Chapter 3 Questions

Question 3.1

Based on the nature of values that each random variable can take, we can have the following classifications:

X: Discrete; since X is essentially count data)

Y: Continuous; since Y is measured data

M: Continuous; since M is measured data

N: Discrete; since N is count data

P: Discrete; since P is count data

Q: Continuous; since Q is measured data

Question 3.3

The results could be summarized into the table below:

Sample point	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
w	3-0 = 3	2-1 = 1	2-1 = 1	1-2 = -1	2-1 = 1	1-2 = -1	1-2 = -1	0-3 = -3

Question 3.5

The principal rule for solving this question is that all probabilities for all sample points within the sample space should add up to 1.

Part (a)

We desire:

$$\sum_{x=0}^{3} c(x^2 + 4) \;= 1$$

So:

$$c \text{ X } [(0 + 4)+ (1 + 4) + (4 + 4) + (9 + 4)] = 1$$

$$c = \frac{1}{30}$$

Part (b)

Following the similar manner of computation:

$$C[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}] = 1$$

$$C = \frac{1}{1+2\times3+1\times3} = \frac{1}{10}$$

Question 3.8

We first compute the probability associated to each sample point, then add up the corresponding sample points to obtain probability associated with each value of W.

P(W=-3)= P(TTT) = 1/3 X 1/3 X 1/3 = 1/27

P(W=-1) = P(HTT) + P(THT) + P(TTH) = 3 X (2/3 X 1/3 X 1/3) = 2/9

P(W=1) = P(HHT) + P(HTH) + P(THH) = 3 X (2/3 X 2/3 X 1/3) = 4/9

P(W=3) = P(HHH) = 2/3 X 2/3 X 2/3 = 8/27

Verify they sum up to one:

1/27 + 2/9 + 4/9 + 8/27 = 1

Thus, above describes a valid probability distribution on discrete random variable W.

Question 3.9

Part (a)

Integrate over the value of X to obtain the area under this part of the density function:

$$P(0 < X < 1) = \int_0^1 \frac{2(x+2)}{5}\,dx$$

$$= \left[\frac{x^2+4x}{5}\right]_0^1$$

= (1+4)/5 – 0

= 1

Part (b)

Still use integration over density function to compute the probability:

$$P(1/4 < X < 1/2) = \int_{1/4}^{1/2} \frac{2(x+2)}{5}\,dx$$

$$= \left[\frac{x^2+4x}{5}\right]_{1/4}^{1/2}$$

= (1/4+2)/5 – (1/16 + 1)/5

= 19/80

Question 3.11

First compute probability distribution of discrete random variable X:

Note that the values X could take on are: 0,1,2.

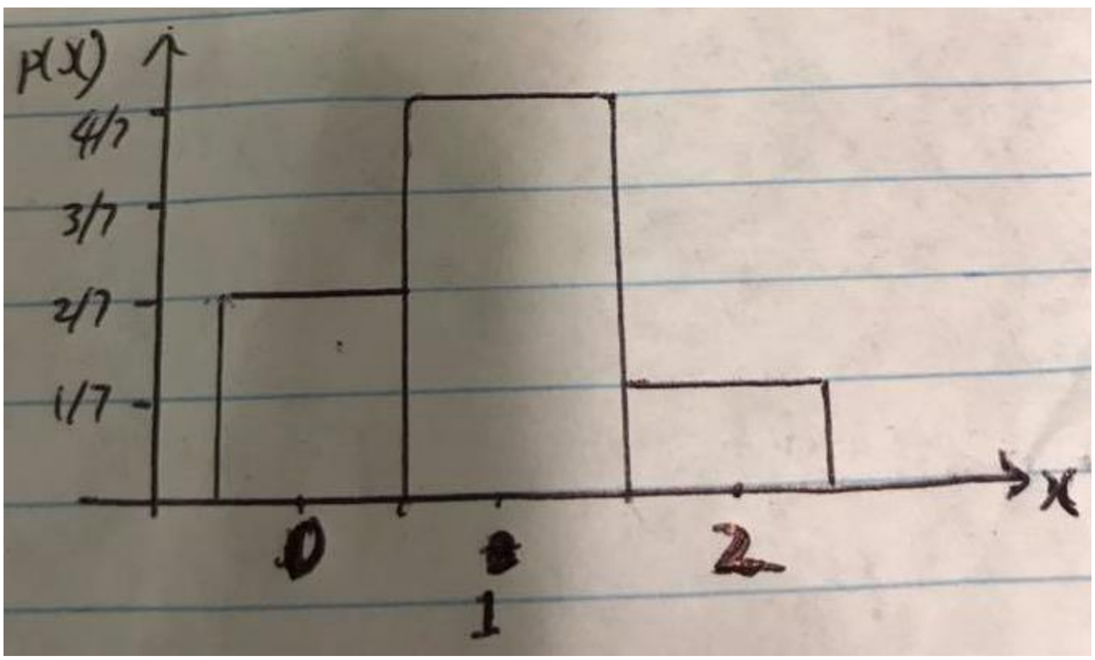
P(X=0) = $\frac{\binom{5}{3}}{\binom{7}{3}}$ = 10/35 = 2/7

