Name: ________________________________ Student No.: ____________________

University of Toronto
Electrical & Computer Engineering
ECE 362, Winter 2013
Thursday, April 4, 2013

Test #2

Professor Deepa Kundur
Duration: 110 minutes

- All work must be performed within the space provided. You may use the back of sheets if necessary.
- No aid sheet allowed. No cell phones, PDAs, etc. allowed. You may use a Type 2 (nonprogrammable) calculator.
- Answer **all** questions.
- If a particular question seems unclear, please explicitly state any reasonable assumptions and proceed with the problem.
- Please properly label all points of interest on sketches and graphs that you are requested to draw, so that there is no ambiguity.
- **For full marks, show all steps and present results clearly**.
- **Important tables are given at the end of the test**.

Question	**Earned Grade**
1	/10
2	/10
3	/10
4	/10
5	/10
Total	**/50**

1. **(10 marks)**

(a) Assume the z-Transform of $x(n)$ is given by $X(z)$ with region of convergence (ROC) $r_1 < |z| < r_2$. Prove the time-reversal property of the z-Transform from first principles:

$$x(-n) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(z^{-1}), \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}.$$

For full points, please show all steps.

(b) Consider a signal $x(n)$ with z-Transform:

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}.$$

For what *general* values of a_0, a_1, a_2, a_3, b_1 and b_2 is $x(n)$ even?

Solution:

(a) We start from the definition of the z-Transform of $x(-n)$. Let $y(n) = x(-n)$ and $m = -n$.

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x(m) z^{m} = \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m} = X(z^{-1}) \end{aligned}$$

For the ROC, where $X(z)$ was finite, now $X(z^{-1})$ is finite, so we have:

$$\begin{array}{rcl} r_1 < & |z| & < r_2 \\ r_1 < & |z^{-1}| & < r_2 \\ r_1 < & \frac{1}{|z|} & < r_2 \\ \dfrac{1}{r_2} < & |z| & < \dfrac{1}{r_1} \end{array}$$

(b) If $x(n)$ is even, then $x(n) = x(-n)$. Therefore,

$$\begin{aligned} X(z) &= X(z^{-1}) \\ \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} &= \frac{b_0 + b_1 z + b_2 z^2 + b_3 z^3}{a_0 + a_1 z + a_2 z^2 + a_3 z^3} \frac{z^{-3}}{z^{-3}} = \frac{b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}}{a_3 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} \frac{\alpha}{\alpha} \\ \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} &= \frac{\alpha b_3 + \alpha b_2 z^{-1} + \alpha b_1 z^{-2} + \alpha b_0 z^{-3}}{\alpha a_3 + \alpha a_2 z^{-1} + \alpha a_1 z^{-2} + \alpha a_0 z^{-3}} \quad \text{for } \alpha \neq 0 \end{aligned}$$

Therefore, we may have the following general relationships amongst the parameters:

$$\begin{array}{ccc} b_0 = \alpha b_3 & \text{and} & b_3 = \alpha b_0 \\ b_1 = \alpha b_2 & \text{and} & b_2 = \alpha b_1 \\ a_0 = \alpha a_3 & \text{and} & a_3 = \alpha a_0 \\ a_1 = \alpha a_2 & \text{and} & a_2 = \alpha a_1 \end{array}$$

We can see from above that a solution only exists for $\alpha^2 = 1$ or $\alpha = \pm 1$, since for for example the first set of equalities: $b_0 = \alpha b_3 = \alpha \cdot \alpha b_0 = \alpha^2 b_0$. Therefore, the signal is even for:

$$b_0 = \alpha b_3 \quad \text{and} \quad b_1 = \alpha b_2$$
$$a_0 = \alpha a_3 \quad \text{and} \quad a_1 = \alpha a_2$$

where $\alpha = \pm 1$.

