Name: ______________________________ Student No.: __________________

University of Toronto
Electrical & Computer Engineering
ECE 362, Winter 2013
Thursday, February 7, 2013

Test #1

Professor Deepa Kundur
Duration: 110 minutes

- All work must be performed within the space provided. You may use the back of sheets if necessary.
- No aid sheet allowed. No cell phones, PDAs, etc. allowed. You may use a Type 2 (nonprogrammable) calculator.
- Answer **all** questions.
- If a particular question seems unclear, please explicitly state any reasonable assumptions and proceed with the problem.
- Please properly label all points of interest on sketches and graphs that you are requested to draw, so that there is no ambiguity.
- **For full marks, show all steps and present results clearly**.

Question	**Earned Grade**
1	/10
2	/10
3	/10
4	/10
5	/10
Total	**/50**

1. **(10 marks)** True or False? No explanation necessary (to save time).

Please circle the appropriate answer for each part.

(a) The closing prices of utility stocks on the New York Stock Exchange represents a <u>discrete-time continuous-amplitude</u> signal.

T or $\boxed{\text{F}}$

Note: the number of stocks sold is a natural number and thus is not continuous-amplitude.

(b) The signals $\cos(2\pi n/6)$ and $\cos(14\pi n/6)$ represent the same discrete-time sinusoid.

$\boxed{\text{T}}$ or F

Note: their cyclic frequencies are apart by an integer ($f_1 = 1/6$ and $f_2 = 7/6$ where $f_2 - f_1 = 1$).

(c) The system represented by

$$y(n) = x^2(n)$$

(where $x(n)$ is the input and $y(n)$ is the output) is a linear time-invariant (LTI) system?

T or $\boxed{\text{F}}$

Note: The system is not linear; for instance it does not obey homogeneity since if you increase the input by a factor α the output is increased by a factor α^2.

(d) The impulse response $h(n)$ of a causal LTI system must always be zero for positive time; that is, $h(n) = 0$ for $n > 0$.

T or $\boxed{\text{F}}$

Note: it is the opposite; $h(n) = 0$ is required for $n < 0$.

(e) The region of convergence for the following signal $x(n) = u(n)$ (where $u(n)$ is the discrete-time unit step function) is a <u>disk</u> (i.e., inside part of a circle) of unit radius.

T or $\boxed{\text{F}}$

Note: the sequence is causal (right-sided). Therefore, the ROC would be the outer part of a circle, not the inner.

2. **(10 marks)**

Show that the fundamental period N_p of the signals

$$s_k(n) = e^{j2\pi kn/N}, \quad k = 0, 1, 2, \ldots$$

is given by $N_p = \frac{N}{\text{GCD}(k,N)}$ where $\text{GCD}(k, N)$ is the greatest common divisor of k and N.

Solution:

First, to check that the discrete-time sinusoidal signal is truly periodic, we make sure that the cyclic frequency is rational. By inspection, the cyclic frequency is given by:

$$f_0 = \frac{k}{N}$$

which is clearly rational since $k, N \in \mathbb{Z}$. If we found f_0 was not rational then, no fundamental period N_p would exist.

Therefore, N_p does exist, and we next aim to find this value. As discussed in the lectures, the fundamental period is given by:

$$N_p = \frac{m}{f_0}$$

where m is the smallest positive integer that makes m/f_0 an integer. Therefore,

$$\begin{aligned}
N_p &= \frac{m}{f_0} = \frac{N}{k}m \\
&= \frac{N'\text{GCD}(k,N)}{k'\text{GCD}(k,N)}m \\
&= \frac{N'}{k'}m \\
&= N' \quad \text{for } m = k' \\
&= \frac{N}{\text{GCD}(k,N)}
\end{aligned}$$

where in the second line, the common integer factor between k and N denoted $\text{GCD}(k, N)$ is factored out leaving k' and N' to be relatively prime, that is, $\text{GCD}(k', N') = 1$.

In the third line above, we see that since $\text{GCD}(k', N') = 1$, m must be equal to k' to make $\frac{N'}{k'}m \in \mathbb{Z}$.

