Introduction to Digital Signal Processing

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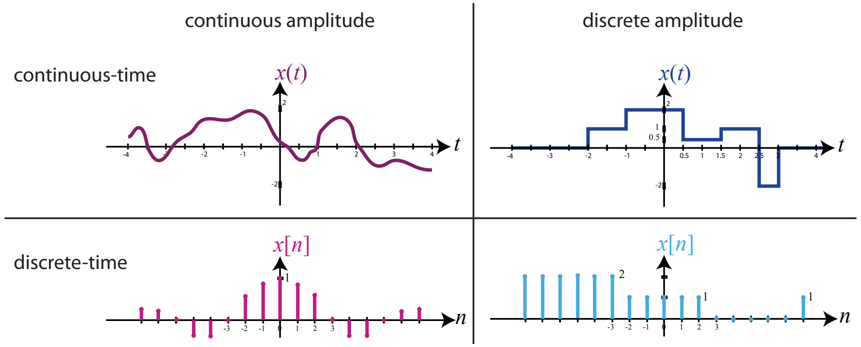
Analog and Digital Signals

- analog signal = continuous-time + continuous amplitude
- digital signal = discrete-time + discrete amplitude



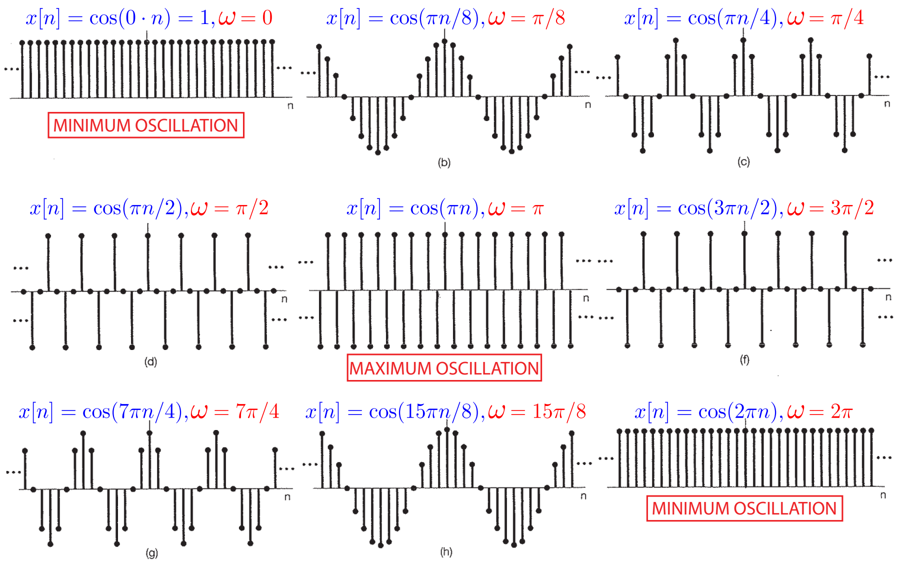
Analog and Digital Signals

- analog system = analog input + analog output
- digital system = digital input + digital output

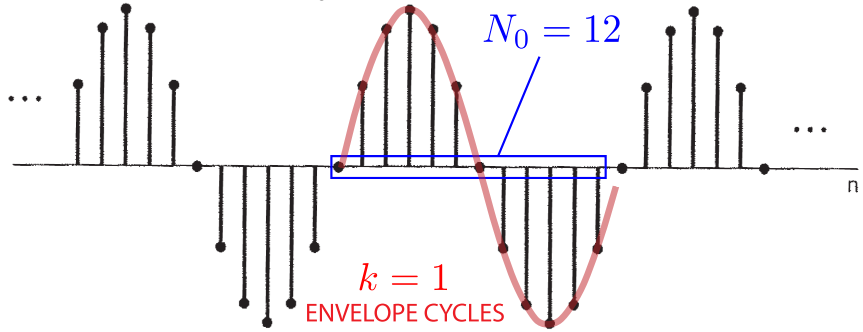
Discrete-time Sinusoids

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta), \quad n \in \mathbb{Z}$$