2. **(10 marks)**

Consider the signal:

$$x(n) = \{ \quad 1, \quad 0, \quad -1, \quad \underset{\uparrow}{2}, \quad 3 \quad \}$$

with discrete-time Fourier transform (DTFT) $X(\omega) = X_R(\omega) + jX_I(\omega)$ where $X_R(\omega)$ and $X_I(\omega)$ are the real and imaginary parts of $X(\omega)$ in rectangular coordinates.

(a) Determine and sketch the signal $y_1(n)$ with DTFT:

$$Y_1(\omega) = X_R(\omega) + X_I(\omega).$$

(b) Determine and sketch the signal $y_2(n)$ with DTFT:

$$Y_2(\omega) = X_R(\omega) - jX_I(\omega).$$

(c) Determine and sketch the signal $y_3(n)$ with DTFT:

$$Y_3(\omega) = j\int_{2\pi} X_R(\phi)X_I(\phi-\omega)d\phi.$$

Solution:

From the tables at the end of this test and given $x(n)$ is real, we can see that

$$\begin{aligned}
\frac{x(n)+x^*(-n)}{2} = \frac{x(n)+x(-n)}{2} &= x_e(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_R(\omega) \\
\frac{x(n)-x^*(-n)}{2} = \frac{x(n)-x(-n)}{2} &= x_o(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad jX_I(\omega) \\
-jx_o(n) &\quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_I(\omega)
\end{aligned}$$

We compute the even and odd components of $x(n)$ to give,

$$\begin{aligned}
x_e(n) &= \frac{1}{2}[\{1,0,-1,2,3,0,0\} + \{0,0,3,2,-1,0,1\}] \\
&= \{ \quad \tfrac{1}{2}, \quad 0, \quad 1, \quad \underset{\uparrow}{2}, \quad 1, \quad 0, \quad \tfrac{1}{2} \quad \} \\
x_o(n) &= \frac{1}{2}[\{1,0,-1,2,3,0,0\} - \{0,0,3,2,-1,0,1\}] \\
&= \{ \quad \tfrac{1}{2}, \quad 0, \quad -2, \quad \underset{\uparrow}{0}, \quad 2, \quad 0, \quad -\tfrac{1}{2} \quad \}
\end{aligned}$$

(a) Therefore, we see that

$$y_1(n) = x_e(n) - jx_o(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_1(\omega) = X_R(\omega) + X_I(\omega)$$

to give

$$y_1(n) = \{ \quad \tfrac{1}{2} - j\tfrac{1}{2}, \quad 0, \quad 1+j2, \quad \underset{\uparrow}{2}, \quad 1-j2, \quad 0, \quad \tfrac{1}{2}+j\tfrac{1}{2} \quad \}$$

(b) Similarly, we see that

$$y_2(n) = x_e(n) - x_o(n) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad Y_2(\omega) = X_R(\omega) - jX_I(\omega)$$

to give

$$y_2(n) = \{ \quad 0, \quad 0, \quad 3, \quad \underset{\uparrow}{2}, \quad -1, \quad 0, \quad 1 \quad \}$$

(c) Again, we see that

$$y_3(n) = 2\pi \cdot x_e(n) \cdot x_o(n) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad Y_3(\omega) = j \int_{2\pi} X_R(\phi) X_I(\phi - \omega) d\phi$$

to give

$$y_3(n) = \{ \quad \tfrac{\pi}{2}, \quad 0, \quad -4\pi, \quad \underset{\uparrow}{0}, \quad 4\pi, \quad 0, \quad -\tfrac{\pi}{2} \quad \}$$

3. **(10 marks)**

Consider the following system for $L \in \mathbb{Z}^+$:

$$y(n) = x(n+L) - x(n-L) \tag{1}$$

(a) Using the following standard building block elements: adders, constant multipliers, unit delay elements, unit advance elements and signal multipliers, determine a realization of the system that uses minimum unit delay/advance elements.

(b) Determine and sketch the magnitude and phase of the frequency response of the system Eq. 1.

