Discrete-Time System Properties

Professor Deepa Kundur

University of Toronto

Terminology: Implication

If “A” then “B”

Shorthand: $A \implies B$

Example 1:

- It is snowing $\implies$ it is at or below freezing temperature

Example 2:

- $\alpha \geq 5.2 \implies \alpha$ is positive

Note: For both examples above, $B \not\implies A$

Terminology: Equivalence

If “A” then “B”

Shorthand: $A \implies B$

If “B” then “A”

Shorthand: $B \implies A$

can be rewritten as

“A” if and only if “B”

Shorthand: $A \iff B$

We can also say:

- $A$ is EQUIVALENT to $B$
- $A \equiv B$

Reference:

Section 2.2

### Terminology: Input-Output Description

- Input-output description (exact structure of system is unknown or ignored):
  \[ y(n) = T[x(n)] \]
- “black box” representation:
  \[ x(n) \xrightarrow{T} y(n) \]

### Classification of Discrete-Time Systems

#### Why is this so important?
- mathematical techniques developed to analyze systems are often contingent upon the general characteristics of the systems being considered
- For a system to possess a given property, the property must hold for every possible input to the system.
  - to disprove a property, need a single counter-example
  - to prove a property, need to prove for the general case

#### Common System Properties:
- static vs. dynamic
- time-invariant vs. time-variant
- linear vs. nonlinear
- causal vs. non-causal
- stable vs. unstable systems
  - 

#### Static vs. Dynamic
- Static system (a.k.a. memoryless): the output at time \( n \) depends only on the input sample at time \( n \); otherwise the system is said to be dynamic

  \[ y(n) = T[x(n), n] \]

  for every time instant \( n \).
Static vs. Dynamic

Consider the general system:

\[ y(n) = T[x(n-N), x(n-N+1), \ldots, x(n-1), x(n), x(n+1), \ldots, x(n+M-1), x(n+M)], \quad N, M > 0 \]

- For \( N = M = 0 \), \( y(n) = T[x(n)] \), the system is **static**.
- For \( 0 < N, M < \infty \), the system is said to be **dynamic** with finite memory of duration \( N + M + 1 \).
- For either \( N \) and/or \( M \) equal to infinite, the system is said to have **infinite** memory.

Examples: memoryless or not?

1. \( y(n) = A x(n), A \neq 0 \)
2. \( y(n) = A x(n) + B, A, B, \neq 0 \)
3. \( y(n) = x(n) \cos\left(\frac{\pi}{25}(n-5)\right) \)
4. \( y(n) = x(-n) \)
5. \( y(n) = x(n+1) \)
6. \( y(n) = \frac{1}{1-x(n+2)} \)
7. \( y(n) = e^{3x(n)} \)
8. \( y(n) = \sum_{k=-\infty}^{n} x(k) \)

Ans: Y, Y, Y, N, N, Y, Y, Y

Time-invariant vs. Time-variant Systems

- **Time-invariant system**: input-output characteristics do not change with time
- A system is time-invariant iff

\[ x(n) \xrightarrow{T} y(n) \implies x(n-k) \xrightarrow{T} y(n-k) \]

for every input \( x(n) \) and every time shift \( k \).

Examples: time-invariant or not?

1. \( y(n) = A x(n), A \neq 0 \)
2. \( y(n) = A x(n) + B, A, B, \neq 0 \)
3. \( y(n) = x(n) \cos\left(\frac{\pi}{25}n\right) \)
4. \( y(n) = x(-n) \)
5. \( y(n) = x(n+1) \)
6. \( y(n) = \frac{1}{1-x(n+2)} \)
7. \( y(n) = e^{3x(n)} \)
8. \( y(n) = \sum_{k=-\infty}^{n} x(k) \)

Ans: Y, Y, N, N, Y, Y, Y, Y
Linear vs. Nonlinear Systems

- **Linear system**: obeys superposition principle

- A system is linear iff
  \[ T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)] \]
  for any arbitrary input sequences \( x_1(n) \) and \( x_2(n) \), and any arbitrary constants \( a_1 \) and \( a_2 \).

Linear Systems: Homogeneity

A system is linear iff

\[ T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)] \]

- **Homogeneity**: Let \( a_2 = 0 \).
  \[ T[a_1 x_1(n)] = a_1 T[x_1(n)] \]

\[ x(n) \xrightarrow{T} y(n) \implies a_1 x(n) \xrightarrow{T} a_1 y(n) \]

for any constant \( a_1 \).

Linear Systems: Additivity

A system is linear iff

\[ T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)] \]

- **Additivity**: Let \( a_1 = a_2 = 1 \).
  \[ T[x_1(n) + x_2(n)] = T[x_1(n)] + T[x_2(n)] \]

\[ x_1(n) \xrightarrow{T} y_1(n) \quad \implies \quad x_1(n) + x_2(n) \xrightarrow{T} y_1(n) + y_2(n) \]

for any input sequences \( x_1(n) \) and \( x_2(n) \).

Therefore:

\[ \text{Linearity} = \text{Homogeneity} + \text{Additivity} \]

Need both!

If a system is not homogeneous, it is not linear.
If a system is not additive, it is not linear.
Linear vs. Nonlinear Systems

Examples: linear or not?

- \( y(n) = A \ x(n), \ A \neq 0 \)
- \( y(n) = A \ x(n) + B, \ A, B \neq 0 \)
- \( y(n) = x(n) \cos(\frac{\pi}{25} n) \)
- \( y(n) = x(-n) \)
- \( y(n) = x(n+1) \)
- \( y(n) = \frac{1}{1-x(n+2)} \)
- \( y(n) = e^{3x(n)} \)
- \( y(n) = \sum_{k=-\infty}^{n} x(k) \)

Ans: Y, N, Y, Y, N, N, Y

Causal vs. Noncausal Systems

- Causal system: output of system at any time \( n \) depends only on present and past inputs

- a system is causal iff

\[ y(n) = T [x(n), x(n - 1), x(n - 2), \ldots] \]

for all \( n \)

Ans: Y, Y, Y, N, N, N, Y

Stable vs. Unstable Systems

- Bounded Input-Bounded output (BIBO) Stable: every bounded input produces a bounded output

- a system is BIBO stable iff

\[ |x(n)| \leq M_x < \infty \implies |y(n)| \leq M_y < \infty \]

for all \( n \).
Discrete-Time System Properties

Discrete-Time Bounded Signals

- $x[n]$ and $y[n]$ represent bounded signals.
- $M_x$ and $M_y$ denote the maximum values.

Stable vs. Unstable Systems

Examples: stable or not?

- $y(n) = A \cdot x(n)$, $A \neq 0$
- $y(n) = A \cdot x(n) + B$, $A, B \neq 0$
- $y(n) = x(n) \cos(\frac{\pi}{25} n)$
- $y(n) = x(-n)$
- $y(n) = x(n+1)$
- $y(n) = \frac{1}{1-x(n+2)}$
- $y(n) = e^{3x(n)}$
- $y(n) = \sum_{k=-\infty}^{n} x(k)$

Ans: Y, Y, Y, Y, Y, N, Y, N

Final Remarks

- For a system to possess a given property, the property must hold for every possible input and parameter of the system.

  - to disprove a property, need a single counter-example

  - to prove a property, need to prove for the general case