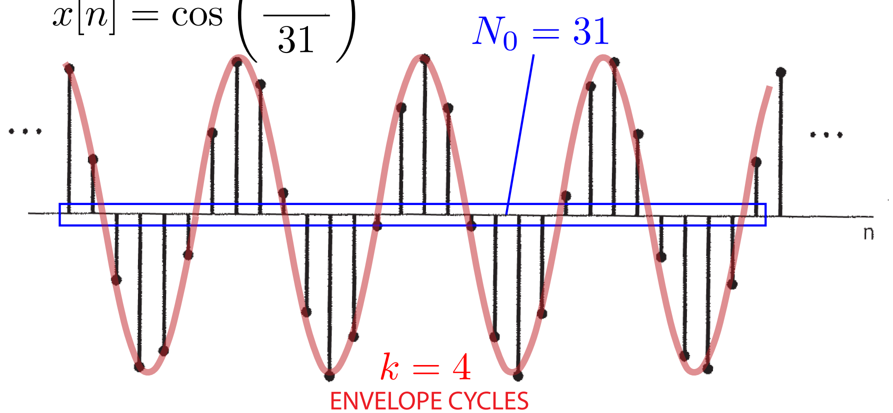
- discrete-time signal (not digital), $\because -A \le x_a(t) \le A$ and $n \in \mathbb{Z}$
- A = amplitude
- ω = frequency in rad/sample
- f = frequency in cycles/sample; note: $\omega = 2\pi f$
- θ = phase in rad



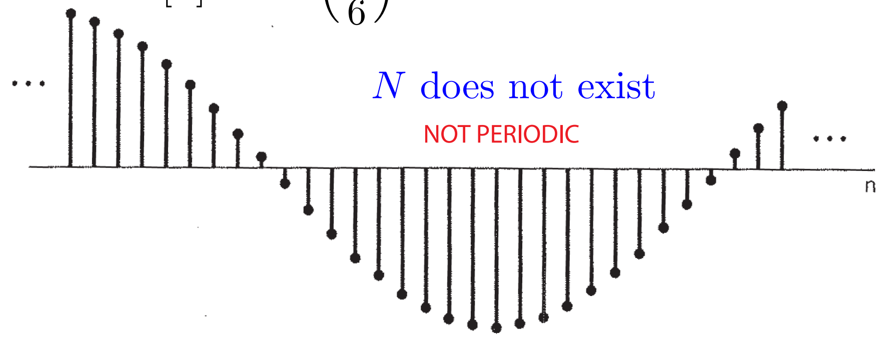
$$x[n] = \cos\left(\frac{\pi n}{6}\right)$$



$$x[n] = \cos\left(\frac{8\pi n}{31}\right)$$



$$x[n] = \cos\left(\frac{n}{6}\right)$$



Uniqueness: Continuous-time

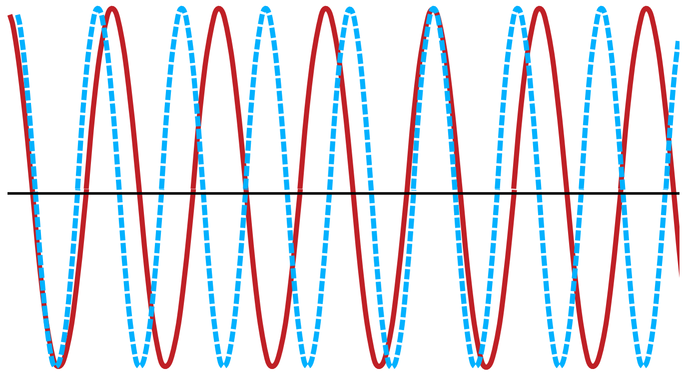
For $F_1 \neq F_2$,

$$A\cos(2\pi F_1 t + \theta) \neq A\cos(2\pi F_2 t + \theta)$$

except at discrete points in time.