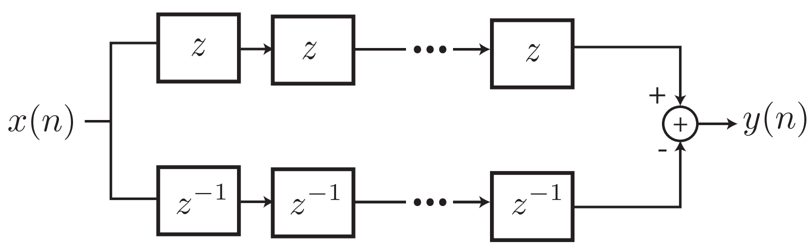
(c) Determine the locations of all spectral nulls, if any.

(d) This system is not causal. Suppose you connect the following system $y(n) = x(n - M)$ with input $x(n)$, output $y(n)$ and $M \in \mathbb{Z}$ in series with the system of Eq. 1. What values of M would make the new overall series connection system causal? For the minimum M, determine the input-output equation of this overall series connected causal system.

(e) Determine the frequency response expression of the system in part (d).

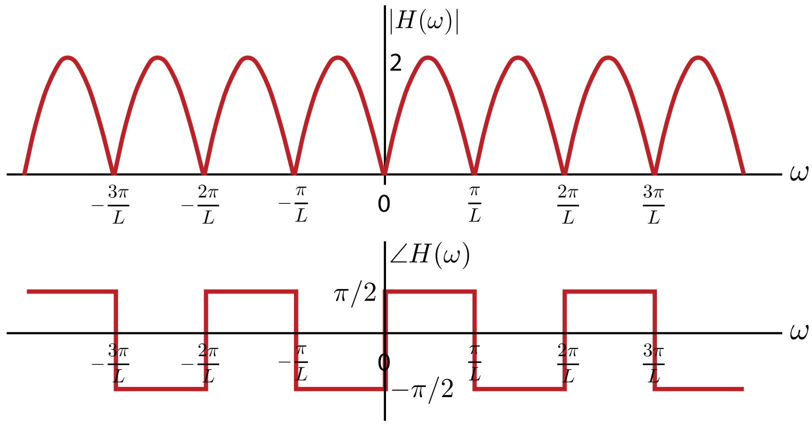
Solution:

(a) The realization is given below where there are L unit delay and unit advance elements in each row.



(b) The frequency response is given by:

$$\begin{aligned}
y(n) &= x(n+L) - x(n-L) \\
Y(\omega) &= e^{j\omega L}X(\omega) - e^{-j\omega L}X(\omega) = [e^{j\omega L} - e^{-j\omega L}]X(\omega) \\
H(\omega) = \frac{Y(\omega)}{X(\omega)} &= [e^{j\omega L} - e^{-j\omega L}] = 2j\sin(\omega L) \\
|H(\omega)| &= 2|\sin(\omega L)| \\
\angle H(\omega) &= \begin{cases} \frac{\pi}{2} & \frac{2k\pi}{L} < \omega < \frac{(2k+1)\pi}{L} \\ -\frac{\pi}{2} & \frac{(2k-1)\pi}{L} < \omega < \frac{2k\pi}{L} \end{cases}, k \in \mathbb{Z}
\end{aligned}$$



(c) The spectral nulls occur when $H(\omega) = 2j\sin(\omega L) = 0$, which occurs when $\omega L = \pi k$ or $\omega = \frac{\pi k}{L}$ for $k \in \mathbb{Z}$.

(d) The overall series connection would be causal for $M \geq L$, the minimum occurring for $M = L$. The overall input-output equation is given by:

$$\begin{aligned} y_1(n) &= x_1(n+L) - x_1(n-L) \\ y_2(n) &= x_2(n-M) = y_1(n-M) = x_1(n-M+L) - x_1(n-M-L) \\ \therefore\ y(n) &= x(n) - x(n-2L) \quad \text{for } M = L \end{aligned}$$

(e) We have that

$$\begin{aligned} H_1(\omega) &= 2j\sin(\omega L) \\ H_2(\omega) &= e^{-j\omega L} \\ H_{\text{overall}}(\omega) &= H_1(\omega)\cdot H_2(\omega) = 2je^{-j\omega L}\sin(\omega L) \end{aligned}$$

4. **(10 marks)**

(a) Compute the N-point discrete Fourier transform (DFT) of the signal: $x_1(n) = \delta(n)$.

