Problem Set 7 Solutions

4.75 light bulbs $\mu = 900$ hours
$\sigma = 50$ hours

Chebyshev's Theorem

$$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2}$$

→ Use when form of distribution unknown

Probability random variable X will assume a value within $\underline{K}$ standard deviations of mean

$$\mu - K\sigma = 700$$

$$K = 4$$

$$P(700 < X < 1100) \geq \underbrace{1 - \frac{1}{4^2}}_{0.9375}$$

$$P(X \leq 700) \leq \left(\tfrac{1}{2}\right)(1 - 0.9375)$$

$$\leq \underline{\underline{0.03125}}$$

4.77(b) Random variable X, $\mu = 10$, $\sigma^2 = 4$ $(\sigma = 2)$

$$P(|X - 10| < 3)$$

$$P(-3 < (X - 10) < 3)$$

$$3 = K\sigma$$
$$= K(2)$$
$$K = 3/2$$

$$P\left[10 - \left(\tfrac{3}{2}\right)(2) < X < 10 + \left(\tfrac{3}{2}\right)(2)\right] \geq \left.1 - \frac{1}{K^2}\right\} \left(1 - \frac{1}{\left(\frac{3}{2}\right)^2}\right)$$

$$\underline{\underline{\geq 5/9}}$$

7.18 $g(x;p) = pq^{x-1}$

Moment generating function of x

$$M_X(t) = E(e^{tX}) \quad (\S\ 7.3,\ \text{DEFINITION } 7.2)$$

$$= \sum_x e^{tX}(pq^{x-1}) \quad \text{DISCRETE}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (e^t q)^x = \frac{pe^t}{1 - qe^t}$$

*Using sum of infinite geometric series beginning at $x=1$

$$\sum_{x=1}^{\infty} r^x = \frac{r}{1-r} \quad |r| < 1$$

MEAN & VARIANCE $\left\{ \mu = \mu_1' \quad \sigma^2 = \mu_2' - \mu^2 \right.$

Using Theorem 7.6 in $\S$ 7.3

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r'$$

r^{th} moment about the origin

$$\mu'_1 = \frac{d(M_X(t))}{dt}\bigg|_{t=0} = \frac{pe^t}{(1-qe^t)} + pe^t\left(\frac{qe^t}{(1-qe^t)^2}\right)\bigg|_{t=0}$$ (using product Rule & chain rule)

$$\mu = \frac{pe^t(1-qe^t) + pqe^{2t}}{(1-qe^t)^2}\bigg|_{t=0} = \frac{(1-q)p + pq}{(1-q)^2} = \underline{\underline{\frac{1}{p}}} \quad \left(\begin{array}{c}\text{since}\\ p+q=1\end{array}\right)$$

$$\mu'_2 = \frac{d^2(M_X(t))}{dt^2} = \frac{(1-qe^t)^2 pe^t + 2pqe^{2t}(1-qe^t)}{(1-qe^t)^4}\bigg|_{t=0} = \frac{2-p}{p^2}$$

$$\sigma^2 = \mu'_2 - \mu^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{(1-p)}{p^2} = \underline{\underline{\frac{q}{p^2}}}$$

7.21 Moment generating function of Random variable X having chi-squared distribution with ν degrees of freedom.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$ moment generating continous

$$\chi^2: f(x;\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, & x>0 \\ 0, & \text{ELSEWHERE} \end{cases}$$

gamma function
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$
for $\alpha > 0$