Uniqueness: Continuous-time

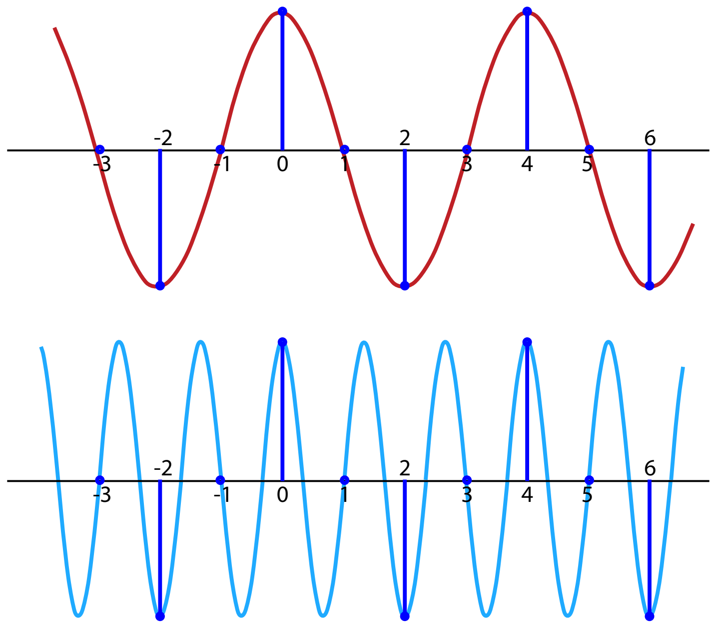
$F_1 \neq F_2$:



Uniqueness: Discrete-time

Let $f_1 = f_0 + k$ where $k \in \mathbb{Z}$,

$$\begin{aligned} x_1(n) &= A\,e^{j(2\pi f_1 n + \theta)} \\ &= A\,e^{j(2\pi (f_0 + k) n + \theta)} \\ &= A\,e^{j(2\pi f_0 n + \theta)} \cdot e^{j(2\pi k n)} \\ &= x_0(n) \cdot 1 = x_0(n) \end{aligned}$$



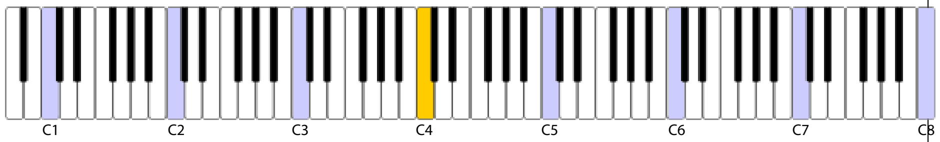
Harmonically Related Complex Exponentials

$$\text{Harmonically related (cts-time)} \quad s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Scientific Designation	Frequency (Hz)	k for $F_0 = 8.176$
C-1	8.176	1
C0	16.352	2
C1	32.703	4
C2	65.406	8
C3	130.813	16
C4	261.626	32
⋮	⋮	
C9	8372.018	1024

Harmonically Related Complex Exponentials

Scientific Designation	Frequency (Hz)	k for $F_0 = 8.176$
C1	32.703	4
C2	65.406	8
C3	130.813	16
C4 (middle C)	261.626	32
C5	523.251	64
C6	1046.502	128
C7	2093.005	256
C8	4186.009	512



Harmonically Related Complex Exponentials

What does the family of harmonically related sinusoids $s_k(t)$ have in common?

$$\text{Harmonically related (cts-time)} \quad s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi (kF_0) t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\begin{aligned} \text{fund. period: } T_{0,k} &= \frac{1}{\text{cyclic frequency}} = \frac{1}{kF_0} \\ \text{period: } T_k &= \text{any integer multiple of } T_0 \\ \text{common period: } T &= k \cdot T_{0,k} = \frac{1}{F_0} \end{aligned}$$

