The z-Transform and Its Properties

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Reference:

Sections 3.1 and 3.2 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

The Direct z-Transform

- Direct z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Notation:

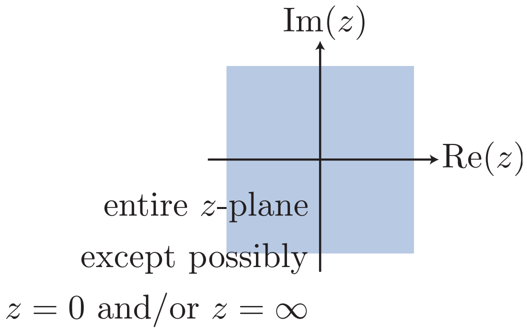
$$X(z) \equiv \mathcal{Z}\{x(n)\}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

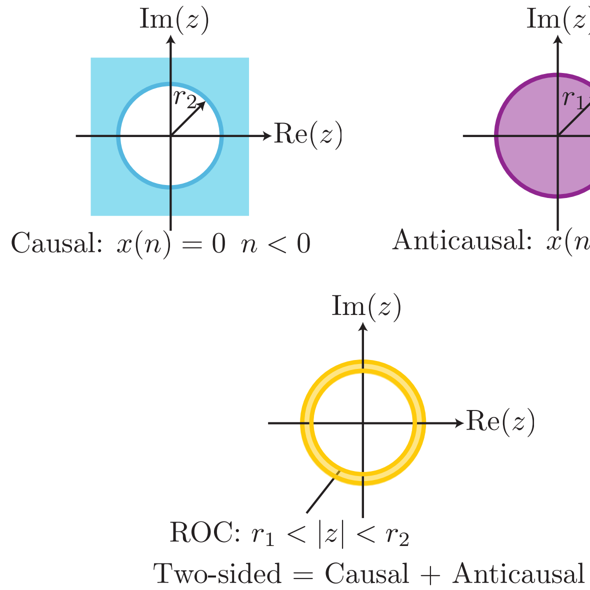
Region of Convergence

- the region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value
- The z-Transform is, therefore, uniquely characterized by:
  1. expression for $X(z)$
  2. ROC of $X(z)$

ROC Families: Finite Duration Signals



ROC Families: Infinite Duration Signals



z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \lvert z \rvert < r_1$
	$x_1(n)$	$X_1(z)$	ROC$_1$
	$x_2(n)$	$X_2(z)$	ROC$_2$
Linearity:	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least ROC$_1 \cap$ ROC$_2$
Time shifting:	$x(n-k)$	$z^{-k} X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1} z)$	$\lvert a \rvert r_2 < \lvert z \rvert < \lvert a \rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z \rvert < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\, x(n)$	$-z \frac{dX(z)}{dz}$	$r_2 < \lvert z \rvert < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At least ROC$_1 \cap$ ROC$_2$

among others ...

Linearity Property

$$x_1(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \text{ROC}_1$$

$$x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \text{ROC}_2$$

$$a_1 x_1(n) + a_2 x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z),$$
$$\text{At least ROC}_1 \cap \text{ROC}_2$$

Linearity Property

Let $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} (a_1 x_1(n) + a_2 x_2(n)) z^{-n} \\
&= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \\
&= a_1 X_1(z) + a_2 X_2(z)
\end{aligned}
$$

ROC: At least $\text{ROC}_1 \cap \text{ROC}_2$

Time Shifting Property

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{ROC}$$

$$x(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k} X(z), \qquad \text{At least ROC}$$

except $z = 0$ $(k > 0)$ or
$z = \infty$ $(k < 0)$

Time Shifting Property

Let $y(n) = x(n-k)$.

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} (x(n-k)) z^{-n}
\end{aligned}
$$

Let $m = n - k$

$$
\begin{aligned}
&= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)} = \sum_{m=-\infty}^{\infty} x(m) z^{-m} z^{-k} \\
&= z^{-k} \left[\sum_{m=-\infty}^{\infty} x(m) z^{-m} \right] = z^{-k} X(z)
\end{aligned}
$$

Time Shifting Property: ROC

$$x(n) = \delta(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = 1, \text{ROC: } \underline{\text{entire } z\text{-plane}}$$

- Example: For $k = -1$

$$y(n) = x(n - (-1)) = x(n+1) = \delta(n+1)$$

$$y(n) = \delta(n+1) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = z, \text{ROC: } \underline{\text{entire } z\text{-plane}}$$

except $z = \infty$

- Example: For $k = 1$

$$y(n) = x(n - (1)) = x(n-1) = \delta(n-1)$$

$$y(n) = \delta(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = z^{-1}, \text{ROC: } \underline{\text{entire } z\text{-plane}}$$

except $z = 0$

Scaling in the z-Domain

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{ROC: } r_1 < |z| < r_2$$

$$a^n x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(a^{-1}z), \qquad \text{ROC: } |a|r_1 < |z| < |a|r_2$$

Scaling in the z-Domain

Let $y(n) = a^n x(n)$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} \\ &= X(a^{-1}z) \end{aligned}$$

ROC: $r_1 < |a^{-1}z| = \frac{|z|}{|a|} < r_2 \;\equiv\; |a|r_1 < |z| < |a|r_2$

z-Transform Properties

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Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_1(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\,x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$

among others . . .

Convolution Property

$$x(n) = x_1(n) * x_2(n) \iff X(z) = X_1(z) \cdot X_2(z)$$

Convolution using the z-Transform

Basic Steps:

1. Compute z-Transform of each of the signals to convolve (time domain $\rightarrow$ z-domain):

$$\begin{aligned} X_1(z) &= \mathcal{Z}\{x_1(n)\} \\ X_2(z) &= \mathcal{Z}\{x_2(n)\} \end{aligned}$$

2. Multiply the two z-Transforms (in z-domain):

$$X(z) = X_1(z)X_2(z)$$

3. Find the inverse z-Transformof the product (z-domain $\rightarrow$ time domain):

$$x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

Common Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

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4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Common Transform Pairs

- z-Transform expressions that are a fraction of polynomials in z^{-1} (or z) are called rational.
- z-Transforms that are rational represent an important class of signals and systems.

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