ECE 316 CHAPTER 5 PROBLEM SET SOLUTIONS

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1 Solution to Problem 5.2

Note that for the summation to converge to the integral we need a normalization factor, $\frac{1}{2W}$. It follows that

$$\frac{1}{2W} G_\delta(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) e^{\frac{-j\pi n f}{W}}, \tag{1}$$

By rewriting (1) in terms of $T_s = \frac{1}{2W}$ we have

$$G_\delta(f) = T_s \sum_{n=-\infty}^{\infty} g(nT_s) e^{-j2\pi f(nT_s)}. \tag{2}$$

On the other hand, for a Reiman integrable function $h(x)$ we have

$$\lim_{\epsilon \to 0} \epsilon \sum_{n=-\infty}^{\infty} h(n\epsilon) \to \int_{-\infty}^{\infty} h(x)dx. \tag{3}$$

By using (3), and defining the function $h(t) = g(t)e^{-j2\pi ft}$ based on (2), we have

$$\lim_{T_s \to 0} T_s G_\delta(f) = \int_{-\infty}^{\infty} h(t)dt = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt = G(f). \tag{4}$$

□

2 Solution to Problem 5.3

First note that from basic calculus we have

$$\int_a^b e^{cx}dx = \frac{e^{bc} - e^{ac}}{c}. \tag{5}$$

For this problem, we have

$$a = -W \tag{6}$$
$$b = W \tag{7}$$
$$c = j2\pi\left(t - \frac{n}{2W}\right) \tag{8}$$
$$x = f. \tag{9}$$

Given (5) and by using (6)-(9) we find that

$$\frac{1}{2W}\int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}df = \frac{1}{2W}\frac{e^{j2\pi\left(t-\frac{n}{2W}\right)(W)} - e^{j2\pi\left(t-\frac{n}{2W}\right)(-W)}}{j2\pi\left(t-\frac{n}{2W}\right)}. \tag{10}$$

Remember that $e^{jx} - e^{-jx} = 2j\sin(x)$. Then, it follows for (10) that

$$\begin{aligned}
\frac{1}{2W}\int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}df &= \frac{1}{2W}\frac{2j\sin\left(j2\pi\left(t-\frac{n}{2W}\right)(W)\right)}{j2\pi\left(t-\frac{n}{2W}\right)} \\
&= \frac{\sin\left(2\pi Wt - n\pi\right)}{2\pi Wt - n\pi} \\
&= \mathrm{sinc}\left(2Wt - n\right). \qquad (11)
\end{aligned}$$

Note that in (11), we had $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. □

3 Solution to Problem 5.5

a) $g(t) = \mathbf{sinc}(200t)$**:** Remember that the Fourier transform of a sinc function is a rectangle. In this case, the rectangle is bandlimited to 100 Hz. Therefore, the Nyquist frequency is 200 Hz corresponding to a Nyquist interval of 5 ms.

b) $g(t) = \mathbf{sinc}^2(200t)$**:** From Fourier transform properties, we known that the Fourier transform of product of two signals in time domain is the convolution of their Fourier transforms in Frequency domains. Therefore, when two sinc functions are multiplied, two rectangles are convolved in the frequency domain. Then, the result of the convolution will be bandlimited to 200 Hz and therefore, the Nyquist frequency and the corresponding Nyquist interval are 400 Hz and 2.5 ms, respectively.

c) $g(t) = \mathbf{sinc}(200t) + \mathbf{sinc}^2(200t)$**:** This is similar to the previous part. Due to the terms $\mathrm{sinc}^2(200t)$, the signal will be bandlimited to 400 Hz.

4 Solution to Problem 5.6

We have

$$g(t) = \cos(\pi t) \iff G(f) = \frac{\delta(f-0.5)+\delta(f+0.5)}{2}. \qquad (12)$$

On the other hand, if a signal g(t) is sampled at $F_s = \frac{1}{T_s}$, assuming G_δ as the Fourier transform of the corresponding signal, we have

$$G_\delta(f) = F_s\sum_{n=-\infty}^{\infty} G(f-nF_s). \qquad (13)$$

We use this formula to obtain the Fourier transforms in each part.

a) $T_s = 0.25$ **s:** For this sampling interval, we have $F_s = 4$ Hz. Then, it follows from (13) that

$$\begin{aligned} G_\delta(f) &= 4 \sum_{n=-\infty}^{\infty} g(f - 4n) \\ &= 2 \sum_{n=-\infty}^{\infty} \left(\delta(f - 0.5 - 4n) + \delta(f + 0.5 + 4n)\right). \end{aligned} \tag{14}$$