Harmonically Related Complex Exponentials

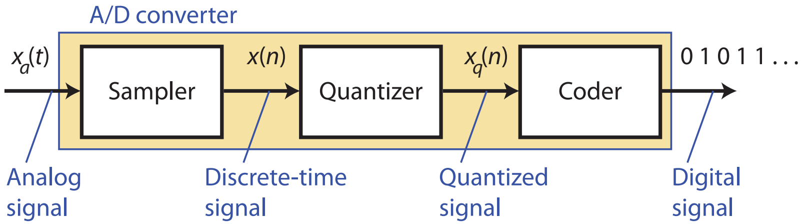
Discrete-time Case:

For periodicity, select $f_0 = \frac{1}{N}$ where $N \in \mathbb{Z}$:

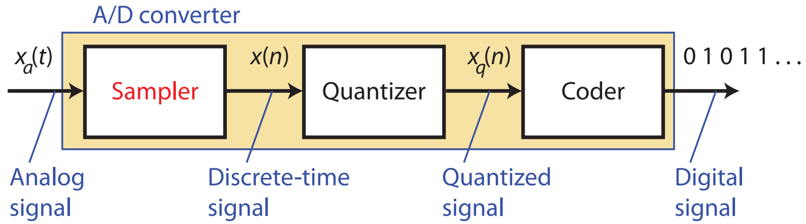
$$\text{Harmonically related (dts-time)} \quad s_k(n) = e^{j2\pi k f_0 n} = e^{j2\pi kn/N}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- There are only N distinct dst-time harmonics: $s_k(n)$, $k = 0, 1, 2, \ldots, N-1$.

Analog-to-Digital Conversion



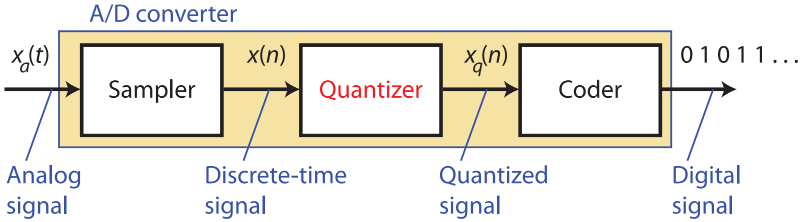
Analog-to-Digital Conversion



Sampling:

- conversion from cts-time to dst-time by taking "samples" at discrete time instants
- E.g., uniform sampling: $x(n) = x_a(nT)$ where T is the sampling period and $n \in \mathbb{Z}$

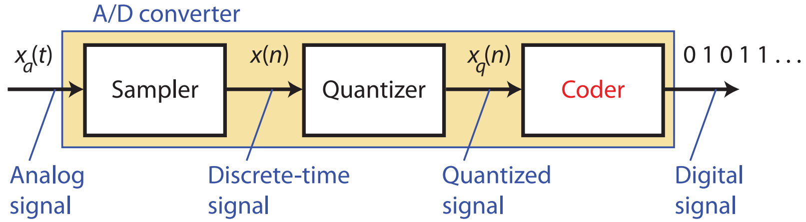
Analog-to-Digital Conversion



Quantization:

- conversion from dst-time cts-valued signal to a dst-time dst-valued signal
- quantization error: $e_q(n) = x_q(n) - x(n)$ for all $n \in \mathbb{Z}$

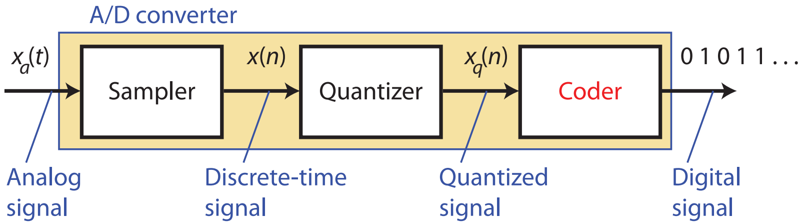
Analog-to-Digital Conversion



Coding:

- representation of each dst-value $x_q(n)$ by a b-bit binary sequence
- e.g., if for any n, $x_q(n) \in \{0, 1, \ldots, 6, 7\}$, then the coder may use the following mapping to code the quantized amplitude:

Analog-to-Digital Conversion



Example coder:

0	000	‖	4	100
1	001	‖	5	101
2	010	‖	6	110
3	011	‖	7	111

Sampling Theorem

If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate

$$F_s > 2F_{max} = 2B$$

then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function

$$g(t) = \frac{\sin(2\pi Bt)}{2\pi Bt}$$

Note: $F_N = 2B = 2F_{max}$ is called the Nyquist rate.