(b) Show

$$\sum_{l=-\infty}^{\infty} \delta(n + lN) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}.$$

Solution:

(a) The N-point discrete Fourier transform (DFT) of $x_1(n) = \delta(n)$ is:

$$\begin{aligned} X(k) &= \sum_{k=0}^{N-1} x(n) e^{j(2\pi/N)kn}, k = 0, 1, \ldots, N-1 \\ &= \sum_{k=0}^{N-1} \delta(n) e^{j(2\pi/N)kn} \\ &= 1, \quad k = 0, 1, \ldots, N-1 \end{aligned}$$

(b) We see that $\sum_{l=-\infty}^{\infty} \delta(n+lN)$ is a periodic repetition of $\delta(n)$ for period N. Therefore, its discrete-time Fourier series (DTFS) coefficients are $c_k = 1$ for all $k \in \mathbb{Z}$ (since the DFT is just a windowed version of the DTFS such that the DTFS is a periodic repetition of the DFT for $k = 0, 1, \ldots, N-1$). The DTFS synthesis expression is given by:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

We also know that:

$$\sum_{l=-\infty}^{\infty} \delta(n + lN) \overset{\text{DTFS}}{\longleftrightarrow} 1$$

which we can plug back in to the DTFS synthesis equation to give:

$$\sum_{l=-\infty}^{\infty} \delta(n + lN) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}.$$