So for χ^2
$$M_X(t) = \underbrace{\frac{1}{\Gamma(\nu/2)2^{\nu/2}}} \int_0^\infty e^{tx} e^{-x/2} x^{\nu/2-1}\,dx$$

$$= \text{“} \quad \text{”} \int_0^\infty x^{\nu/2-1} e^{-(1/2-t)x}\,dx$$

$$= \text{“} \quad \text{”} \int_0^\infty \left(\frac{2y}{1-2t}y\right)^{\nu/2-1} e^{-y} \frac{2}{(1-2t)}\,dy$$

$$= \text{“} \quad \text{”} \left(\frac{2}{1-2t}\right)^{\nu/2} \underbrace{\int_0^\infty y^{\nu/2-1} e^{-y}\,dy}_{\Gamma(\nu/2)}$$

$$= \frac{1}{\cancel{\Gamma(\nu/2)}2^{\nu/2}}\left(\frac{2}{1-2t}\right)^{\nu/2} \cancel{\Gamma(\nu/2)}$$

$$= \underline{\underline{(1-2t)^{-\nu/2}}}$$

Let $y = \left(\frac{1}{2}-t\right)x$

$$\frac{dy}{dx} = \left(\frac{1}{2}-t\right)$$

$$dx = \frac{dy}{(\frac{1}{2}-t)} = \frac{2dy}{(1-2t)}$$

$$x = \frac{y}{(\frac{1}{2}-t)} = \frac{2y}{(1-2t)}$$

$a^n a^m = a^{n+m}$

$(ab)^n = a^n b^n$

7.23 If both X & Y are distributed independently, following exponential distributions with mean parameter 1, find distribution of

$U = X + Y$ $\qquad$ $f(x;1) = e^{-x}, \quad x > 0$

Exponential distribution

→ Using Theorem 7.10

$M_U(t) = M_X(t)\, M_Y(t)$

$$\underbrace{M_X(t)}_{\text{Same result for } M_Y(t)} = E(e^{tx}) = \int_0^\infty e^{tx} e^{-x}\, dx, \quad x > 0$$

$$= \int_0^\infty e^{x(t-1)}\, dx$$

MGF exists if $M_X(t)$ converges; $t < 1$

$$= \frac{1}{t-1} e^{x(t-1)} \Big|_0^\infty$$

$$= -\left(\frac{1}{t-1}\right) = \frac{1}{1-t}$$

$$M_U(t) = \left(\frac{1}{1-t}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}$$

This is equal to the moment generating function of a gamma distribution with: $\alpha = 2 \quad \beta = 1 \qquad f(x; \alpha = 2, \beta = 1)$ gamma distribution

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* The MGF of the gamma distribution is:

$$\left(\frac{1}{1-\beta t}\right)^{\alpha}$$

Gamma pdf $f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$ $\quad \alpha > 0 \; \beta > 0$, $x > 0$

Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \quad \alpha > 0$

Use a change of variable technique

$y = x\left(\frac{1}{\beta} - t\right) \quad x = \frac{\beta}{1-\beta t} y \quad dx = \frac{\beta}{1-\beta t} dy$

$$M_X(t) = \int_0^\infty e^{tx}\, \overbrace{f(x)}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^\infty x^{\alpha-1} e^{tx} e^{-x/\beta}\, dx$$

$$= \text{''} \quad \text{''} \int_0^\infty x^{\alpha-1} e^{-x(1/\beta - t)}\, dx$$

$$\int_0^\infty \left(\frac{\beta}{1-\beta t} y\right)^{\alpha-1} e^{-y} \left(\frac{\beta}{1-\beta t}\right) dy$$

$$= \text{''} \quad \text{''} \underbrace{\int_0^\infty y^{\alpha-1} e^{-y}}_{\Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^{\alpha} \cancel{\left(\frac{1-\beta t}{\beta}\right)} \cancel{\left(\frac{\beta}{1-\beta t}\right)} dy$$

$$= \frac{1}{\cancel{\Gamma(\alpha)\beta^{\alpha}}} \frac{\cancel{\beta^{\alpha}}\, \cancel{\Gamma(\alpha)}}{(1-\beta t)^{\alpha}}$$

$$= \frac{1}{(1-\beta t)^{\alpha}}$$

8.1 a) Population could be all residents of Richmond (with phones)

b) Population - infinite number of coin tosses

c) All pairs of new shoe tested on professional tour.

d) All time intervals to drive from home to office.