P(X=1) = $\frac{\binom{5}{2}\binom{2}{1}}{\binom{7}{3}}$ = (10X2)/35 = 4/7

P(X=2) = $\frac{\binom{5}{1}\binom{2}{2}}{\binom{7}{3}}$ = (5X1)/35 = 1/7

Note that they sum up to 1, thus the above distribution is a valid probability distribution.

The probability histogram is as following:



Question 3.13

To obtain the cumulative distribution function, we just need to performing a cumulative running sum of the probability mass function on X.

Thus, the cumulative distribution function is shown as following:

X	$x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$	$3 \leq x < 4$	$x \geq 4$
F(x)	0	0.41	0.41+0.37 =0.78	0.78+0.16=0.94	0.94+0.05=0.99	0.99+0.01 = 1

Note the ending value for the cumulative function always equals to 1.

Question 3.14

Part (a)

We just need to directly plot in the value of x into F(x):

With 12 minutes = 0.2 hours, we have:

$$F(x = 0.5) = 1 - e^{-8\times 0.2} = 0.7981$$

Part (b)

First, compute probability density function:

$$f(x) = F'(x) = 8e^{-8x}$$

Then, compute the probability using integration over the density function:

$$p = \int_0^{0.2} f(x)dx$$

$$= 8\int_0^{0.2} e^{-8x}dx$$

$$= 8 \times \frac{-1}{8}[e^{-8x}]_0^{0.2}$$

$$= 1 - e^{-8\times 0.2}$$

= 0.7981

Question 3.15

First, find the cumulative distribution function F(x):

Note that from question 3.11:

P(X=0) = 2/7

P(X=1) = 4/7

P(X=2) = 1/7

Then the cumulative function for F(x) is shown as following:

x	$x < 0$	$0 \le x < 1$	$1 \le x < 2$	$x \ge 2$
F(x)	0	2/7	2/7 + 4/7 = 6/7	6/7 + 1/7 = 1

Then use F(x) for asked quantities.

Part (a)

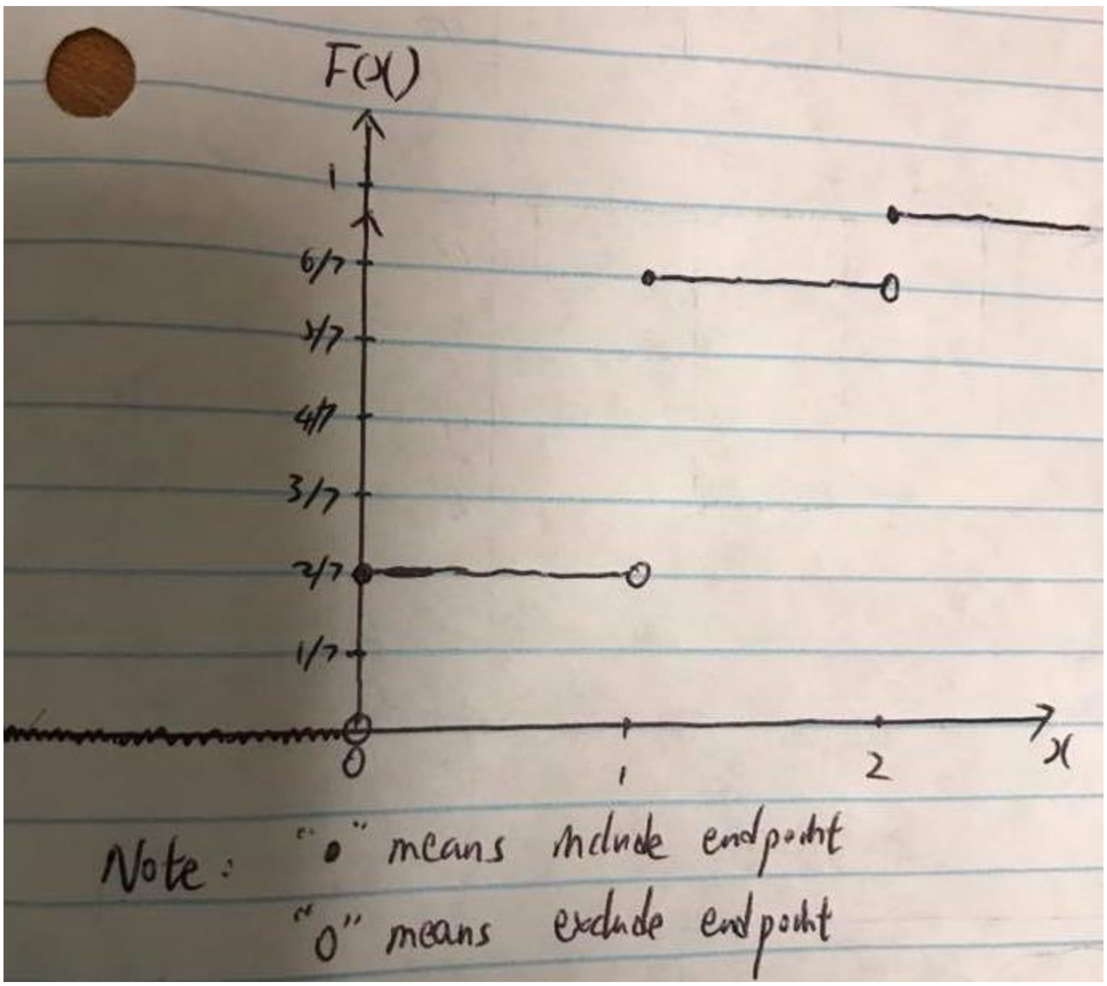
P(X = 1) = F($X = 1$) – F($X = 0$) = 6/7 – 2/7 = 4/7 (since there is no probability mass within (0,1))