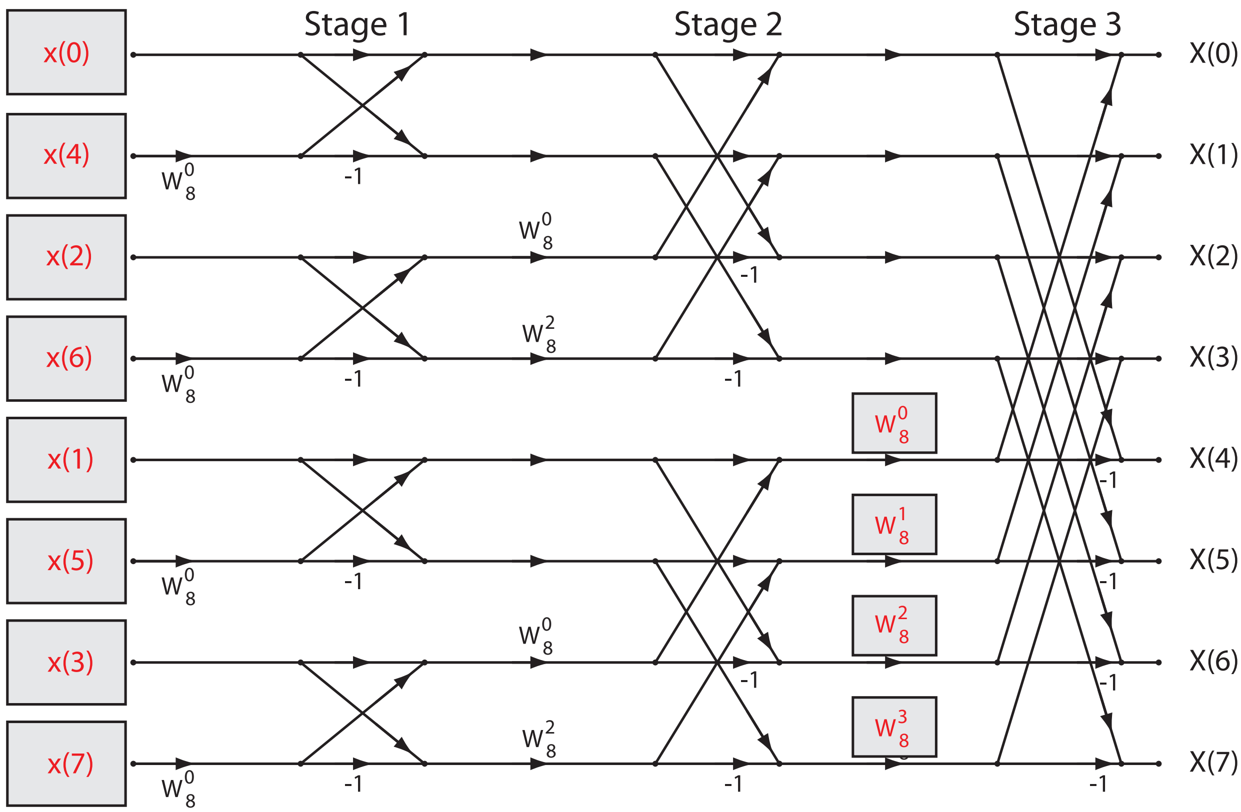
5. **(10 marks)**

Since the class seemed concerned about drawing butterflies, the butterfly representation for a radix-2 decimation-in-time 8-point FFT is provided below. Please fill in the blanks (i.e., the slightly shaded boxes, specifically representing the input values into the butterfly network and the four twiddle factors between Stage 2 and Stage 3). *Hint: the missing twiddle factors can be deduced by taking a first-level radix-2 decimation-in-time decomposition that we did in the lectures and finding the W_N factor that comes out of one of the summations. Here is how you would start:*

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} = \sum_{n \text{ even}} x(n)W_N^{kn} + \sum_{n \text{ odd}} x(n)W_N^{kn} \quad k = 0, 1, \ldots, N-1$$

Solution:

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)W_N^{kn} \quad k = 0, 1, \ldots, N-1 \\
&= \sum_{n \text{ even}} x(n)W_N^{kn} + \sum_{n \text{ odd}} x(n)W_N^{kn} \\
&= \sum_{m=0}^{(N/2)-1} x(2m)W_N^{k(2m)} + \sum_{m=0}^{(N/2)-1} x(2m+1)W_N^{k(2m+1)} \\
&= \sum_{m=0}^{(N/2)-1} \underbrace{x(2m)}_{\equiv f_1(m)} W_N^{2km} + \sum_{m=0}^{(N/2)-1} \underbrace{x(2m+1)}_{\equiv f_2(m)} W_N^{2km} W_N^k \\
&= \underbrace{\sum_{m=0}^{(N/2)-1} f_1(m)W_{N/2}^{km}}_{\frac{N}{2}-\text{DFT of } f_1(m)} + W_N^k \underbrace{\sum_{m=0}^{(N/2)-1} f_2(m)W_{N/2}^{km}}_{\frac{N}{2}-\text{DFT of } f_2(m)} \\
&= F_1(k) + W_N^k F_2(k), \quad k = 0, 1, \ldots, N-1
\end{aligned}$$



Stage 1
Stage 2
Stage 3
x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
W_8^0
W_8^2
W_8^1
W_8^3
-1
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)

Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\ x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$

DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \|X(\omega)\|^2$

DTFT Symmetry Properties

Time Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(-n)$	$X(-\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
	$X(\omega) = X^*(-\omega)$
	$X_R(\omega) = X_R(-\omega)$
$x(n)$ real	$X_I(\omega) = -X_I(-\omega)$
	$\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$
	$\angle X(\omega) = -\angle X(-\omega)$
$x'_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x'_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$

DFT Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(k)$
Periodicity:	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift:	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift:	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate:	$x^*(n)$	$X^*(N-k)$
Circular convolution:	$x_1(n) \otimes x_2(n)$	$X_1(k)X_2(k)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \otimes X_2(k)$
Parseval's theorem:	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Note: The following tables are courtesy of Ashish Khisti and Ravi Adve and were developed originally for ECE355.