For this case, there is no overlapping and the signal can be recovered by using an ideal interpolating filter with a bandwidth slightly larger than 0.5 Hz.

b) $T_s = 1$ **s:** For this scenario, we have $F_s = 1$ s which is exactly equal to the Nyquist frequency of the signal. Note that from the sampling theorem, the sufficient condition for the sampled signal to be fully recovered is that it is sampled at a frequency *higher* than the Nyquist frequency. If the sampling frequency is equal to the Nyquist frequency, the sampling theorem might not hold.

Let's define $g(t) = \cos(pit) = g_+(t) + g_-(t)$ where $g_+(t) = 0.5e^{j\pi t}$ and $g_-(t) = 0.5e^{-j\pi t}$. In line with this, also define $G(f) = G_+(f) + G_-(f)$ where $G_+(f) = 0.5\delta(f-0.5)$ and $G_-(f) = 0.5\delta(f+0.5)$. Finally, define $G_\delta(f) = G_{\delta,+}(f) + G_{\delta,-}(f)$ in the same manner. It follow for $G_{\delta,+}(f)$ that (using (13))

$$G_{\delta,+}(f) = F_s \sum_{n=-\infty}^{\infty} G_+(f - nF_s) = \sum_{n=-\infty}^{\infty} 0.5\delta(f - 0.5 - n). \tag{15}$$

Similarly, we have for $G_-(f)$

$$G_{\delta,-}(f) = \sum_{n=-\infty}^{\infty} 0.5\delta(f + 0.5 - n) = \sum_{m=-\infty}^{\infty} \delta(f + 0.5 + m). \tag{16}$$

It is easy to verify that $G_{\delta,-}(f) = G_{\delta,+}(f)$. Then, For $G_\delta(f)$ it follows that

$$G_\delta(f) = 2G_{\delta,+}(f) = \sum_{n=-\infty}^{\infty} \delta(f - 0.5 - n). \tag{17}$$

Remark: I broke the function $G_\delta(f)$ into two functions one corresponding to the left spike and the other for the right spike. The reason to make this complicated was to help you guys try $g(t) = \sin(pit)$. For such a case, all the steps taken are the same but we have $g_+(f) = -g_-(-f)$ and therefore, you will see that $G_\delta(f) = 0$ in this case. This is why the sampling frequency, in general, has to be larger than the Nyquist frequency. In most cases the equality holds, but when the question is like this one, and you have a delta function at the bandwidth of the signal, there might be some complications.

c) $T_s = 1.5$ **s:** For this case, we have $F_s = \frac{2}{3}$ Hz. Clearly, we will experience overlapping.

$$G_\delta(f) = \frac{1}{3} \sum_{n=-\infty}^{\infty} \left(\delta\left((f - 0.5 - \frac{2}{3}n\right) + \delta\left(f + 0.5 - \frac{2}{3}n\right)\right). \tag{18}$$

After filtering the interval $[-0.5-\epsilon, 0.5+\epsilon]$ we get

$$\hat{G}(f) = \frac{1}{3}\left(\delta(f+0.5) + \delta(f+\frac{1}{6}) + \delta(f-\frac{1}{6}) + \delta(f-0.5)\right), \tag{19}$$

Leading to

$$\hat{g}(t) = \frac{2}{3}\left(\cos(\pi t) + \cos(\frac{\pi t}{3})\right), \tag{20}$$

i.e., an extra cosine term is added due to overlapping. □

5 Solution to Problem 5.7

The given signal is $g(t) = \frac{\sin(\pi t)}{\pi t}$, i.e., a sinc function with a bandwidth of 0.5 Hz. From the Nyquist theorem, any sampling rate $F_s > 1$ Hz will work. For this case, $F_s = 1$ Hz also works.

In Frequency Domain: $G(f)$ is a rectangle, i.e., $G(f) = 1$ if $f \in [-0.5, 0.5]$ and zero otherwise. By sampling at $F_s = 1$, the result will be $G_\delta(f) = \sum_n G(f-1)$. You can verify that $G_\delta(f) = 1$ everywhere. Now to recover the signal, we need the interpolation filter $h(t) = \mathrm{sinc}(t)$, i.e., the original signal itself is the interpolating filter. As a consequence, the output will equal the initial signal.

