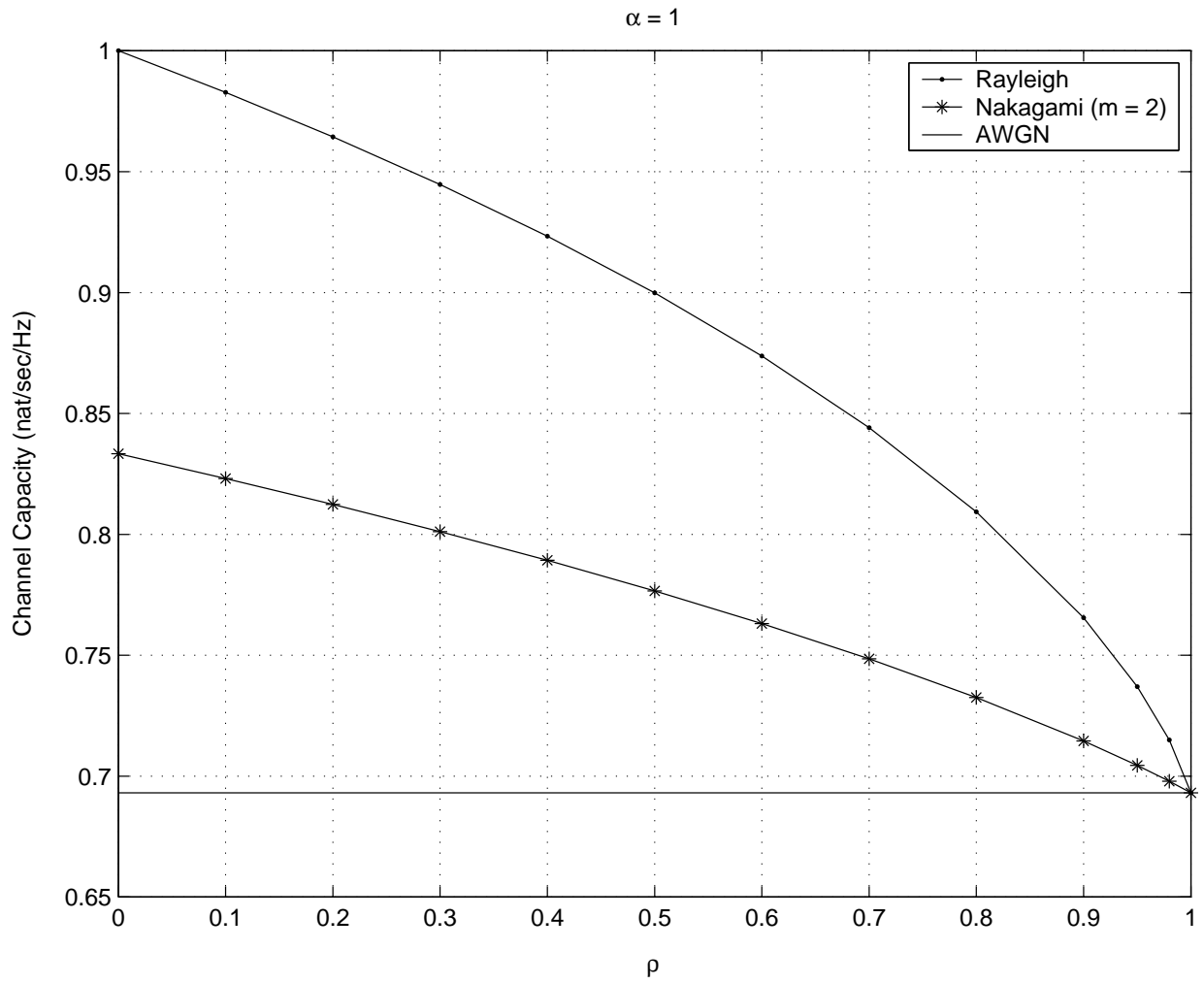


Figure 5: Capacity under peak received-power constraint vs. fading correlation coefficient, ρ , with $\alpha = 1$

