ECE 316 CHAPTER 2 PROBLEM SET SOLUTIONS

1. (a) $(1+j)^3 = (\sqrt{2}\,e^{j\pi/4})^3 = (\sqrt{2})^3 e^{j3\pi/4}$.

(b) $(\sqrt{3}+j^3)(1-j) = (\sqrt{3}-j)(1-j)$

$z_1 = (\sqrt{3}-j) = 2e^{-j\pi/6}$

$z_2 = (1-j) = \sqrt{2}\,e^{-\pi/4\,j}$

$\Rightarrow \quad z_1 z_2 = 2\sqrt{2}\,e^{-j(\pi/4+\pi/6)} = 2\sqrt{2}\,e^{-j5\pi/12}$.

(c) $$\frac{2-j(6/\sqrt{3})}{2+j(6/\sqrt{3})} = \frac{\left[2-j(6/\sqrt{3})\right]^2}{4+36/3} = \frac{(4e^{-j\pi/3})^2}{16} = e^{-j2\pi/3}.$$

(d) $$j(1+j)e^{j\pi/6} = e^{j\pi/2}(\sqrt{2}\,e^{j\pi/4})\,e^{j\pi/6} =$$
$$\sqrt{2}\,e^{j(\pi/2+\pi/4+\pi/6)} = \sqrt{2}\,e^{j\,11/12\,\pi}.$$

(e) $$\frac{e^{j\pi/3}-1}{1+j\sqrt{3}} = \frac{(\cos\pi/3 + j\sin\pi/3)-1}{1+j\sqrt{3}} = \frac{1/2 + j\sqrt{3}/2 - 1}{1+j\sqrt{3}}$$
$$= \frac{-1/2 + j\sqrt{3}/2}{1+j\sqrt{3}} \cdot \frac{(1-j\sqrt{3})}{(1-j\sqrt{3})} = \frac{1+\sqrt{3}\,j}{4} = \frac{2e^{j\pi/3}}{4}$$
$$= 1/2\, e^{j\pi/3}.$$

2. Euler's relation: $e^{j\theta} = \cos\theta + j\sin\theta$.

(a) $\sin\theta \sin\varphi = \frac{1}{2}\cos(\theta-\varphi) - \frac{1}{2}\cos(\theta+\varphi)$.

$$\sin\theta\sin\varphi = \left(\frac{e^{j\theta}-e^{-j\theta}}{2j}\right)\left(\frac{e^{j\varphi}-e^{-j\varphi}}{2j}\right) =$$

$$-\tfrac{1}{4}\left[e^{j(\theta+\varphi)} - e^{j(\theta-\varphi)} - e^{j(\varphi-\theta)} + e^{-j(\theta+\varphi)}\right]$$

$$= -\tfrac{1}{4}\left[\left(e^{j(\theta+\varphi)} + e^{-j(\theta+\varphi)}\right) + (-1)\left(e^{j(\theta-\varphi)} + e^{-j(\theta-\varphi)}\right)\right]$$

$$= \tfrac{1}{4}\left[e^{j(\theta-\varphi)} + e^{-j(\theta-\varphi)}\right] - \tfrac{1}{4}\left[e^{j(\theta+\varphi)} + e^{-j(\theta+\varphi)}\right]$$

$$= \tfrac{1}{4}2\cos(\theta-\varphi) - \tfrac{2}{4}\cos(\theta+\varphi)$$

$$= \underline{\tfrac{1}{2}\cos(\theta-\varphi) - \tfrac{1}{2}\cos(\theta+\varphi)}.$$

(b) $\sin(\theta+\varphi) = \sin\theta\cos\varphi + \sin\varphi\cos\theta$.

$$\sin(\theta+\varphi) = \frac{e^{j(\theta+\varphi)} - e^{-j(\theta+\varphi)}}{2j} =$$

$$\frac{e^{j\theta}e^{j\varphi} - e^{-j\theta-j\varphi}}{2j} =$$

$$\frac{1}{2j}\left[(\cos\theta + j\sin\theta)(\cos\varphi + j\sin\varphi) - (\cos\theta - j\sin\theta)(\cos\varphi - j\sin\varphi)\right]$$

$$= \frac{1}{2j}\left[\cos\theta\cos\varphi + j\cos\theta\sin\varphi + j\sin\theta\cos\varphi - \sin\theta\sin\varphi - \cos\theta\cos\varphi + j\cos\theta\sin\varphi + j\sin\theta\cos\varphi + \sin\theta\sin\varphi\right]$$

$$= \frac{1}{2j}\left[2j\cos\theta\sin\varphi + 2j\sin\theta\cos\varphi \right] = \sin\theta\cos\varphi + \sin\varphi\cos\theta .$$

