The Discrete Fourier Transform

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Discrete Fourier Transform

- Frequency analysis of discrete-time signals must conveniently be performed on a computer or DSP.
- Recall:

$$\text{aperiodic in time} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \text{continuous in frequency}$$
$$x(n) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(\omega)$$

- $X(\omega)$ cannot therefore be computed for the entire set $\omega \in \mathbb{R}$; for practicality, $X(\omega)$ must be computed for a discrete and finite set of values in $\omega \in \mathbb{R}$.

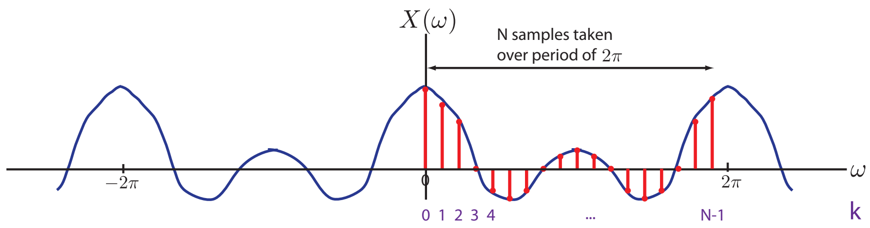
Discrete Fourier Transform

- Strategy to compute $X(\omega)$:
  1. Compute $X(\omega)$ for equally spaced samples.
  2. Compute $X(\omega)$ samples for one period only (recall, $X(\omega)$ is periodic with period 2π)
- Assuming we compute N samples of $X(\omega)$ over one period of 2π, the resulting computed frequency signal would effectively be a sampled version of $X(\omega)$ such that:

$$\omega = \frac{2\pi k}{N}$$

Frequency Domain Sampling

$$\omega = \frac{2\pi k}{N}$$



Note: spacing between each sample is $\frac{2\pi}{N}$.

Frequency Domain Sampling

- Recall, the discrete-time Fourier transform (DTFT):

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- Suppose we sample $X(\omega)$ according to: $\omega = \frac{2\pi k}{N}$.

Frequency Domain Sampling

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\
X\left(\frac{2\pi}{N}k\right) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi}{N}kn}, \\
&= \sum_{l=-\infty}^{\infty}\sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi k\frac{n}{N}} \quad \text{Let } n' = n - lN \\
&= \sum_{l=-\infty}^{\infty}\sum_{n'=0}^{N-1} x(n'+lN)\underbrace{e^{-j2\pi k\frac{n'+lN}{N}}}_{=e^{-j2\pi k\frac{n'}{N}}\underbrace{e^{-j2\pi k\frac{lN}{N}}}_{=1}}
\end{aligned}$$

Frequency Domain Sampling

For $k = 0, 1, 2, \ldots, N-1$,

$$\begin{aligned}
X\left(\frac{2\pi}{N}k\right) &= \sum_{l=-\infty}^{\infty}\sum_{n=0}^{N-1} x(n+lN)e^{-j2\pi k\frac{n}{N}} \\
&= \sum_{n=0}^{N-1}\sum_{l=-\infty}^{\infty} x(n+lN)e^{-j2\pi k\frac{n}{N}} \\
&= \sum_{n=0}^{N-1}\underbrace{\left[\sum_{l=-\infty}^{\infty} x(n+lN)\right]}_{\text{equivalent signal } x_p(n)} e^{-j2\pi k\frac{n}{N}} \\
&= \sum_{n=0}^{N-1} x_p(n)e^{-j2\pi k\frac{n}{N}}
\end{aligned}$$

where $x_p(n)$ is the periodic repetition of $x(n)$.

Frequency Domain Sampling

For $k = 0, 1, 2, \ldots, N-1$,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)e^{-j2\pi k\frac{n}{N}}$$

- Characteristics of $x_p(n)$: (1) discrete-time, (2) period $= N$, (3) has as a DTFS:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi k\frac{n}{N}}$$

- $X\left(\frac{2\pi}{N}k\right)$ looks like a scaled DTFS of $x_p(n)$,

$$c_k = \frac{1}{N}X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \ldots, N-1.$$

except we only take the coefficient values at $k = 0, 1, \ldots, N-1$.