8.14 Sample variance unchanged if constant c $\pm$ each value

a) Sample variance $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

adding constant c to each X_i $\bar{X} \to \bar{X} + c$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left((X_i + c) - (\bar{X} + c)\right)^2$$

c is cancelled out for each X_i, S^2 remains unchanged

b) Each observation multiplied by c:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (cX_i - c\bar{X})^2$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} c^2 (X_i - X)^2 = \underline{c^2 S^2}$$

8.17 Samples of size $n = 16$ drawn from normal population with $\mu = 50$ $\sigma = 5$

Probability sample mean $(\mu_{\bar{x}} - \underline{1.9}\sigma_{\bar{x}} < \bar{X} < \mu_{\bar{x}} - \underline{0.4\sigma_{\bar{x}}})$

$$P(-1.9 < Z < -0.4) = 0.3446 - 0.0287 = \underline{\underline{0.3159}}$$

Table A.3 in text

8.20 $f(x) = \begin{cases} 1/3 & x = 2, 4, 6, \\ 0 & \text{elsewhere} \end{cases}$ Random sample $n = 54$

$$\mu = E(X) = \sum_x x f(x) = 2\left(\tfrac{1}{3}\right) + 4\left(\tfrac{1}{3}\right) + 6\left(\tfrac{1}{3}\right) = 4 = \mu_{\bar{x}}$$

$$\sigma^2 = E[(X-\mu)^2] = (-2)^2\left(\tfrac{1}{3}\right) + (2)^2\left(\tfrac{1}{3}\right) = 8/3$$

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{(8/3)}{54} \qquad \sigma_{\bar{x}} = 2/9$$

$$Z_1 = \frac{4.15 - 4}{2/9} = 0.68 \quad Z_2 = \frac{4.35 - 4}{2/9} = 1.58 \quad \to P(0.68 < Z < 1.58)$$

$$= 0.9429 - 0.7517 = 0.1912$$

correction for discrete distribution measured to nearest tenth (0.05)

8.32 $\sigma^2 = 1$ Two different machines

$n = 36$ A $\bar{x}_A = 4.5$ (ounces)

B $\bar{x}_B = 4.7$

a) $P(\bar{X}_B - \bar{X}_A \geq 0.2)$ using CLT if $\mu_A = \mu_B$

$\sigma_A^2 = \sigma_B^2 = 1$

using Theorem 8.3

$$Z = \frac{(\bar{X}_A - \bar{X}_B) - \cancel{(\mu_A - \mu_B)}}{\sqrt{(\sigma_A^2/n_A) + (\sigma_B^2/n_B)}} = \frac{-0.2}{\sqrt{1/36 + 1/36}} = -0.85 \rightarrow P(Z \leq -0.85)$$

$= 0.1977$

b) Are the population means for the 2 machines actually different?

→ Not necessarily, probability in a) is not negligible

8.33 $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ if $\bar{x} = \mu$ (population mean) $n = 25$

a) $P = 0.5$

b) Observed $\bar{x} = 7960$

$P(\bar{X} > 7960 \mid \mu = 7950)$

$$Z = \frac{7960 - 7950}{100/\sqrt{25}} = 0.5$$

$P(Z \geq 0.5)$

$= 1 - 0.6915$

$= \underbrace{0.3085}_{}$

not strong evidence that population mean exceeds limit

8.35 $P(\bar{X} \leq 775 \mid \mu = 800)$ $Z = \frac{775 - 800}{40/\sqrt{16}} = -2.5$ $P(Z \leq -2.5) = 0.0062$

$P(\bar{X} \leq 775 \mid \mu = 760)$ $Z = \frac{775 - 760}{40/\sqrt{16}} = 1.5$ $P(Z \leq 1.5) = 0.9332$

From ex. 8.4 $n = 16$, $\sigma = 40$

The event of observing a sample mean $\bar{X} \leq 775$ is very rare if population mean is 800, making the claim $\mu = 800$ questionable

8.36 X_1, X_2, X_n } random sample from distribution that can only take on positive values

show → $Y = X_1 X_2 \cdots X_n$ has appx. lognormal distribution

Random variable X has lognormal distribution if $Y = \ln(X)$ has normal distribution with mean μ & s.d. σ