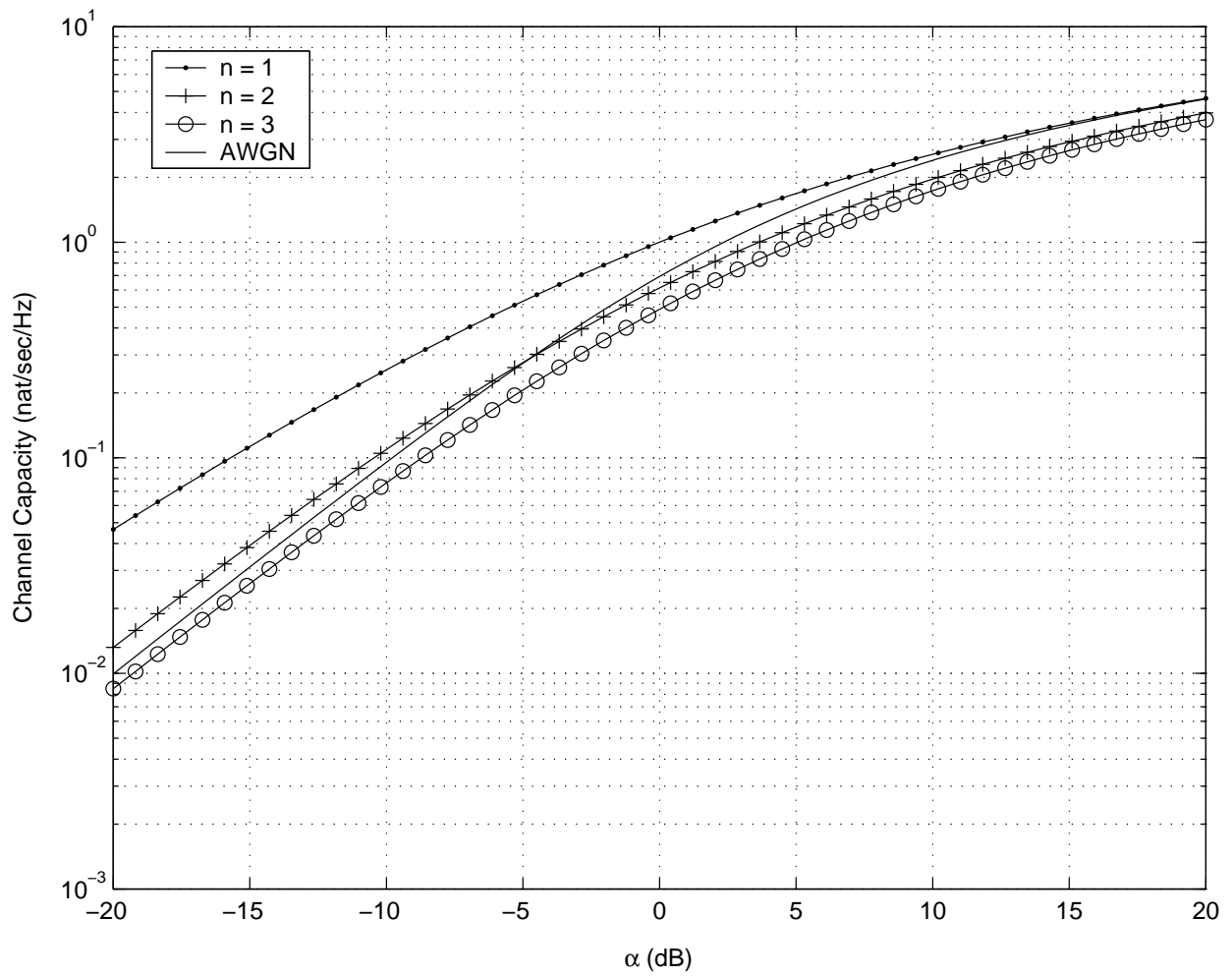


Figure 6: Capacity under peak received-power constraint and Rayleigh fading for different numbers of primary users