In Time Domain: For any integer n, we have $g(n) = 1$ is $n = 1$ and $g(n) = 0$ of $n \neq 0$. Therefore, after sampling, we have $g_\delta(t) = \delta(t)$, i.e., the sampled signal is a single delta function in time domain and at $t = 0$. Since the interpolating filter is a rectangle in frequency domain covering the integra $[-0.5, 0.5]$, the impulse response of the filter is the original signal. Then, the result of the convolution is the original signal.

Conclusion: Any $F_s \geq 1$ Hz works.

1 Solution to Problem 5.8

From the Fourier series table we have if $g(t) = \mathrm{rect}(t)$, i.e., a rectangle in time domain which is equal to 1 in $[-0.5, 0.5]$ and zero everywhere else, then $G(f) = \mathrm{sinc}(f)$. For this question, $h(t)$ which is defined in (5.9) of the book, can be assumed to be $g(t)$ which is first scaled by $\frac{1}{T}$ in time domain, and then shifted to the right by $\frac{T}{2}$ seconds. Assuming $g_T(t)$ as the scaled version of $g(t)$ by $\frac{1}{T}$, we simple have $h(t) = g_T(t - 0.5T)$. It follows that

$$g_T(t) = g(\frac{t}{T}) \Rightarrow G_T(f) = T\mathrm{sinc}(fT). \tag{1}$$

On the other hand, $H(f) = G_T(f)e^{-j\pi fT} = T\mathrm{sinc}(fT)e^{-j\pi fT}$.

For the second part we have

$$\lim_{T\to 0} \frac{H(f)}{T} = \lim_{T\to 0} \mathrm{sinc}(fT)e^{-j\pi fT} = 1, \tag{2}$$

i.e., the Fourier transform tends to that of the unit delta function $\delta(t)$. Note that the pule $h(t)/T$ converges to a delta function as $T \to 0$.

2 Solution to Problem 5.12

a) : By definition, the PAM signal is defined as

$$\begin{aligned} s_{PAM}(t) &= \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) = \left(\sum_{n=\infty}^{\infty} m(nT_s)\delta(t - nT_s) \right) * h(t) \\ &= s_\delta(t) * h(t). \end{aligned} \tag{3}$$

where $s_\delta(t)$ is the signal sampled using a spike train. For $S_\delta(f)$ we simply have

$$S_\delta(f) = F_s \sum_{n=-\infty}^{\infty} M(f - nF_s) = \sum_{n=-\infty}^{\infty} \frac{A_m}{2} \left(\delta(f - 0.2 - n) + \delta(f + 0.2 - n) \right). \tag{4}$$

On the other hand, we have

$$S_{PAM}(f) = \frac{A_m}{2} \sum_{n=-\infty}^{\infty} \left(H(0.2 + n)\delta(f - 0.2 - n) + H(-0.2 - n)\delta(f + 0.2 - n) \right). \tag{5}$$

with $H(f) = T\mathrm{sinc}(fT)$ and $T = 0.45$.

b) : Without the equalizer, the ideal low-pass filter passes the frequencies between -0.2 to 0.2 Hz. Then the output, namely, $\hat{m}(t)$, is

$$\hat{m}(t) = H(0.2)A_m cos(0.4\pi t) = 0.44A_m cos(0.4\pi t). \tag{6}$$

If we use an equalizer, the original signal is obtained, i.e., $\hat{m}(t) = m(t)$.