Sampling Theorem

$$\text{Sampling Period} = T = \frac{1}{F_s} = \frac{1}{\text{Sampling Frequency}}$$

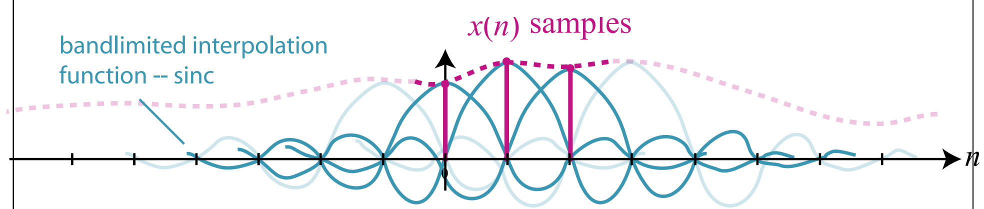
Therefore, given the interpolation relation, $x_a(t)$ can be written as

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g(t - nT)$$

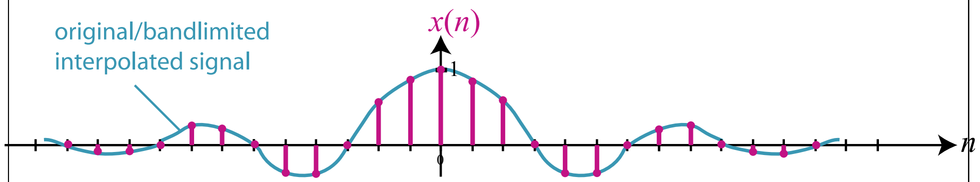
$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \; g(t - nT)$$

where $x_a(nT) = x(n)$; called bandlimited interpolation.

Bandlimited Interpolation

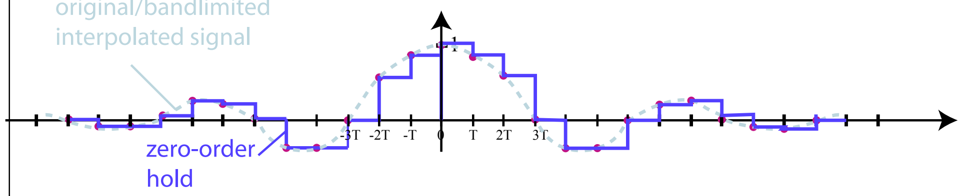


Digital-to-Analog Conversion



- Common interpolation approaches: bandlimited interpolation, zero-order hold, linear interpolation, higher-order interpolation techniques, e.g., using splines
- In practice, “cheap” interpolation along with a smoothing filter is employed.

Digital-to-Analog Conversion



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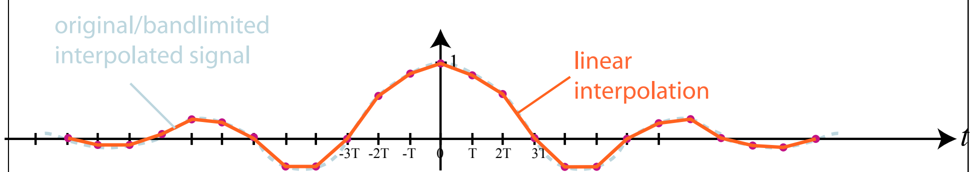
Digital-to-Analog Conversion



- Common interpolation approaches: bandlimited interpolation, zero-order hold, linear interpolation, higher-order interpolation techniques, e.g., using splines
- In practice, “cheap” interpolation along with a smoothing filter is employed.