Intuition

aperiodic + dst in time $\overset{\text{DTFT}}{\longleftrightarrow}$ cts + periodic in freq

$\downarrow$ periodic repetition $\qquad \downarrow$ sampling

periodic + dst in time $\overset{\text{DTFS}}{\longleftrightarrow}$ dst + periodic in freq

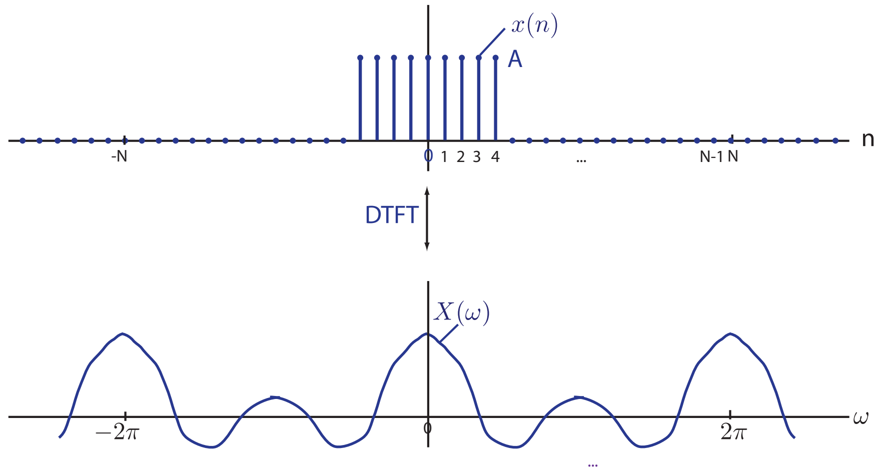
one period of dst-time samples $\overset{\text{DFT}}{\longleftrightarrow}$ one period of dst-freq samples

$n = 0, 1, \ldots, N-1 \qquad k = 0, 1, \ldots, N-1$

Therefore, we define the Discrete Fourier Transform (DFT) as being a computable transform that approximates the DTFT.