(3) Problem 2.1.

$$g(t) = e^{-t}\sin(2\pi f_c t)\, u(t).$$

$$G(f) = \int_0^\infty e^{-t}\sin(2\pi f_c t)\, e^{-j2\pi f t}\, dt = \int_0^\infty e^{-t}\frac{1}{2j}\left(e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right) e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2j}\,\frac{1}{-1+j(\omega_c-\omega)}\, e^{(-1+j(\omega_c-\omega))t}\Big]_0^\infty - \frac{1}{2j}\,\frac{1}{-1-j(\omega+\omega_c)}\, e^{(-1-j(\omega+\omega_c))t}\Big]_0^\infty$$

$$\omega_c = 2\pi f_c$$
$$\omega = 2\pi f$$

$$= \frac{1}{2j}\,\frac{1}{j(\omega-\omega_c)+1} - \frac{1}{2j}\,\frac{1}{1+j(\omega+\omega_c)} = \frac{\omega_c}{1+2j\omega+(\omega_c^2-\omega^2)}$$

$$\Rightarrow G(f) = \frac{2\pi f_c}{1+j2\pi f+4\pi^2(f_c^2-f^2)}.$$

(4) Problem 2.2.

$$g(t) = \int_{-W}^{0} e^{j\pi/2}\, e^{j2\pi f t}\, df + \int_0^W e^{-j\pi/2}\, e^{j2\pi f t}\, df =$$

$$e^{j\pi/2}\frac{1}{j2\pi t}\, e^{j2\pi f t}\Big]_{-W}^{0} + e^{-j\pi/2}\frac{1}{j2\pi t}\, e^{j2\pi f t}\Big]_0^W =$$

$$\frac{e^{j\pi/2}}{j2\pi t}\left(1 - e^{-j2\pi W t}\right) + e^{-j\pi/2}\frac{1}{j2\pi t}\left(e^{j2\pi W t} - 1\right)$$

$$= \frac{1}{j2\pi t}\left[e^{j\pi/2} - e^{j(\pi/2 - 2\pi wt)} + e^{-j\pi/2 + j2\pi wt} - e^{-j\pi/2} \right] =$$

$$\frac{1}{j2\pi t}\left[j - (-j) + e^{-j(\pi/2 - 2\pi wt)} - e^{j(\pi/2 - 2\pi wt)} \right] =$$

$$\frac{1}{2j\pi t}\left[2j + 2j \sin(2\pi wt - \pi/2) \right] = \underline{\frac{1}{\pi t}\left[1 + \sin(2\pi wt - \pi/2) \right]}$$

(6) problem 2.19

(a) ① $G(f) = \int_{-T/2}^{T/2} A\cos(\omega_0 t)\, e^{-j\omega t}\, dt$ $\qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{2T} = \frac{\pi}{T}$

$\omega = 2\pi f$

(direct Method).

② $g(t) = \text{rect}(t/T) \cdot A\cos\left(\frac{\pi t}{T}\right)$

$\Rightarrow G(f) = F\left(\text{rect}(t/T)\right) * F\left(A\cos\left(\frac{\pi t}{T}\right)\right).$

$$F(A\cos(\pi t/T)) = A/2\left[\delta(f-f_0) + \delta(f+f_0)\right] \quad f_0 = \frac{1}{2T}.$$

$$F(\mathrm{rect}(t/T)) = \frac{T\,\mathrm{Sin}(\pi f T)}{\pi f T} = T\,\mathrm{Sinc}(fT)$$

$$\Rightarrow G(f) = \left(T\,\mathrm{Sinc}((f-f_0)T) + T\,\mathrm{Sinc}((f+f_0)T)\right) A/2$$

$$= AT/2\left(\mathrm{Sinc}(fT - 1/2) + \mathrm{Sinc}(fT + 1/2)\right).$$

(b) it is shifted $T/2$ to the riget :

$$\Rightarrow \quad g(t - T/2) \xrightarrow{F} G(f)e^{-j\pi f T}$$

$$= AT/2\left[\mathrm{Sinc}(fT - 1/2) + \mathrm{Sinc}(fT + 1/2)\right] e^{-j\pi f T}.$$

1. Suppose $g(t) \longleftrightarrow G(f)$

$h(t) = g(t) + G(t) + g(-t) + G(-t)$ (1)

show : $F[h(t)] = h(f)$. $F(\cdot)$: Fourier transform.

$$F(h(t)) = F(g(t) + G(t) + g(-t) + G(-t))$$
$$= F(g(t)) + F(G(t)) + F(g(-t)) + F(G(-t))$$

$F(g(t)) = G(f)$

$F(g(-t)) = G(-f)$ using property 2.