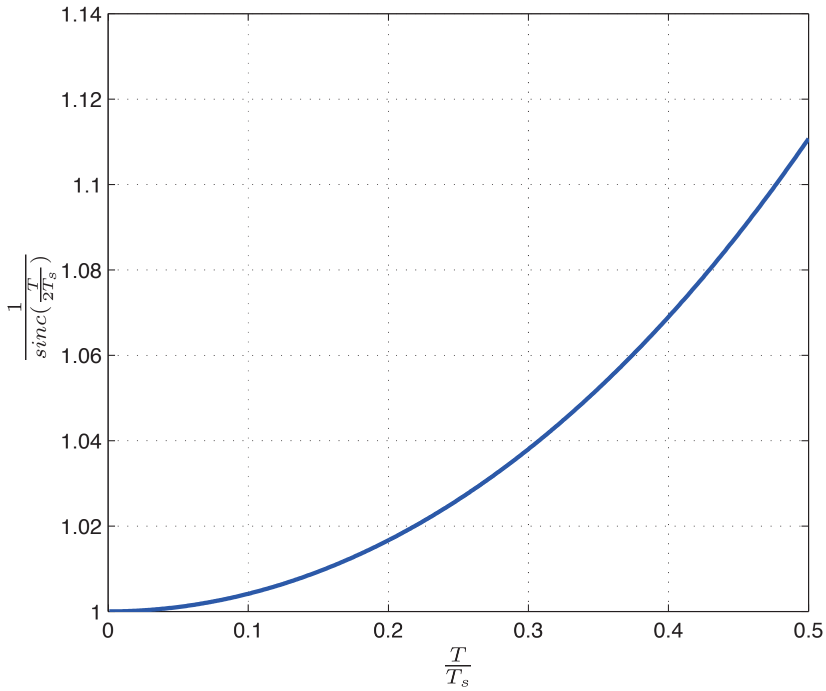

Figure 1: Equalizer gain for the PAM signal.

3 Solution to Problem 5.13

From page 201, following equation (5.17), the distortion for PAM system can be equalized by assuming the inverse filter for the pulse over the signal bandwidth. We known that (see Problem 5.8), for the rectangular pulse of duration T, i.e., $g_T(t)$, the Fourier transform is $G_T(f) = T\text{sinc}(fT)$. Then, at the highest frequency of the signal which is $F_s/2$, we have $G_T(F_s/2) = T\text{sinc}(F_sT/2) = T\text{sinc}(\frac{T}{2T_s}$. The ideal low-pass filter has a gain $1/T$ and therefore, the part equalizer has to fix is $\text{sinc}(\frac{T}{2T_s}$. For the given value of $T/T_s = 0.25$, the distortion is $\text{sinc}(0.125) = 0.97$ and therefore, the filter gain has to be $H(F_s/2) \simeq 1/0.97 \simeq 1.03$. The required equalizing gain for different values of T/T_s is plotted in Fig. 1

4 Solution to Problem 5.14

a) : The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore, $\frac{2400}{2^R} > 160$ Hz leads to $R \leq 3$.

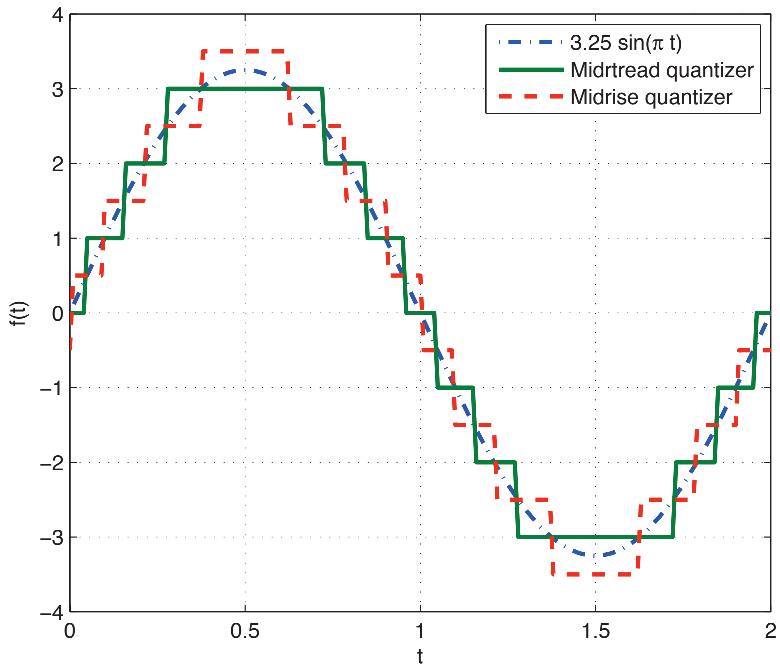

Figure 2: Quantization of a sinusoid.

b) : The multiplexing should take place in the following order: every $\frac{1}{2400*3} = \frac{1}{7200}$ s there is a transmission as follows

- Time 0 to $\frac{1}{2400}$ s: Transmit s_3, s_4, s_1.
- Time $\frac{1}{2400}$ s to $\frac{2}{2400}$ s: Transmit s_3, s_4, s_2.
- For the next 6 intervals of length $\frac{1}{2400}$ s reaching time $\frac{1}{300}$ s: Transmit s_3, s_4, X, where X is null.
- Repeat the same procedure every $\frac{1}{300}$ s.

