STA286 Problem Set 4 Solution

Problem 4.6

Let random variable "X" denote earning, then according to expectation definition:

$$E(X) = \sum_{x \in X} xp(x)$$

$$= \frac{1}{12} \times 7 + \frac{1}{12} \times 9 + \frac{1}{4} \times 11 + \frac{1}{4} \times 13 + \frac{1}{6} \times 15 + \frac{1}{6} \times 17$$

$$= \frac{38}{3}$$

Problem 4.7

According to definition of expectation, the expected gain is computed as following:

$$E(X) = \sum_{x \in X} xp(x)$$

$$= 4000 \times 0.3 + (-1000) \times 0.7$$

$$= 500$$

Problem 4.12

Taking expectation over continuous probability density function:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

$$= \int_{0}^{1} x \times 2(1 - x)dx$$

$$= \left[x^2 - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

Thus, the profit in money figure is $\frac{1}{3} \times \$5000 = \$1666.67$.

Problem 4.17

$$\mu_{g(X)} = \sum_{x \in X} g(x)f(x)$$

$$= (2 \times (-3) + 1)^2 \times \frac{1}{6} + (2 \times 6 + 1)^2 \times \frac{1}{2} + (2 \times 9 + 1)^2 \times \frac{1}{3}$$

$$= 209$$

Problem 4.18

Expected value of the random variable g(X) is:

$$E[g(X)] = \sum_{x \in X} g(x)f(x)$$

$$= \sum_{x \in X} x^2 f(x)$$

$$= \sum_{x \in X} x^2 \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x}$$

$$= 0^2 \binom{3}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{3-0} + 1^2 \binom{3}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{3-1} + 2^2 \binom{3}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{3-2} + 3^2 \binom{3}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{3-3}$$

$$= 0 + 1 \times 3 \times \frac{1}{4} \times \frac{9}{16} + 4 \times 3 \times \frac{1}{16} \times \frac{3}{4} + 9 \times 1 \times \frac{1}{64} \times 1$$

$$= \frac{9}{8}$$

Problem 4.21

The average profit is computed by expectation of the random variable $g(X) = X^2$:

$$E[g(X)] = \int_0^1 x^2 f(x) dx$$

$$= \int_0^1 x^2 \times 2(1 - x) dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1 = \frac{1}{6}$$

Thus, the profit in money figure is $\frac{1}{6} \times \$5000 = \$833.33$.

Problem 4.23

Part (a)

$$\mathbb{E}[g(X,Y)] = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} g(x,y)\, p(x,y)$$

$$= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} x y^2\, p(x,y)$$

$$= 2\times1^2\times0.1 + 4\times1^2\times0.15 + 2\times3^2\times0.2 + 4\times3^2\times0.3$$
$$+ 2\times5^2\times0.1 + 4\times5^2\times0.15$$

$$= 35.2$$

Part (b)

$$\mu_X = \sum_{x \in \mathcal{X}} x\, p(x) \qquad \mu_Y = \sum_{y \in \mathcal{Y}} y\, p(y)$$

We need the marginal probabilities $p(x)$ and $p(y)$

Note: $p(x) = \sum_{y \in \mathcal{Y}} p(x,y)$ and $p(y) = \sum_{x \in \mathcal{X}} p(x,y)$.

Thus we have:

x	2	4
$p(x)$	0.4	0.6

y	1	3	5
$p(y)$	0.25	0.5	0.25

Then, we can proceed to compute:

$$\mu_X = 2\times0.4 + 4\times0.6 = 3.2$$

$$\mu_Y = 1\times0.25 + 3\times0.5 + 5\times0.25 = 3.0$$

Problem 4.35

According to the Theorem 4.2, we have:

$$\sigma^2 = E(X^2) - \mu^2$$

$$\mu = \sum_{x \in \mathcal{X}} x f(x) = 2\times0.01 + 3\times0.25 + 4\times0.4 + 5\times0.3 + 6\times0.04 = 4.11$$

$$E(X^2) = \sum_{x \in \mathcal{X}} x^2 f(x) = 2^2\times0.01 + 3^2\times0.25 + 4^2\times0.4 + 5^2\times0.3 + 6^2\times0.04 = 17.63$$