Fourier Properties

Property	DTFS	CTFS	DTFT	CTFT
Synthesis	$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$
Analysis	$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\Omega_0 n}$	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$	$X(e^{j\Omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\Omega n}$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
Linearity	$\alpha x[n] + \beta y[n] \leftrightarrow \alpha a_k + \beta b_k$	$\alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$	$\alpha x[n] + \beta y[n] \leftrightarrow \alpha X(e^{j\Omega}) + \beta Y(e^{j\Omega})$	$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(j\omega) + \beta Y(j\omega)$
Time Shifting	$x[n-n_0] \leftrightarrow a_k e^{-j2\pi n_0 k/N}$	$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$	$x[n-n_0] \leftrightarrow e^{-j\Omega n_0}X(e^{j\Omega})$	$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$
Frequency Shift	$x[n]e^{j2\pi mn/N} \leftrightarrow a_{k-m}$	$x(t)e^{jm\omega_0 t} \leftrightarrow a_{k-m}$	$x[n]e^{j\Omega_0 n} \leftrightarrow X(e^{j(\Omega-\Omega_0)n})$	$x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega-\omega_0))$
Conjugation	$x^*[n] \leftrightarrow a^*_{-k}$	$x^*(t) \leftrightarrow a^*_{-k}$	$x^*[n] \leftrightarrow X^*(e^{-j\Omega})$	$x^*(t) \leftrightarrow X^*(-j\omega)$
Time Reversal	$x[-n] \leftrightarrow a_{-k}$	$x(-t) \leftrightarrow a_{-k}$	$x[-n] \leftrightarrow X(e^{-j\Omega})$	$x(-t) \leftrightarrow X(-j\omega)$
Convolution	$\sum_{r=0}^{N-1} x[r]y[n-r] \leftrightarrow Na_k b_k$	$\int_T x(\tau)y(t-\tau)d\tau \leftrightarrow Ta_k b_k$	$x[n]*y[n] \leftrightarrow X(e^{j\Omega})Y(e^{j\Omega})$	$x(t)*y(t) \leftrightarrow X(j\omega)Y(j\omega)$
Multiplication	$x[n]y[n] \leftrightarrow \sum_{r=0}^{N-1} a_r b_{k-r}$	$x(t)y(t) \leftrightarrow a_k * b_k$	$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\Omega-\theta)})d\theta$	$x(t)y(t) \leftrightarrow \frac{1}{2\pi}X(j\omega)*Y(j\omega)$
First Difference/ Derivative	$x[n]-x[n-1] \leftrightarrow (1-e^{-j2\pi k/N})a_k$	$\frac{dx(t)}{dt} \leftrightarrow jk\omega_0 a_k$	$x[n]-x[n-1] \leftrightarrow (1-e^{-j\Omega})X(e^{j\Omega})$	$\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$
Running Sum/ Integration	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{a_k}{1-e^{-j2\pi k/N}}$	$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{a_k}{jk\omega_0}$	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{X(e^{j\Omega})}{1-e^{-j\Omega}} + \pi X(e^{j0})\delta(\Omega)$	$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$
Parseval's Relation	$\frac{1}{N}\sum_{n=0}^{N-1}\|x[n]\|^2 = \sum_{k=0}^{N-1}\|a_k\|^2$	$\frac{1}{T}\int_T \|x(t)\|^2 dt = \sum_{k=-\infty}^{\infty}\|a_k\|^2$	$\sum_{n=-\infty}^{\infty}\|x[n]\|^2 = \frac{1}{2\pi}\int_{2\pi}\|X(e^{j\Omega})\|^2 d\Omega$	$\int_{-\infty}^{\infty}\|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\|X(j\omega)\|^2 d\omega$
Real and even signals	Real and even in frequency domain			
Real and odd signals	Purely imaginary and odd in frequency domain			

Additional Property: A real-valued time-domain signal $x(t)$ or $x[n]$ will have a conjugate-symmetric Fourier representation.