Let $W_i = \ln(X_i)$

Since:
$\ln(xy) = \ln(x) + \ln(y)$

Using CLT $Z = \dfrac{\bar{W} - \mu_W}{\sigma/\sqrt{n}} \sim n(Z, 0, 1)$

$$\bar{W} = \frac{1}{n}\sum_{i=1}^{n} \ln(X_i) = \frac{1}{n}\ln\Big(\underbrace{\prod_{i=1}^{n} X_i}_{Y}\Big) = \frac{1}{n}\ln(Y)$$

8.40 X^2 DISTRIBUTION, find X_a^2 such that

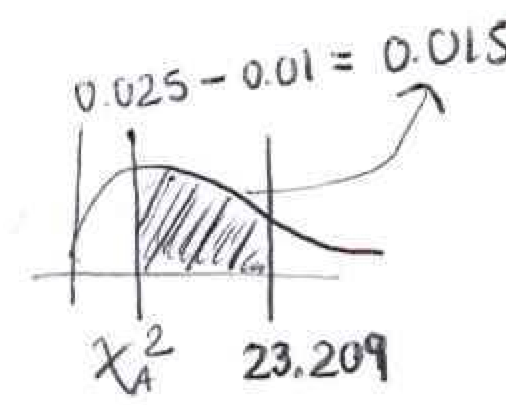
From Table A.5

find X_α^2 for corresponding ν & α

a) $P(X^2 > X_a^2) = 0.01$ when $\nu = 21$ $\underline{38.932}$ $a = 0.01$ $\nu = 21$

b) $P(X^2 < X_a^2) = 0.95$ when $\nu = 6$ $\underline{12.592}$ $a = 0.05$ $\nu = 6$

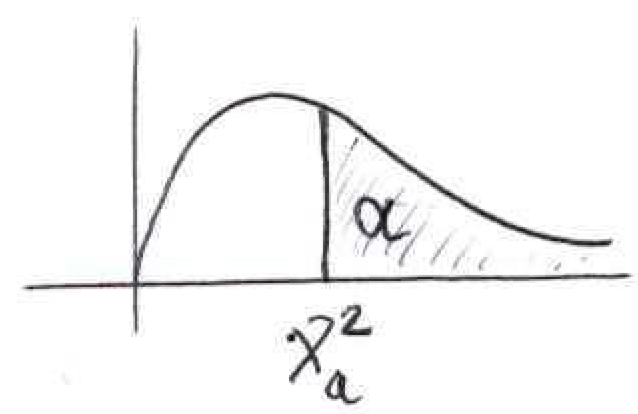
c) $P(\underline{X_\alpha^2} < X^2 < \underbrace{23.209}_{\substack{\nu = 10 \\ \alpha = 0.01}}) = 0.015$ when $\nu = 10$ $\alpha = 0.01 + 0.15$
$= 0.025$



$X_{0.025}^2 = \underline{20.483}$

* Theorem 8.4

(p 224) $X^2 = \dfrac{(n-1)S^2}{\sigma^2}$



Probability that a sample produces $X^2 >$ specified value $= \alpha$ (Table A.5, page 740)

8.41 Sample variances are continous measurements
Random sample of 25 observations from normal population $\sigma^2 = 6$

a) $P(S^2 > 9.1) = P\left(\underbrace{\dfrac{(n-1)S^2}{\sigma^2}}_{X^2} > \underbrace{\dfrac{(24)(9.1)}{6}}_{36.4}\right) = P(X^2 > 36.4) = 0.05$

24 d.f.
find 36.4 in table A.5 (page 740)
(36.415)

b) $P(3.462 < S^2 < 10.745) = P\left(\dfrac{(24)(3.462)}{6} < \dfrac{(n-1)S^2}{\sigma^2} < \dfrac{(24)(10.745)}{6}\right)$

$= P(13.848 < X^2 < 42.980) = 0.95 - 0.01$
$= 0.94$

6/6