Part (b)

P(0 < X≤ 2) = P($X \leq 2$) – P($X \leq 0$) = F(X = 2) – F(X = 0) = 1 – 2/7 = 5/7

Question 3.16

The graph is shown as following:



Question 3.29

Part (a)

For this density function to be valid, it has to satisfy two conditions:

1. f(x) $\geq 0\ \forall x$

2. $\int_{-\infty}^{+\infty} f(x)dx = 1$

Condition 1 is easily verified as exponential function never takes on negative values.

For condition 2 verification:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{1}^{+\infty} 3x^{-4}dx$$

$$= -[x^{-3}]_{1}^{+\infty}$$

$$= 1$$

Thus this density function is indeed valid.

Part (b)

F(x) is computed by the integration of f(x):

$$F(X) = \int_{-\infty}^{X} f(a)da$$

$$= \int_{1}^{x} 3a^{-4}da$$

$$= -[a^{-3}]_{1}^{x}$$

$$= 1 - x^{-3} \quad (for\ x \geq 1)$$

For x<1, as f(x)=0 everywhere, F(x)=0.

Part (c)

The probability can be computed using compliment:

$$P = 1\text{-}F(4)$$

$$= 1 - (1 - 4^{-3})$$

$$= 1/64$$

Question 3.36

Given the questions asked are about probabilities of certain range, compute the cumulative distribution function first:

$$F(x) = \int_{-\infty}^{x} f(a)da$$

$$= \int_{0}^{x} 2(1 - a)da$$

$$= [-a^2 + 2a]_0^x$$

$$= -x^2 + 2x \text{ (if } 0 <= x <= 1)$$

If x < 0: trivially F(x) = 0

If x > 1: F(x) = 1

Part (a)

Use cumulative function:

$$P(X \leq 1/3) = F(x = 1/3) = -\left(\frac{1}{3}\right)^2 + 2/3 = 5/9$$

Part (b)

Use compliment for computing the desired probability, with cumulative function:

$$P = 1-P(x<=0.5)$$

$$= 1-F(x=0.5)$$

$$= 1 - (-0.5^2 + 2 \times 0.5)$$

$$= 0.25$$

Part (c)

$$P(x < 0.75|x \geq 0.5) = \frac{p(0.5 \leq x < 0.75)}{p(x \geq 0.5)}$$

$$= \frac{F(x=0.75) - F(x=0.5)}{1 - F(x=0.5)}$$

$$= \frac{-0.75^2 + 2 \times 0.75 - (-0.5^2 + 2 \times 0.5)}{1 - (-0.5^2 + 2 \times 0.5)}$$

$$= \frac{0.9375 - 0.75}{0.25}$$

$$= 0.75$$

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Problem 3.38

Part (a)

$$p(X \leq 2, Y = 1)$$
$$= p(X = 0, Y = 1) + p(X = 1, Y = 1) + p(X = 2, Y = 1)$$
$$= f(0,1) + f(1,1) + f(2,1)$$
$$= \frac{0+1}{30} + \frac{1+1}{30} + \frac{2+1}{30} = \frac{1}{5}$$