3. **(10 marks)**

A discrete-time system can be:

(a) Static or dynamic
(b) Linear or nonlinear
(c) Time invariant or time varying
(d) Causal or noncausal
(e) Stable or unstable

Examine the following system (that outputs the even part of the input) with respect to the properties above:

$$y(n) = \frac{x(n) + x(-n)}{2}$$

Please note that for full marks, you must prove or disprove the properties of linearity, time-invariance and stability. For static or causality, you can refer to the definition to discuss your conclusion.

Solution:

(a) To static, the system must depend only on the present time input. It is clear from the integral that $y(n)$ depends on inputs values at other time instants than n (specifically $y(n)$ depends on inputs at time $-n$). Thus, by definition, the system is $\boxed{\text{DYNAMIC}}$.

(b) For linearity, if we have:

$$\begin{aligned}
x_1(n) &\longrightarrow && y_1(n) = \frac{x_1(n) + x_1(-n)}{2} \\
x_2(n) &\longrightarrow && y_2(n) = \frac{x_2(n) + x_2(-n)}{2} \\
& && \text{then we should have the following} \\
\alpha x_1(n) + \beta x_2(n) &\longrightarrow && \alpha y_1(n) + \beta y_2(n)
\end{aligned}$$

FOR ALL possible $x_1(n)$, $x_2(n)$, and $\alpha, \beta \in \mathbb{R}$.
If we let $x(n) = \alpha x_1(n) + \beta x_2(n)$, we obtain:

$$\begin{aligned}
y(n) &= \frac{x(n) + x(-n)}{2} \\
&= \frac{\alpha x_1(n) + \beta x_2(n) + \alpha x_1(-n) + \beta x_2(-n)}{2} \\
&= \alpha \frac{x_1(n) + x_1(-n)}{2} + \beta \frac{x_2(n) + x_2(-n)}{2} \\
&= \alpha y_1(n) + \beta y_2(n)
\end{aligned}$$

Therefore, $\alpha x_1(n) + \beta x_2(n) \longrightarrow \alpha y_1(n) + \beta y_2(n)$, and the system is $\boxed{\text{LINEAR}}$.

(c) For time invariance, if we have:

$$\begin{array}{rcl} x_1(n) & \longrightarrow & y_1(n) = \dfrac{x_1(n) + x_1(-n)}{2} \\ & & \text{then we should have the following} \\ x_1(n-n_0) & \longrightarrow & y_1(n-n_0) = \dfrac{x(n-n_0) + x(-(n-n_0))}{2} = \dfrac{x(n-n_0) + x(-n+n_0)}{2} \end{array}$$

FOR ALL possible $x_1(n)$ and $n_0 \in \mathbb{Z}$.
Therefore, suppose we inject $x(n) = x_1(n-n_0)$ into the system, we therefore have:

$$\begin{aligned} y(n) &= \frac{x(n) + x(-n)}{2} \\ &= \frac{x_1(n-n_0) + x_1((-n) - n_0)}{2} \\ &= \frac{x_1(n-n_0) + x_1(-n-n_0)}{2} \end{aligned}$$

Since $x(m) = x_1(m - n_0)$ and we substitute $m = n$ for the first term above to get $x(n) = x_1(n - n_0)$, and we substitute $m = -n$ for the second term above to get $x(-n) = x_1((-n) - n_0)$.

Therefore, $x_1(n-n_0) \not\longrightarrow y_1(n-n_0)$, so the system is $\boxed{\text{TIME VARYING}}$.

(d) For causality, the output at any time instant must only depend on present and past time values of the input FOR ALL integer n. Since

$$y(n) = \frac{x(n) + x(-n)}{2},$$

we see that $y(-3) = \frac{x(-3)+x(-(-3))}{2} = \frac{x(-3)+x(3)}{2}$ which means that for at least one time instant, the output depends on a future time value of input. (i.e., $y(-3)$ depends on $x(3)$). Therefore, the system is $\boxed{\text{NONCAUSAL}}$.

(e) For stability, if the input is bounded then it must be guaranteed that the output is bounded. Analytically, this means that

$$\text{If } \exists \text{ an } M_x > 0 \text{ such that } |x(n)| \le M_x < \infty \Longrightarrow \exists \text{ a } M_y > 0 \text{ such that } |y(n)| \le M_y < \infty$$

If $|x(n)| \le M_x$, then we see the following:

$$\begin{aligned} |y(n)| &= \left| \frac{x(n) + x(-n)}{2} \right| \\ &\le \frac{|x(n)|}{2} + \frac{|x(-n)|}{2} \\ &\le \frac{M_x}{2} + \frac{M_x}{2} = M_x < \infty \end{aligned}$$

Therefore, the system is $\boxed{\text{STABLE}}$.
Many people actually do not do this question properly because they don't take the absolute value of $y(n)$. If you do not do it, you have not proved the result properly. Simplify stating the result is bounded is not correct.