Notes:

1. For the CTFS, the signal $x(t)$ has a period of T, fundamental frequency $\omega_0 = 2\pi/T$; for the DTFS, the signal $x[n]$ has a period of N, fundamental frequency $\Omega_0 = 2\pi/N$. a_k and b_k denote the Fourier coefficients of $x(t)$ (or $x[n]$) and $y(t)$ (or $y[n]$) respectively.
2. Periodic convolutions can be evaluated by summing or integrating over *any* single period, not just those indicated above.
3. The "Running Sum" formula for the DTFT above is valid for Ω in the range $-\pi < \Omega \leq \pi$.

Fourier Pairs

FOURIER SERIES COEFFICIENTS OF PERIODIC SIGNALS*			
Continuous-Time		**Discrete-Time****	
Time Domain – $x(t)$	Frequency Domain – a_k	Time Domain – $x[n]$	Frequency Domain – a_k
$Ae^{j\omega_0 t}$	$a_1 = A$ $a_k = 0,\ k \neq 1$	$Ae^{j\Omega_0 n}$	$a_1 = A,$ $a_k = 0, k \neq 1$
$A\cos(\omega_0 t)$	$a_1 = a_{-1} = A/2$ $a_k = 0, k \neq 1$	$A\cos(\Omega_0 n)$	$a_1 = a_{-1} = A/2$ $a_k = 0, k \neq 1$
$A\sin(\omega_0 t)$	$a_1 = a_{-1}^* = \frac{A}{2j}$ $a_k = 0, k \neq 1$	$A\sin(\Omega_0 n)$	$a_1 = a_{-1}^* = \frac{A}{2j}$ $a_k = 0, k \neq 1$
$x(t) = A$	$a_0 = A, a_k = 0$ otherwise	$x[n] = A$	$a_0 = A, a_k = 0$ otherwise
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$a_k = \frac{1}{T}$	$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$a_k = \frac{1}{N}$
Periodic square wave $x(t) = \begin{cases} 1 & \lvert t\rvert < T_1 \\ 0 & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t) = x(t+T)$	$a_0 = \dfrac{2T_1}{T}$ $a_k = \dfrac{\sin(k\omega_0 T_1)}{k\pi},\ k \neq 0$		

FOURIER TRANSFORM PAIRS			
Continuous-Time		**Discrete-Time****	
Time Domain – $x(t)$	Frequency Domain – $X(j\omega)$	Time Domain – $x[n]$	Frequency Domain – $X(e^{j\Omega})$
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\dfrac{2\sin(\omega T_1)}{\omega}$	$x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\dfrac{\sin(\Omega(N_1 + 1/2))}{\sin(\Omega/2)}$
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin Wn}{\pi n}$	$X(e^{j\Omega}) = \begin{cases} 1, & \lvert\Omega\rvert \leq W \\ 0, & \text{otherwise} \end{cases}$
$\delta(t)$	1	$\delta[n]$	1
1	$2\pi\delta(\omega)$	1	$2\pi\delta(\Omega)$
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$	$u[n]$	$\dfrac{1}{1 - e^{-j\Omega}} + \pi\delta(\Omega)$
$e^{-at}u(t), \mathrm{Re}(a) > 0$	$\dfrac{1}{a + j\omega}$	$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1 - ae^{-j\Omega}}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t), \mathrm{Re}(a) > 0$	$\dfrac{1}{(a + j\omega)^n}$	$\dfrac{(n + r - 1)!}{n!(r-1)!}a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{(1 - ae^{-j\Omega})^r}$

*In the Fourier series table, $\omega_0 = \frac{2\pi}{T}$ and $\Omega_0 = \frac{2\pi}{N}$, where T and N are the periods of $x(t)$ and $x[n]$ respectively.
**For the DTFS, a_k is given only for k in the range $-N/2+1 \leq k \leq N/2$ for even N, $-(N-1)/2 \leq k \leq (N-1)/2$ for odd N, and $a_k = a_{k+N}$; for the DTFT $X(e^{j\Omega})$ is given only for Ω in the range $-\pi < \Omega \leq \pi$, and $X(e^{j\Omega}) = X(e^{j(\Omega+2\pi)})$.

Fourier Transform for Periodic Signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \leftrightarrow X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$