5 Solution to Problem 5.15

The first quantizer is a midtread quantizer with 7 levels. The second quantizer is a midrise quantizer with 8 levels. Both quantizers require 3 bits for a binary representation. The results from two quantizers are illustrated in Fig. 2

6 Solution to Problem 5.25

a) : The physical signals are limited in duration, which leads to unlimited bandwidth. Therefore, by assuming a finite sampling frequency, there is always some distortions due to overlapping. However, before sampling the signal, the signal is subject to a low pass filter. Therefore, the distortion caused in the reconstructed signal is due to canceling the high frequency components.

b) : Most signals, such as multimedia signals, can be well approximated by a bandlimited version. Then, by sampling at a high enough rate, the signal can be reconstructed with minimal errors which are not discernable by a human user, e.g., voice signal transmission over a cellular network, PCM, etc.

7 Solution to Problem 5.26

We are multiplying the signal $g(t)$ by a pulse train $c(t)$ where the period of $c(t)$ is $T_s = \frac{1}{f_s}$ and the duration of pulse is T. Naturally, $T < \frac{1}{f_s}$, so $f_sT < 1$ (I think there is a typo in the book). So we proceed with the assumption that $f_sT \ll 1$.

a) : You can represent $c(t)$ as $c(t) = p(t) * d(t)$ where $p(t)$ is a single pulse with duration T and $d(t)$ is a delta train at intervals T_s, i.e., $d(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$. Then, by the properties of Fourier transform $C(f) = P(f)D(f)$. We know that (problem 5.8) $P(f) = T\text{sinc}fT$. On the other hand, $D(f) = f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$. Then, we find that $C(f) = \sum_{k=-\infty}^{\infty} a_k \delta(f - kf_s)$ with $a_k = f_sT\text{sinc}(kf_sT)$.

On the other hand, we are interested in the spectrum of $g_c(t) = g(t)c(t)$. Again, from the Fourier transform, we have $G_c(f) = G(f) * C(f)$. Given that $C(f)$ is a series of delta functions in the frequency domain, the result of the convolution is immediate. We obtain $G_c(f) = \sum_{k=-\infty}^{\infty} a_k G_c(f - kf_s)$.

b) : Naturally, if f_s satisfies the Nyquist theorem, and $a_0 \neq 0$, the signal can be recovered by using the ideal low-pass filter with a bandwidth $f_s/2$. Note that the pulse train $c(t)$ has a nonzero DC value with $a_0 = T/T_s$, i.e., the duty cycle. The scenario with a zero DC value for $c(t)$ occurs in the next problem.

8 Solution to Problem 5.27

The output input relation between $y(t)$ and $x(t)$ can be expressed as $y(t) = x(t) \times c(t)$ where $c(t)$ is a periodic square waveform, where $c(t) = 1, t \in [2kT_s, 2kT_s + T_s]$, $c(t) = -1, t \in [2kT_s + T_s, 2kT_s + 2T_s]$. First note that this is some form of modulation. The easiest way to get $x(t)$ from $y(t)$ is to multiply $y(t)$ with the same modulating waveform $c(t)$, i.e., $y(t)c(t) = x(t)c(t)^2 = x(t)$.

To find the spectrum of $Y(f)$ note that $Y(f) = X(f) * C(f)$. For $C(f)$ we have

$$c(t) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{jk\pi}{T_s} t} \Rightarrow C(f) = \sum_{k=-\infty}^{\infty} a_k \delta(f - \frac{2k}{T_s}). \tag{7}$$

For the given square wave, it turns out that $a_{2k} = 0$ and a_{2k+1} is imaginary. Then, we have

$$Y(f) = \sum_{k=-\infty}^{\infty} a_{2k+1} X(f - \frac{k}{T_s}). \tag{8}$$

9 Solution to Problem 5.28

The input signal is bandlimited and has sinusoid tones at frequencies, $f = kf_0$ where $k = 1, 2, ..., m$. Assume $X(f) = \sum_{k=-m}^{m} a_k \delta(f - kf_0)$.

a) : Consider $f_s = (1 - a)f_0$ and assume we have a single-tone signal $z(t)$ such that $Z(f) = \delta(f - kf_0)$. After sampling $z(t)$ we have

$$Z_\delta(f) = \sum_{n=-\infty}^{\infty} \delta(f - kf_0 - nf_s). \tag{9}$$

We have $kf_0 + nf_s = (k + n)f_0 - naf_0$. Then, for $n = -k$ we will have a spike at $f = -naf_0 = kf_0$. Similarly, if we assume $Z(f) = \delta(f + kf_0)$, then the same argument says that there is a spike at $f = -kf_0$. Since we assumed an arbitrary k, this is true for all $k = 1, 2, 3, ...$ Therefore, for very small a, we have spikes at akf_0.