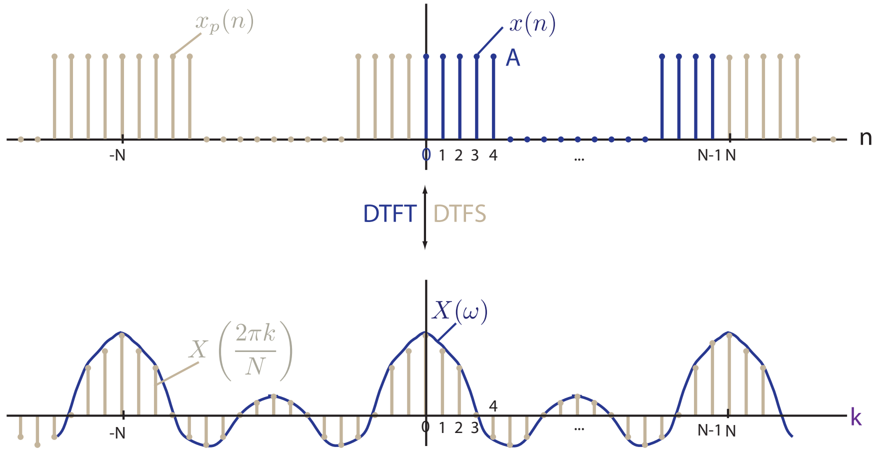
Intuition

Example



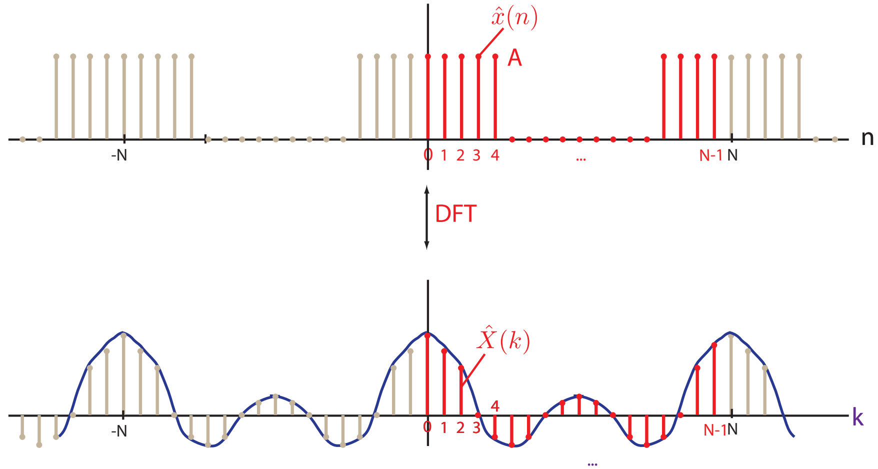
Intuition

Example



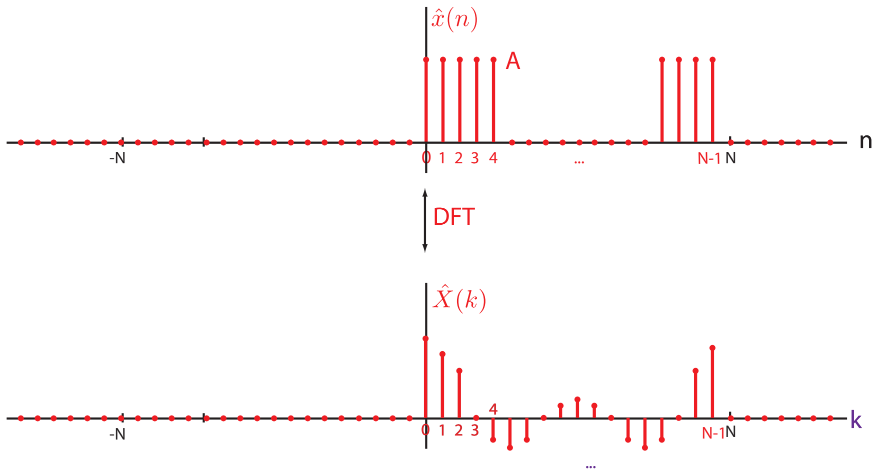
Intuition

Example



Intuition

Example



DTFT, DTFS and DFT

$$x(n) \text{ for all } n \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad X(\omega) \text{ for all } \omega$$

$$\downarrow \text{ periodic repetition} \qquad \downarrow \text{ sampling}$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n+lN) \text{ for all } n \quad \overset{\text{DTFS}}{\longleftrightarrow} \quad X(k) = X(\omega)|_{\omega=\frac{2\pi}{N}k} \text{ for all } k$$

$$\hat{x}(n) \quad \overset{\text{DFT}}{\longleftrightarrow} \quad \hat{X}(k)$$

where

$$\hat{x}(n) = \begin{cases} x_p(n) & \text{for } n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{X}(k) = \begin{cases} X(k) & \text{for } k = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Frequency Domain Sampling

- Recall, sampling in time results in a periodic repetition in frequency.

$$x(n) = x_a(t)|_{t=nT} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(\omega + \frac{2\pi}{T}k)$$

- Similarly, sampling in frequency results in periodic repetition in time.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n+lN) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(k) = X(\omega)|_{\omega=\frac{2\pi}{N}k}$$

Frequency Domain Sampling and Reconstruction

- Therefore,

$$x(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\omega)$$
$$x_p(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(k)$$

- Implications:
  - The samples of $X(\omega)$ can be used to reconstruct $x_p(n)$.

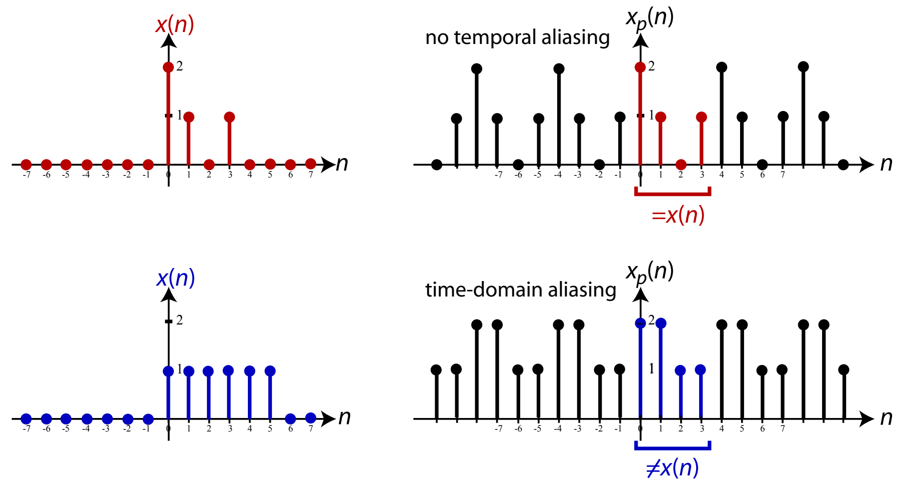
Frequency Domain Sampling and Reconstruction