$F(G(t)) = g(-f)$ using property 4 (Duality)

$F(G(-t)) = g(-(-f)) = g(f)$ property 2 & property 4.

$\Rightarrow F(h(t)) = G(f) + G(-f) + g(-f) + g(f)$ (2)

Compare (1) & (2) $\Rightarrow \underline{F(h(t)) = h(f)}$.

2. We know : $F(e^{-\pi t^2}) = e^{-\pi f^2}$

Find $\int_{-\infty}^{\infty} \exp(-a(x-m)^2)\,dx$.

if $\exp(-a(x-m)^2) = h(x)$ $\int_{-\infty}^{\infty} h(x)\,dx = H(0) = H(f)\Big|_{\text{at } f=0}$.

where $H(f) = F(h(x))$

property 7 (2.31).

if $g(t) = e^{-\pi t^2} \longleftrightarrow G(f) = e^{-\pi f^2}$

we need to write $h(x)$ in terms of $g(x)$.

$h(x) = g\left((x-m)\sqrt{a/\pi}\right) \Rightarrow H(f) = \frac{1}{\sqrt{a/\pi}}\, G\left(\frac{f}{\sqrt{a/\pi}}\right) e^{-j2\pi m f}$

if $H(f) = \sqrt{\frac{\pi}{a}}\, G\left(\sqrt{\frac{\pi}{a}} f\right) e^{-j2\pi m f}$, $G(f) = e^{-\pi f^2}$

$$\Rightarrow H(f) = \sqrt{\frac{\pi}{a}}\, e^{-\pi\left(\sqrt{\frac{\pi}{a}} f\right)^2} e^{-j2\pi m f}$$

$$\Rightarrow \underbrace{H(0) = \sqrt{\frac{\pi}{a}}}. \quad \text{or} \quad \int_{-\infty}^{\infty} \exp\left(-a(x-m)^2\right) dx = \sqrt{\frac{\pi}{a}}.$$

3. Drill problem 2.13.

freq. response $= F$(impulse response)

$$H(f) = F(h(t))$$

$h(t) = e^{-t^2/2\tau^2}$ using question #2. (in this tutorial).

if $g(t) = e^{-\pi t^2} \rightarrow h(t) = g\left(\sqrt{a/\pi}\, t\right)$ where $a = \frac{1}{2\tau^2}$

$$\Rightarrow H(f) = \frac{1}{\sqrt{a/\pi}}\, G\left(\frac{f}{\sqrt{a/\pi}}\right) = \sqrt{\frac{\pi}{a}}\, G\left(\sqrt{\frac{\pi}{a}} f\right)$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\pi\left(\sqrt{\pi/a}\, f\right)^2} = \underbrace{\sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2}{a} f^2}}$$

Freq. response $= \sqrt{2\pi}\, |\tau|\, e^{-2\pi^2\tau^2 f^2}$.

(Q 4.)

2.22. Solution based on the definition of the Fourier Transform has been given in prob. set 1. Here's another way to show:

$$g(t) \xrightarrow{F} G(f)$$

- By property 3 (Conjugation Rule)

$$g^*(t) \longrightarrow G^*(-f) \quad \text{but } g \text{ is real} \Rightarrow g(t) = g^*(t)$$

$$\Rightarrow \quad g(t) \longrightarrow G^*(-f) \quad \text{or} \quad G(f) = G^*(-f) \quad (1)$$

$$\text{or} \quad G^*(f) = G(-f)$$

- By property 2 (Dialation)

$$g(at) \longrightarrow \frac{1}{|a|} G(f/a) \quad \Rightarrow \text{if } a = -1$$

$$g(-t) \rightarrow G(-f) \quad \text{but } g \text{ is even} \Rightarrow g(t) = g(-t)$$

$$\Rightarrow \quad g(t) \rightarrow G(-f) \quad \text{or} \quad G(f) = G(-f) \quad (2)$$

using (1) and (2) for a signal which is both real and even:

$G^*(f) = G(f) \Rightarrow$ its Fourier transform is purely <u>real</u>.

if the signal is odd $\Rightarrow$

- By property 2

$$g(-t) \rightarrow G(-f) \quad g \text{ is odd} \Rightarrow g(-t) = -g(t)$$

$$\Rightarrow -g(t) \rightarrow G(-f) \quad \text{or} \quad g(t) \rightarrow -G(-f) \Rightarrow G(f) = -G(-f) \quad (3)$$

using (1) and (3) for a real & odd signal: $G^*(f) = -G(f)$

$\Rightarrow$ Fourier Transform is purely <u>imaginary</u>.