Now if we filter-out only these spikes, and call the signal $y(t)$, we have $Y(f) = \sum_k a_k \delta(f - kaf_0)$. If we normalize $Y(f)$ by $\frac{1}{a}$, then we can assume that $Y(f)$ is the output to the system $y(t) = x(at)$ (it follows from the properties of the Fourier transform).

b) : The highes frequency of $Y(f)$ is at mf_0. We can obtain $Y(f)$ by filtering $X(f)$ after sampling using an ideal filter with a bandwidth of $f_s/2$. Then, the highest frequency component of $Y(f)$ must be smaller than filter bandwidth, i.e.,

$$mf_0 \leq \frac{f_s}{2} = \frac{(1 - a)f_0}{2} \Rightarrow a \leq \frac{1}{2m + 1}. \tag{10}$$

c) : This follows from the argument of part a. Note that when a signal is expanded in time, then it is compressed in frequency.

10 Solution to Problem 5.30

a) : You can find the transform by direct integration of the given waveform. I explain an alternative method (which is much faster, although it seems otherwise). First reconstruct the signal as follows:

- Take a rectangular pulse of duration T, say, $p(t)$. Define the rectangular pulse $A(t) = p(t) * p(t)$.
- Note that in time domain, we have $A(t) = (t+T)u(t+T) - 2tu(t) + (t-T)u(t-T)$.
- Add the signal $A(t)$ with T times its derivative. Note that $A'(t) = u(t+T) - 2u(t) + u(t-T)$.
- Shift the result, in time, to the right, by T.
- Divide the result of previous steps by T.
- You find the signal in Fig. 5.29 of the book. Then, we have $h(t) = \frac{1}{T}(A(t-T) + TA'(t-T))$.

Using this method of constructing $h(t)$ you can easily find the Fourier transform. We simply have

$$\begin{aligned}
H(f) &= \frac{1}{T}\left(A(f)e^{-j2\pi fT} + j2\pi fTA(f)e^{-j2\pi fT}\right) \\
&= \frac{A(f)}{T}\left(1 + j2\pi fT\right)e^{-j2\pi fT} \\
&= \frac{(P(f))^2}{T}\left(1 + j2\pi fT\right)e^{-j2\pi fT} \\
&= T^{-1}\left(T\mathrm{sinc}(fT)\right)^2\left(1 + j2\pi fT\right)e^{-j2\pi fT} \\
&= T^{-1}\left(T\mathrm{sinc}(fT)e^{-j\pi fT}\right)^2\left(1 + j2\pi fT\right) \\
&= T\left(\frac{1 - \exp - j2\pi fT}{j2\pi fT}\right)^2\left(1 + j2\pi fT\right). \qquad (11)
\end{aligned}$$

Note that in the last step, I simply used the fact that by shifting the pulse from $-T/2$ to $T/2$, by $T/2$ to the right, you get a pulse starting from 0 and ending at T. For such a pulse, the Fourier transform is $\frac{1-\exp -j2\pi fT}{j2\pi fT}$.

b) : The magnitude response is sketched in Fig. 3 and the phase response in Fig. 4. The plots are for the case $T = 1$.

c) : At the frequency $f = \frac{f_s}{2} = \frac{1}{2T_s}$, we need to compensate for $H(\frac{T}{2T_s}$. For $T/T_s = 0.1$ we have $|H(0.05)| \simeq 1.04$ which is roughly the same as the zero-order hold (Actually worse). There is also a phase compensation requirement as well. As you can see from Fig. 4, the phase around origin is nonlinear!

Remark: If you look up the first-order hold filters in the literature, specifically, in the digital control systems literature where these types of sampling and reconstruction are crucial, you realize that the first-order-hold is defined differently! In fact, the first-order hold filter remembers the $A(t)$ explained in part a.