4. **(10 marks)**

Let $h(n)$ be the impulse response and $s(n)$ be the step response of a given linear time invariant (LTI) system. Given that step response is given by $s(n) = h(n) * u(n)$.

In any way you deem appropriate, show that

$$h(n) = s(n) - s(n-1).$$

Please note that you must explain any non-obvious steps for full marks either through a small derivation of logical reasoning.

Hint: $\delta(n) = u(n) - u(n-1)$.

Solution:

$$\begin{aligned}
\delta(n) &= u(n) - u(n-1) \\
\delta(n) * h(n) &= [u(n) - u(n-1)] * h(n) \\
h(n) &= u(n) * h(n) - u(n-1) * h(n) \\
&= h(n) * u(n) - h(n) * u(n-1) \\
&= s(n) - s(n-1)
\end{aligned}$$

Note: $s(n-1) = h(n) * u(n-1)$ since $s(n) = h(n) * u(n)$ and an LTI system with imp use response $h(n)$ is time-invariant. Therefore if $u(n)$ is injected and the output is $s(n)$, when $u(n-1)$ is injected the output is $s(n-1)$.

5. **(10 marks)**

(a) Determine the Direct Form II realization of the following LTI system:

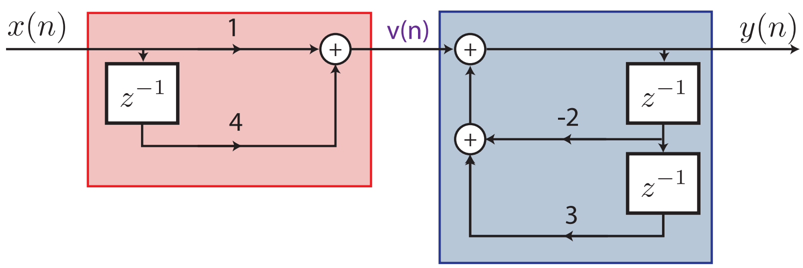
$$y(n) + 2y(n-1) - 3y(n-2) = x(n) + 4x(n-1)$$

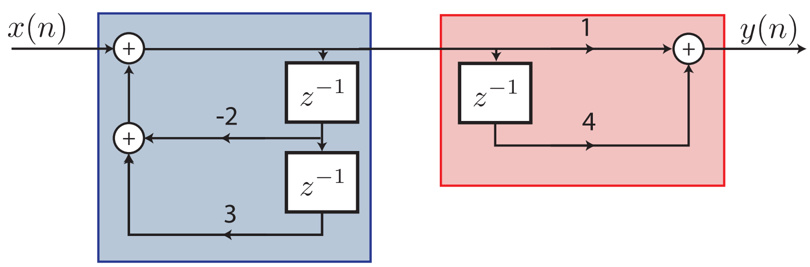
(assuming that it is relaxed; i.e., zero initial conditions)

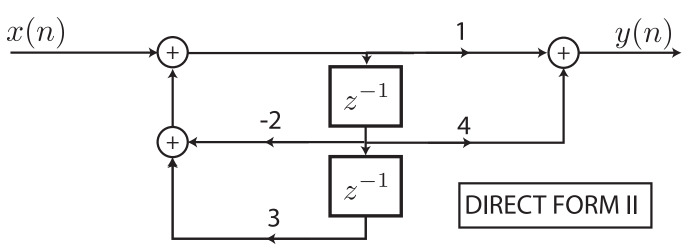
(b) What is the advantage of the Direct Form II realization over the Direct Form I realization?

(a) The Direct Form II realization can be determined as follows:

$$\begin{aligned} v(n) &= x(n) + 4x(n-1) \\ y(n) + 2y(n-1) - 3y(n-2) &= v(n) \end{aligned}$$







(b) The Direct Form II realization has fewer memory elements than the Direct Form I.