- **Q:** Can we reconstruct $x(n)$ from the samples of $X(\omega)$?
  - Can we reconstruct $x(n)$ from $x_p(n)$?
- **A:** Maybe.

$$x_p(n) = \left[\sum_{l=-\infty}^{\infty} x(n + lN) \right]$$

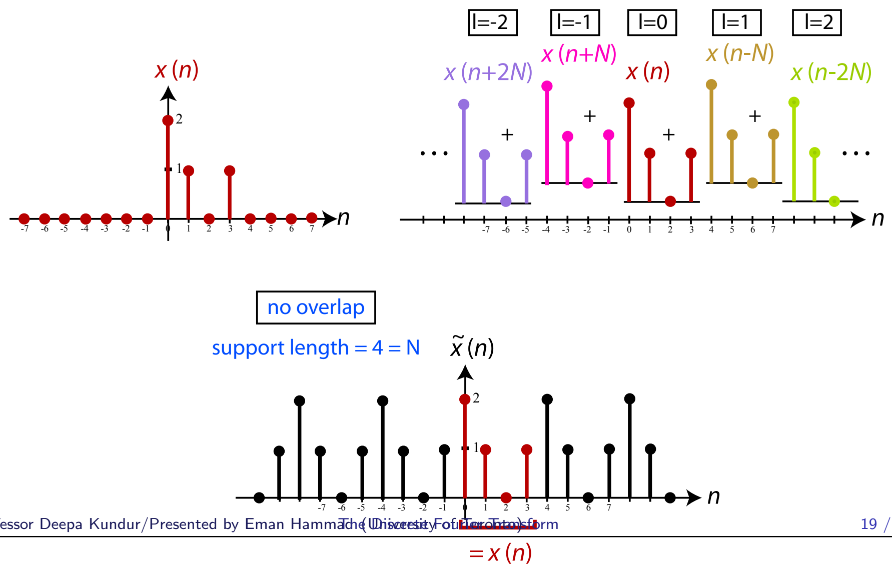
Frequency Domain Sampling and Reconstruction

$N = 4$



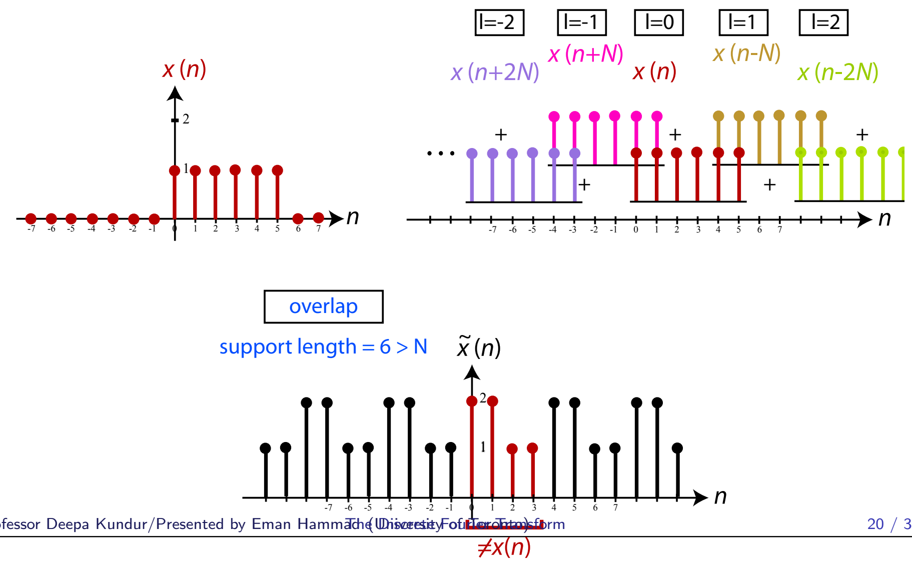
Frequency Domain Sampling and Reconstruction

