STA 286 - Tutorial #10

Q9.1. The population standard deviation is $\sigma = 5.8$ m

From Theorem 9.2,

$$n = \left[\frac{(2.575)(5.8)}{2} \right]^2 \quad \because \left[n = \left(\frac{z_{\alpha/2}\, \sigma}{e} \right)^2 \right]$$

The z-value having an area of 0.005 to the right $\{ \because$ area of 0.995 to the left, is $z_{0.005} = 2.575$.

$$\Rightarrow n = 55.76 \simeq \underline{56}$$

Q9.2. $n = 30$ 96% confidence-interval:

$\bar{x} = 780$ hrs $z_{0.02} = 2.054$

$\sigma = 40$ hrs

For a $100(1-\alpha)\%$ confidence interval,

$$\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow 780 - (2.054)\left(\frac{40}{\sqrt{30}}\right) < \mu < 780 + (2.054)\left(\frac{40}{\sqrt{30}}\right)$$

$$\Rightarrow \underline{765 < \mu < 795}$$

Q9.6 Here, the standard deviation, σ, remains the same : $\sigma = 40$ hrs

whereas $e = 10$

From Theorem 9.2,

$$n = \left(\frac{z_{\alpha/2}\,\sigma}{e}\right)^2$$

$$n = \left(\frac{(2.054)(40)}{10}\right)^2$$

$n \approx 68$ (round up)

Q.9.9 $n = 20$

$\bar{x} = 11.3\,g$

$s = 2.45\,g$ (sample standard deviation)

Recall, if we have a random sample from a normal distribution, then the random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a Student t-distribution, w $n-1$ DOF.

(95% confidence interval)

$t_{0.025} = 2.093$ w 19 DOF

$\therefore$ For a $100(1-\alpha)\%$ confidence interval

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

$$\Rightarrow \quad 11.3 - (2.093)(2.45/\sqrt{20}) < \mu < 11.3 + (2.093)(2.45/\sqrt{20})$$

$\Rightarrow \quad 10.13 < \mu < 12.45.$

Q9.17. (Recall, Q9.9 The distribution was a student t - distribution).

The 95% prediction interval is given by:

$$\bar{x} - t_{\alpha/2}\, s\sqrt{1 + 1/n} < x_0 < \bar{x} + t_{\alpha/2}\, s\sqrt{1+1/n}$$

$\Rightarrow 11.3 \pm (2.093)(2.45)\sqrt{1 + 1/20}$

$\Rightarrow 11.3 \pm 5.25\ g$

Q.9.36. From Theorem 8.3, we expect that the sampling distribution $f(\bar{X}_1 - \bar{X}_2)$ to be approximately normally distributed, with mean $\mu_{\bar{X}_1 - \bar{X}_2} = \mu_1 - \mu_2$ and standard deviation given by:

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Here, we have:

$n_A = 50$ $\qquad$ $n_B = 50$

$\bar{x}_A = 78.3$ kg $\qquad$ $\bar{x}_B = 87.2$ kg

$\sigma_A = 5.6$ kg $\qquad$ $\sigma_B = 6.3$ kg.

For a 95% confidence interval, $z_{0.025} = 1.96$.

$$\Rightarrow (87.2 - 78.3) \pm 1.96\sqrt{\frac{5.6^2}{50} + \frac{6.3^2}{50}}$$

$= 8.9 \pm 2.34$ kg

$\therefore \quad 6.56 \text{ kg} < \mu_A - \mu_B < 11.24 \text{ kg}$

Q. 9.81.

The likelihood function is given by:

$$L(x_1, \ldots, x_n; p) = \prod_{i=1}^{n} f(x_i; p)$$

For a Bernoulli process w parameter 'p' as the prob. of success, we have:

$$L(x_1, \ldots, x_n; p) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$$

$$= p^{n\bar{x}} (1-p)^{n(1-\bar{x})}$$

Taking logarithms, we get.

$$\ln L(x_1, \ldots, x_n; p) = \ln \left(p^{n\bar{x}} (1-p)^{n(1-\bar{x})} \right)$$

$$= n \ln \left[p^{\bar{x}} (1-p)^{(1-\bar{x})} \right]$$

$$= n \left[\bar{x} \ln p + (1-\bar{x}) \ln (1-p) \right]$$

$$\frac{\partial \ln L}{\partial p} = n \frac{\partial}{\partial p} \left[\bar{x} \ln p + (1-\bar{x}) \ln(1-p) \right]$$

$$= n \left(\frac{\bar{x}}{p} + \frac{(1-\bar{x})}{(1-p)} \right)$$

In order to maximize the derivative, set it equal to zero

$$\Rightarrow \quad n \left(\frac{\bar{x}}{p} + \frac{(1-\bar{x})}{(1-p)} \right) = 0$$

$$\Rightarrow \quad \frac{\bar{x}}{p} - \frac{1-\bar{x}}{1-p} = 0$$

$$\Rightarrow \quad \frac{\bar{x}}{p} = \frac{1-\bar{x}}{1-p}$$

$$\Rightarrow \quad \bar{x} - \cancel{\bar{x}p} = p - \cancel{\bar{x}p}$$

$$\therefore \quad \hat{p} = \bar{x}$$

Q9.84. Weibull distribution:

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases} \quad \forall \alpha, \beta > 0$$

(a)
$$L(x_1, \ldots, x_n; \alpha, \beta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta)$$

$$= (\alpha\beta)^n \prod_{i=1}^{n} x_i^{\beta-1} e^{-\alpha x_i^{\beta}}$$

$$= (\alpha\beta)^n e^{-\alpha \sum_{i=1}^{n} x_i^{\beta}} \left(\prod_{i=1}^{n} x_i \right)^{\beta-1}$$

(b) Taking the logarithms, we get:

$$\ln L = n[\ln\alpha + \ln\beta] - \alpha \sum_{i=1}^{n} x_i^{\beta} + (\beta-1) \sum_{i=1}^{n} \ln x_i$$

In order to solve for the maximum likelihood estimate, solve the following two equations:

$$\frac{\partial \ln L}{\partial \alpha} = 0 \quad \& \quad \frac{\partial \ln L}{\partial \beta} = 0$$

Q9.41.