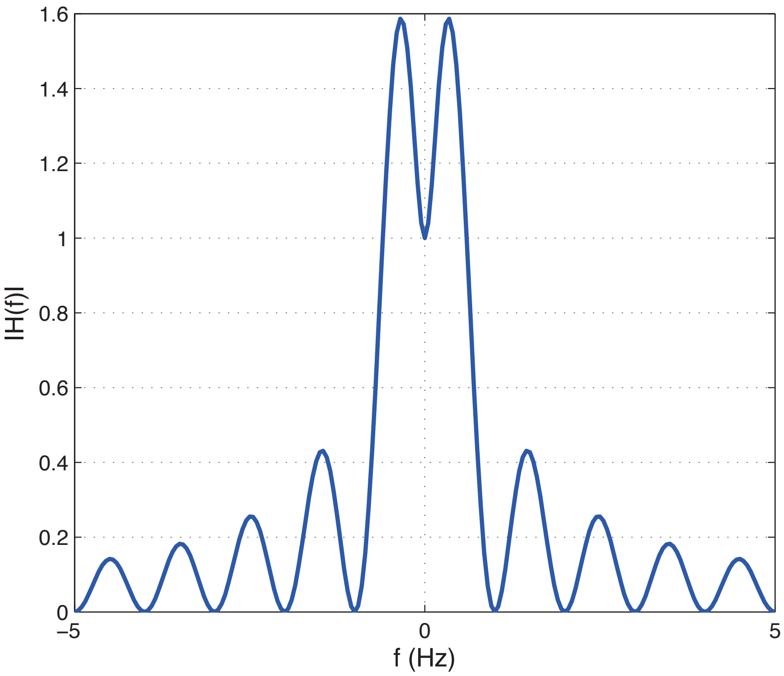

Figure 3: Magnitude response.

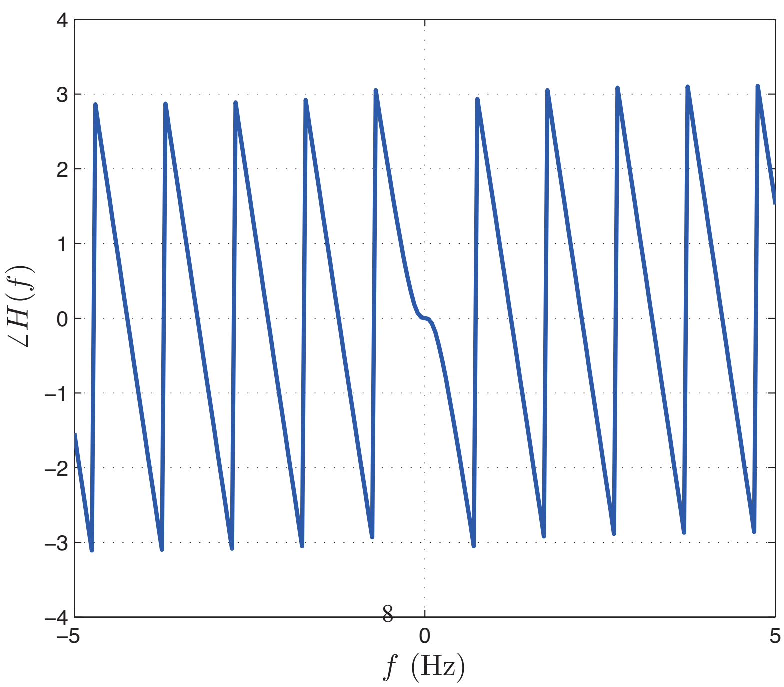

Figure 4: Phase response.

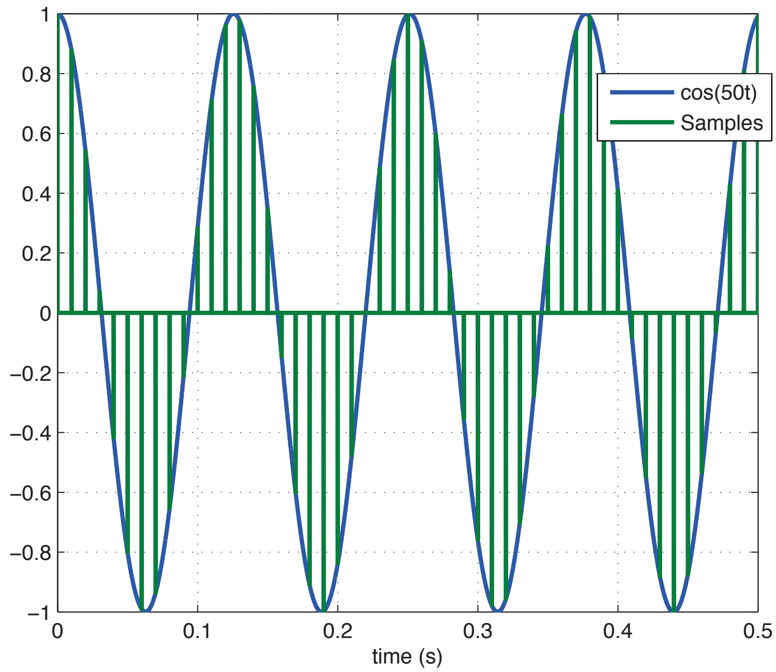

Figure 5: Signal and its samples.