(Q.5)
2.23. $$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt \Rightarrow |G(f)|=\left|\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt\right|$$

since $|e^{-j2\pi ft}|=1 \Rightarrow \leq \int_{-\infty}^{\infty} |g(t)|\,dt$. $\Rightarrow$ if a signal is absolutely integrable, its Fourier Transform must exist.

$\left|\int f(t)dt\right| \leq \int |f(t)|dt$

(Q.6)
2.25. $y=x^2(t)$ $X(f)$ is limited $-w \leq f \leq w$.

$y(t)=x(t)\cdot x(t) \Rightarrow$ By preperty 11. $Y(f)=X(f)*X(f)$

$$Y(f)=\int_{-\infty}^{\infty} X(f-\tau)X(\tau)\,d\tau = \int_{-w}^{w} X(f-\tau)X(\tau)\,d\tau. \quad (1)$$

To show $Y(f)$ is limited to to $-2w \leq f \leq 2w$, it suffices to show that $\forall f$ out of $[-2w,2w]$, $Y(f)=0$.

in (1) $-w \leq \tau \leq w \Rightarrow f-w \leq f-\tau \leq f+w$

- if $f>2w \Rightarrow$ $f-\tau \geq f-w$ $> 2w-w=w$ $\Rightarrow f-\tau>w$

since $f-\tau>w \Rightarrow X(f-\tau)=0$ since X is bounded to $[-w,w]$.

Therefore the integral $\int_{-\infty}^{\infty} X(f-\tau)X(\tau)\,d\tau=0$.

- Also if $f<-2w \Rightarrow$ $f-\tau \leq f+w$ $<-2w+w=-w \Rightarrow f-\tau<-w$

since $f-\tau<w \Rightarrow X(f-\tau)=0$ since X is bounded to $[-w,w]$.

$\Rightarrow$ intergral in (1) is 0 $\forall f<-2w$ and $\forall f>2w$.

Therefore Y is bounded to $[-2w,2w]$.

(Q.7)
2.26.

(a) $$\delta(t) = \lim_{T \to 0} \frac{1}{T} \text{rect}(t/T) \qquad \text{if } g(t) = \text{rect}(t)$$

$$G(f) = \text{Sinc}(f) \Rightarrow \frac{1}{T}\text{rect}(t/T) \xrightarrow{F} \frac{T}{T} G(fT)$$
$$= G(fT)$$
$$= \text{Sinc}(fT)$$

$$\lim_{T \to 0} G(f) = \lim_{T \to 0} \text{Sinc}(fT) = 1.$$

$$\Rightarrow \quad F(\delta(t)) = 1.$$

(b) $$\delta(t) = \lim_{T \to 0} \frac{1}{T} \text{Sinc}(t/T)$$

$$\text{sinc}(t) \xrightarrow{F} \text{rect}(f) \Rightarrow \frac{1}{T} \text{Sinc}(t/T) \xrightarrow{F} \frac{1}{T} T\, \text{rect}(fT)$$
$$= \text{rect}(fT)$$

$$\lim_{T \to 0} \text{rect}(fT) = 1.$$

$$\Rightarrow F(\delta(t)) = 1.$$

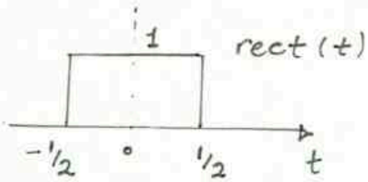
in both cases the, following property has been applied:

$$F(\lim_{T \to 0} f(t)) = \lim_{T \to 0} F(f(t))$$

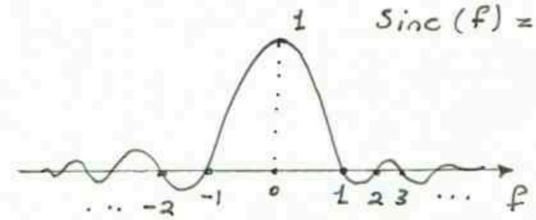
Fourier Transform.

Consider signal $g(t) = \text{rect}(t/T)$ with $T = 1\,\mu s$.