$N = 4$



Frequency Domain Sampling and Reconstruction

$N = 4$



Frequency Domain Sampling and Reconstruction

- $x(n)$ can be recovered from $x_p(n)$ if there is no overlap when taking the periodic repetition.
- If $x(n)$ is finite duration and non-zero in the interval $0 \le n \le L-1$, then

$$x(n) = x_p(n), \quad 0 \le n \le N-1 \quad \text{when } N \ge L$$

- If $N < L$ then, $x(n)$ cannot be recovered from $x_p(n)$.
  - or equivalently $X(\omega)$ cannot be recovered from its samples $X\left(\frac{2\pi}{N}k\right)$ due to time-domain aliasing

The Discrete Fourier Transform Pair

- DFT and inverse-DFT (IDFT):

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \quad k = 0, 1, \ldots, N-1 \\ x(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k \frac{n}{N}}, \quad n = 0, 1, \ldots, N-1 \end{aligned}$$

Note: we drop the $\hat{}$ notation from now on.

DFT Example

Q: Determine the N-point DFT of the following sequence for $N \ge L$:

$$x(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

A: The DTFT of $x(n)$ is given by:

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{L-1} x(n) e^{-j\omega n} \\ &= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \end{aligned}$$

DFT Example

Thus,

$$\begin{aligned} X(\omega) &= \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \\ X(k) &= \frac{\sin(\frac{2\pi k}{N} L/2)}{\sin(\frac{2\pi k}{N}/2)} e^{-j\frac{2\pi k}{N}(L-1)/2} \\ &= \frac{\sin(\pi k L/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N} \end{aligned}$$

DFT Properties

- The properties of the DFT are different from those typical of the DTFS and DTFT because they are circular in nature.
- That is, they apply to the periodic repetition of the signal.

Important DFT Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(k)$
Periodicity:	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift:	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift:	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate:	$x^*(n)$	$X^*(N-k)$
Circular convolution:	$x_1(n) \otimes x_2(n)$	$X_1(k)X_2(k)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \otimes X_2(k)$
Parseval's theorem:	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Circular Properties

- Circular operations: apply the transformation on the periodic repetition of $x(n)$ and then obtain the final result by taking points for $n = 0, 1, \ldots, N-1$
- Often use the modulo notation:

$$(n)_N = n \mod N = \text{remainder of } n/N$$

Modulo Indices and Periodic Repetition

$$\boxed{(n)_N = n \mod N = \text{remainder of } n/N}$$

Example: $N = 4$

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$(n)_4$	0	1	2	3	0	1	2	3	0	1	2	3	0

$$\frac{n}{N} = \text{integer} + \frac{\boxed{\text{nonneg integer} < N}}{N}$$

$$\frac{5}{4} = 1 + \frac{1}{4} \qquad \frac{-2}{4} = -1 + \frac{2}{4}$$

Modulo Indices and Periodic Repetition

$$\boxed{(n)_N = n \mod N = \text{remainder of } n/N}$$

Example: $N = 4$

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$(n)_4$	0	1	2	3	0	1	2	3	0	1	2	3	0

$x((n))_4$ will be periodic with period 4. The repeated pattern will be consist of: $\{x(0), x(1), x(2), x(3)\}$.

Thus, $x((n))_N$ is a periodic signal comprised of the following repeating pattern: $\{x(0), x(1), \cdots x(N-2), x(N-1)\}$.

Circular Symmetry and Convolution

- Circular Symmetry:
  - circular time reversal: $x((-n))_N = x(N-n)$
  - circularly even: $x(N-n) = x(n)$
  - circularly odd: $x(N-n) = -x(n)$

- Circular Convolution:

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N, \quad m = 0, 1, \ldots, N-1$$

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