Part (b)

$$p(X > 2, Y \leq 1)$$
$$= p(X = 3, Y = 0) + P(X = 3, Y = 1)$$
$$= f(3,0) + f(3,1)$$
$$= \frac{3+0}{30} + \frac{3+1}{30} = \frac{7}{30}$$

Part (c)

$$p(X > Y)$$
$$= f(1,0) + f(2,0) + f(2,1) + f(3,0) + f(3,1) + f(3,2)$$
$$= \frac{(1+0) + (2+0) + (2+1) + (3+0) + (3+1) + (3+2)}{30}$$
$$= \frac{3}{5}$$

Part (d)

$$p(X + Y = 4)$$
$$= f(2,2) + f(3,1)$$
$$= \frac{(2+2) + (3+1)}{30}$$
$$= \frac{4}{15}$$

Problem 3.40

Part (a)

Integrate the joint density function over all values of Y to get marginal density of X:

$$g(x) = \int_0^1 f(x,y)dy$$

$$= \int_0^1 \frac{2}{3}(x + 2y)dy$$

$$= \frac{2}{3}[xy + y^2]_0^1$$

$$= \frac{2x + 2}{3}$$

Part (b)

Integrate the joint density function over all values of X to get marginal density of Y:

$$h(y) = \int_0^1 f(x,y)dy$$

$$= \int_0^1 \frac{2}{3}(x + 2y)dx$$

$$= \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1$$

$$= \frac{4y + 1}{3}$$

Part (c)

This event corresponds to the condition on random variable: $X \leq 0.5$.

As only the drive-through facility is in the picture of the problem, we only need to consider the marginal density function g(x).

Integrate over the density function to get the desired event probability:

$$p(X \leq 0.5) = \int_0^{0.5} g(x)dx$$

$$= \int_0^{0.5} \frac{2x + 2}{3}dx$$

$$= \left[\frac{x^2 + 2x}{3}\right] \begin{matrix} 0.5 \\ 0 \end{matrix}$$

$$= \frac{1.25}{3}$$

$$= \frac{5}{12}$$

Problem 3.56

Part (a)

Note that for values of f(x,y), Y's range is actually defined based on x, we can have the following observation:

Given a fixed value of Y = 0.5:

Case 1: x = 0.3. Then y < 1-x, and thus f(0.3, 0.5) = 6X0.3 = 1.8 (*Note that since it's density function, so a value greater than 1 is possible*).

Case 2: x = 0.7. Then y > 1-x, and thus f(0.7, 0.5) = 0

Thus we have f(0.3, 0.5)≠f(0.7, 0.5), meaning the probability density for y = 0.5 is dependent on x. Thus, X and Y are not independent.

Part (b)

Note that for Y = 0.5, when X ≥ 0.5, y < 1-x doesn't hold and leads to probability density 0. Thus, we only need to consider domains of X where the density is non-zero.

$$P(X > 0.3 \mid Y = 0.5)$$

$$= \int_{0.3}^{1} f(x, 0.5)dx$$

$$= \int_{0.3}^{0.5} f(x, 0.5)dx$$

$$= \int_{0.3}^{0.5} 6xdx$$

$$= [3x^2]_{0.3}^{0.5}$$

$$= 0.48$$

Problem 3.57

Part (a)

The requirement for probability density function to be valid is that it would be integrated to 1 over all domains.

$$\int_0^2 \int_0^1 \int_0^1 f(x, y, z) dxdydz$$

$$= \int_0^2 \int_0^1 \int_0^1 kxy^2 z dxdydz$$

$$= \int_0^2 \int_0^1 \frac{ky^2 z}{2} dydz$$

$$= \int_0^2 \frac{kz}{6} dz$$

$$= \frac{k}{3}$$

$$= 1 \ (by\ definition)$$

Thus, k = 3.

Part (b)

Integrate the corresponding region with respect to each of the variable, we have:

$$p(X < \frac{1}{4}, Y > \frac{1}{2}, 1 < Z < 2)$$

$$= \int_1^2 \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{4}} f(x, y, z) dxdydz$$

$$= \int_1^2 \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{4}} kxy^2 z dxdydz$$

$$= \int_1^2 \int_{\frac{1}{2}}^1 \frac{ky^2 z}{32} dydz$$

$$= \int_1^2 \frac{7kz}{768} dz$$

$$= \left[\frac{7kz^2}{1536}\right] \begin{matrix} 2 \\ 1 \end{matrix}$$

$$= \frac{63}{1536}$$

$$= \frac{21}{512}$$