$n_1 = 14$ $\qquad$ $n_2 = 16$

$\bar{x}_1 = 17$ $\qquad$ $\bar{x}_2 = 19$

$S_1^2 = 1.5$ $\qquad$ $S_2^2 = 1.8$

Calculating the pooled estimate of s.d., we have:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

$$S_p^2 = \frac{(13)(1.5) + (15)(1.8)}{14 + 16 - 2}$$

$$S_p^2 = 1.6607$$

$$\Rightarrow S_p = 1.289.$$

For a 99% confidence interval, & 28-DOF

$t_{0.005} = 2.763$, $\nu = 28$.

(Assuming equal variances).

A $100(1-\alpha)\%$ confidence interval is given by:

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$\Rightarrow (19 - 17) \pm (2.763)(1.289)\sqrt{\frac{1}{16} + \frac{1}{14}}$$

$$= 2 \pm 1.30$$

$$\Rightarrow 0.70 < \mu_1 - \mu_2 < 3.30$$

You aren't responsible for knowing the below method for the exam.

(Assuming unequal variances)

An approx. $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by:

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where $t_{\alpha/2}$ is the t-value with

$$v = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{[(s_1^2/n_1)^2/(n_1-1)] + [(s_2^2/n_2)^2/(n_2-1)]}$$

$$\Rightarrow \quad v = \frac{(1.5/14 + 1.8/16)^2}{[(1.8/16)^2/(15)] + [(1.5/14)^2/(13)]}$$

$$v = 27.938 \simeq \underline{28}$$

$\therefore$ [illegible]

$\therefore$ For a 95% confidence interval & $v = 28$,

$$t_{0.025} = 2.048$$

$$\Rightarrow (19 - 17) \pm (2.048)\sqrt{\frac{(1.8)}{16} + \frac{(1.5)}{14}}$$

$$= 2 \pm 0.994.$$

$$\Rightarrow \underline{1.006 < \mu_1 - \mu_2 < 2.994}$$

[BONUS EXERCISES]

Q 9.29. $S'^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / n$

$$E(S'^2) = E\left[\frac{(n-1)}{n} S^2\right]$$
$$= \frac{(n-1)}{n} E(S^2)$$
$$= \frac{(n-1)}{n}\left[E\left\{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right\}\right]$$
$$= \frac{(n-1)}{n}\left[\frac{1}{n-1}\left\{\sum_{i=1}^{n} E(X_i - \mu)^2 - nE(\bar{X}-\mu)^2\right\}\right]$$
$$= \frac{(n-1)}{n}\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} \sigma^2_{X_i} - n\sigma_{\bar{X}}^2\right)\right]$$

However,

$$\sigma_{X_i}^2 = \sigma^2 \quad \forall i = 1, 2, \ldots, n$$
$$\& \quad \sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}.$$

$$\therefore E(S'^2) = \frac{(n-1)}{n}\left[\frac{1}{n-1}\left(n\sigma^2 - n\frac{\sigma^2}{n}\right)\right]$$
$$= \frac{(n-1)}{n}\sigma^2$$

Q.9.33. From Theorem 8.4, we know:

$$X^2 = \frac{(n-1)S^2}{\sigma^2}$$

follows a chi-sq. distribution w $n-1$ degrees of freedom with variance $2(n-1)$.

$$\therefore Var(S^2) = Var\left(\frac{\sigma^2}{n-1} X^2\right)$$

$$= \frac{2}{n-1}\sigma^4$$

$$Var(S'^2) = Var\left(\frac{n-1}{n} S^2\right)$$

$$= \left(\frac{n-1}{n}\right)^2 Var(S^2)$$

$$= \frac{2(n-1)}{n^2}\sigma^4$$

$\therefore$ Variance of S'^2 is smaller.

Q9.99 The likelihood fn. is given by:

$$L(x_1, \ldots, x_n; \mu) = \prod_{i=1}^{n} \frac{e^{-\mu}\mu^{x}}{x!}$$

Now, the maximum likelihood estimator is given by:

$$\ln L = \ln\left(\frac{e^{-n\mu}\mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\right)$$

$$\ln L = -n\mu + \ln(\mu)\sum_{i=1}^{n} x_i - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

(Taking the derivative & setting it equal to 0)

$$\frac{\partial \ln L}{\partial \mu} = 0$$

$$-n + \sum_{i=1}^{n} x_i / \mu = 0$$

$$\Rightarrow \quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \quad \hat{\mu} = \bar{x}$$

On the other hand, using the method of moments (as described in the question) we also ~~get~~ get:

$$\hat{\mu} = \bar{x}$$

QED ☒