d) : I assumed three different types of filters: 1) zero-order hold, 2) first-order hold as defined here, and 3) the actual first-order hold in my opinion. Fig. **??** shows the signal along with its samples only. Fig. 7, you see the signal, the samples, the output of zero-order hold, and the first-order hold *according to the definition of this problem* and in the following plot, the first-order hold system used in control systems literature. As anticipated, the actual first-order hold system should *outperform* the zero-order hold system.

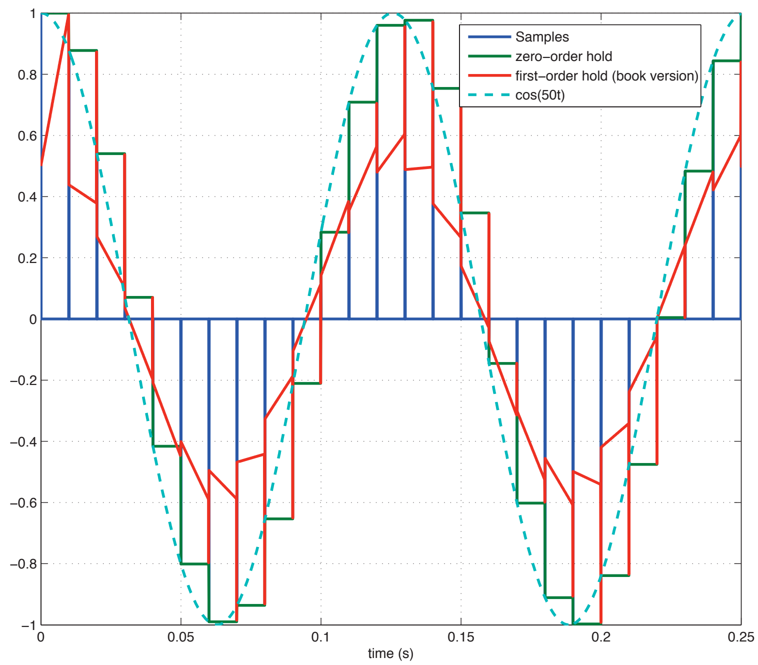

Figure 6: The output of reconstruction filter for zero and first-order hold systems (according to the book).

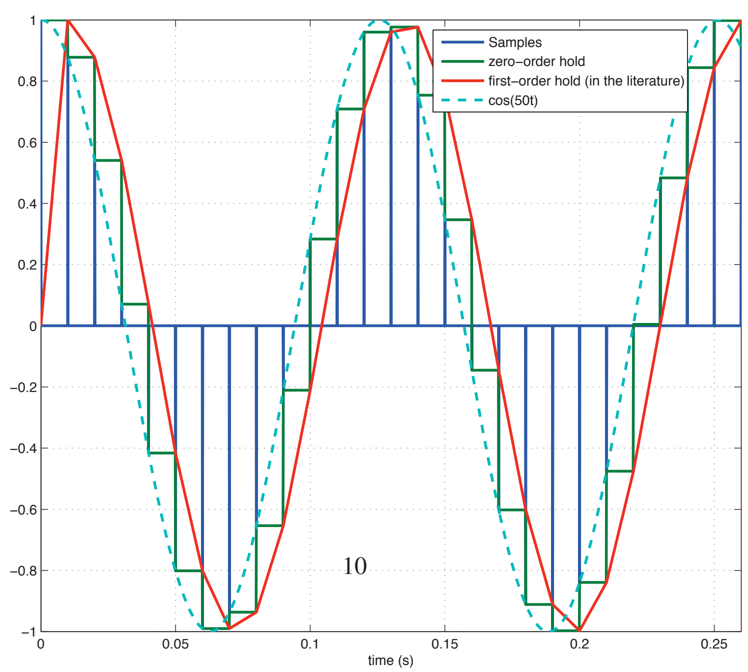

Figure 7: The output of reconstruction filter for zero and first-order hold systems (according to the literature).