Thus $\sigma^2 = 17.63 - 4.11^2 = \cancel{13.52}\ 0.738$

Problem 4.41

To compute standard deviation: $\sigma = \sqrt{E[g(x)^2] - \mu_{g(x)}^2}$

From Problem 4.17, we have obtained:

$$\mu_{g(x)} = 209$$

To compute $E[g(x)^2]$:

$$E[g(x)^2] = \sum_{x \in X} g(x)^2 f(x)$$

$$= \sum_{x \in X} (2x+1)^4 f(x)$$

$$= [2\times(-3)+1]^4 \tfrac{1}{6} + [2\times 6+1]^4 \tfrac{1}{2} + [2\times 9+1]^4 \tfrac{1}{3}$$

$$= 57825$$

Therefore:

$$\sigma = \sqrt{57825 - 209^2} = \sqrt{14144} = 8\sqrt{221}$$

Problem 4.50

Variance: $\sigma_x^2 = E[x^2] - \mu^2$

$\cancel{\mu = \sum}$

$$\mu = \int_0^1 f(x) \cdot x \, dx \qquad E[x^2] = \int_0^1 f(x) \cdot x^2 \, dx$$

$$= \int_0^1 2(1-x) \cdot x \, dx \qquad = \int_0^1 2(1-x)\, x^2 \, dx$$

$$= \left[x^2 - \tfrac{2}{3}x^3\right]_0^1 \qquad = \left[\tfrac{2}{3}x^3 - \tfrac{1}{2}x^4\right]_0^1$$

$$= \frac{1}{3} \qquad = \frac{1}{6}$$

Thus: $\sigma_x^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$

Standard deviation: $\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{1}{18}} = \frac{\sqrt{2}}{6}$

Problem 4.58

According to Theorem 4.6, we have:

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$$

Thus, computing expectation of Y as summing over two functions of X:

$$EY = E[60X^2 + 39X] = E(60X^2) + E(39X)$$
$$= 60\,E(X^2) + 39\,E(X)$$

$$E(X^2) = \int_0^2 x^2 f(x)\,dx$$
$$= \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx$$
$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_1^2$$
$$= \frac{1}{4} + \frac{11}{12} = \frac{7}{6}$$

$$E(x) = \int_0^2 x\,f(x)\,dx$$
$$= \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx$$
$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$
$$= \frac{1}{3} + \frac{2}{3} = 1$$

Thus: $EY = 60 \cdot \frac{7}{6} + 39 \cdot 1 = 109$

Problem 4.60

Part (a)

$$E(2X - 3Y)$$
$$= E(2X) - E(3Y)$$
$$= 2E(X) - 3E(Y)$$

Marginal distribution:

x	2	4
$p(x)$	0.4	0.6

y	1	3	5
$p(y)$	0.25	0.5	0.25

$E(X) = \sum_{x \in X} x\,p(x) = 2 \times 0.4 + 4 \times 0.6 = 3.2$

$E(Y) = \sum_{y \in Y} y\,p(y) = 1 \times 0.25 + 3 \times 0.5 + 5 \times 0.25 = 3.0$

So: $E(2X - 3Y)$
$= 2 \times 3.2 - 3 \times 3.0$
$= -2.6$

Part (b) Since X and Y are independent, we have:

$$E(XY) = E(X)\,E(Y)$$

From part (a), we have $E(X) = 3.2$ and $E(Y) = 3.0$

So: $E(XY) = 3.2 \times 3.0 = 9.6$

Problem 4.65

Part (a) Note that red die toss event is independent of the green die toss event. Thus X and Y are independent random variables.

$$E(X+Y) = EX + EY$$

For a die toss, there is $\frac{1}{6}$ probability for outcome of $1 \sim 6$, thus

$$EX = EY = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

So: $E(X+Y) = 3.5 + 3.5 = 7.0$

Part (b) $E(X-Y) = EX - EY = 3.5 - 3.5 = 0$

Part (c) Because X, Y are independent random variables:

$$E(XY) = EX\,EY = 3.5 \times 3.5 = 12.25$$

Problem 4.66

Similar to Problem 4.65, X and Y are independent.

Part (a) $\sigma^2(2X-Y) = 2^2\sigma^2 X + (-1)^2\sigma^2 Y = 4\sigma^2 X + \sigma^2 Y$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \frac{1}{6}[1^2+2^2+3^2+4^2+5^2+6^2] - \left[\frac{1}{6}[1+2+3+4+5+6]\right]^2$$

$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Similarity $\sigma_Y^2 = \frac{35}{12}$ (same computation, on a different random variable)

So: $\sigma^2(2X-Y) = 4\cdot\frac{35}{12} + \frac{35}{12} = \frac{175}{12}$

Part (b) $\sigma^2(X+3Y-5) = \sigma^2 X + 3^2\sigma^2 Y + 0 = \sigma^2 X + 9\sigma^2 Y$

$$= \frac{35}{12} + 9\cdot\frac{35}{12} = \frac{175}{6}$$