(a) Sketch $G(f)$.

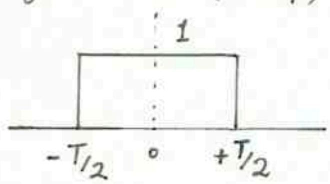


$\xrightarrow{F}$

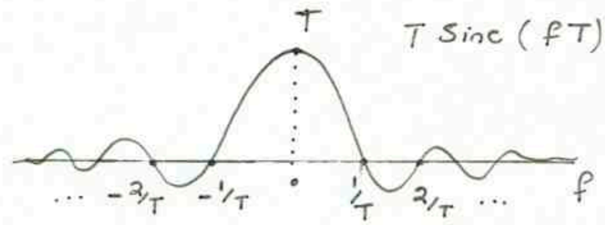


$\text{Sinc}(f) = \frac{\sin \pi f}{\pi f}$

$g(t) = \text{rect}(t/T)$



$\xrightarrow{F}$



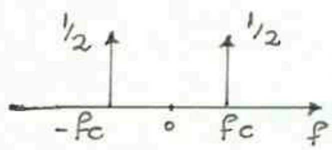
$T\,\text{Sinc}(fT)$

$\text{Sinc}(fT) = \frac{\sin(\pi f T)}{\pi f T} = 0 \rightarrow f = \frac{k}{\pi}$

$k = 1, 2, \ldots$ or $k = -1, -2, \ldots$

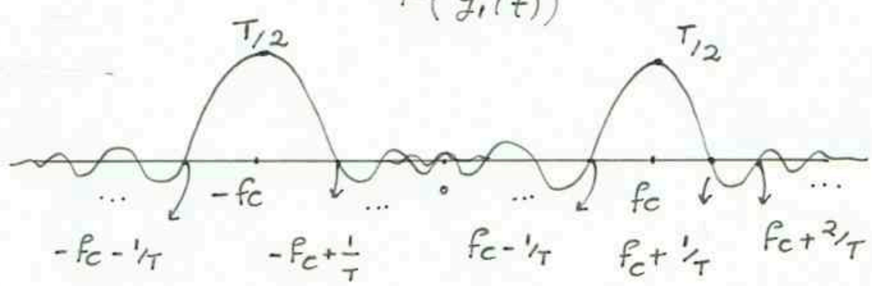
(b) $g_1(t) = g(t)\cos(2\pi f_c t)$ $f_c = 1\,GHz$.

$g_1(t)$:



$F(\cos(2\pi f_c t)) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$

$F(g_1(t))$

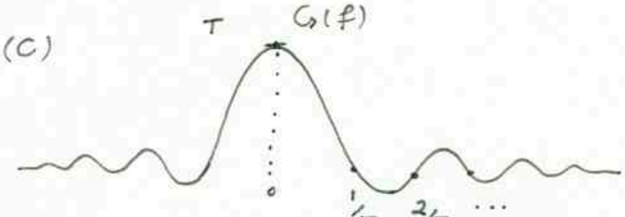


$$F(g_1(t)) = G(f) * \frac{\delta(f - f_c)}{2} + G(f) * \frac{\delta(f + f_c)}{2}.$$

First null bandwidth : for g_1 : $\frac{2}{T} = 2\,MHz$.

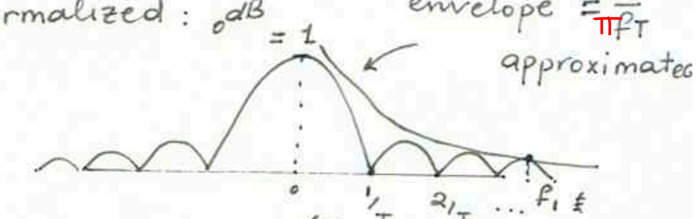
for g : $\frac{1}{T} = 1\,MHz$

(c)



$|G(f)|$ normalized : $_0 dB = 1$

envelope $\cong \frac{1}{\pi f T}$ approximated



BW 60 dB - BW for g : $\frac{10^9}{\pi}$ GHz

BW 60dB - BW for g_1 : $\frac{2 \cdot 10^9}{\pi}$ GHz.

$$0 - 20\log\left(\frac{1}{f_1 T}\right) = 60 \rightarrow \frac{1}{f_1 T} = 10^{-60/20} = 10^{-3}$$

$$\Rightarrow \pi f_1 = \frac{1000}{T} = 10^3 \cdot 10^6 = 1\,GHz.$$

$$f_1 = \frac{10^9}{\pi}$$