Elementary Discrete-Time Signals

1. unit sample sequence (a.k.a. Kronecker delta function):

$$\delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

2. unit step signal:

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

3. unit ramp signal:

$$u_r(n) = \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

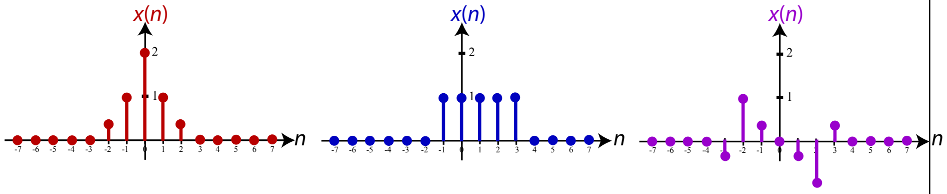
Note:

$$\begin{aligned} \delta(n) &= u(n) - u(n-1) = u_r(n+1) - 2u_r(n) + u_r(n-1) \\ u(n) &= u_r(n+1) - u_r(n) \end{aligned}$$

Signal Symmetry

Even signal: $x(-n) = x(n)$

Odd signal: $x(-n) = -x(n)$



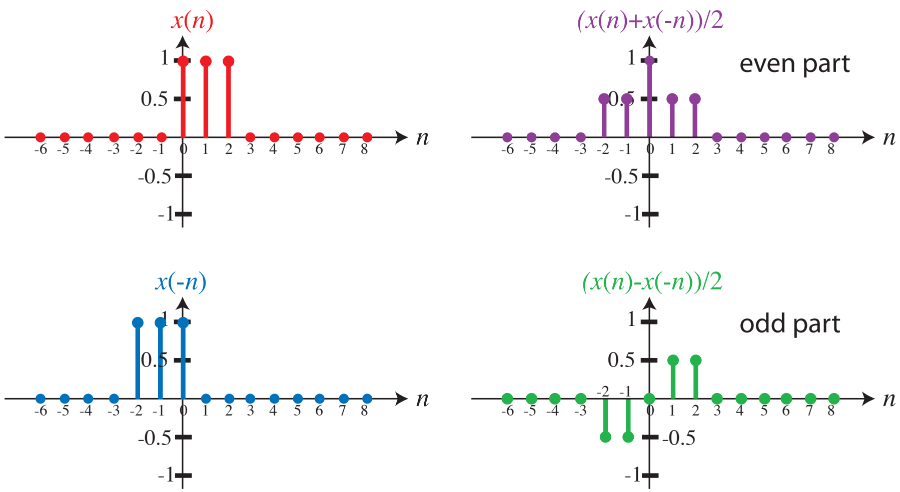
Signal Symmetry

Even signal component: $x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$

Odd signal component: $x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$

Note: $x(n) = x_e(n) + x_o(n)$

Signal Symmetry

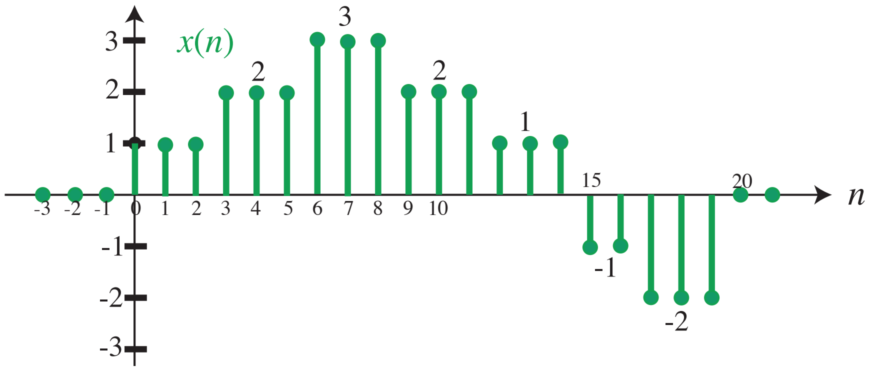


Simple Manipulation of Discrete-Time Signals

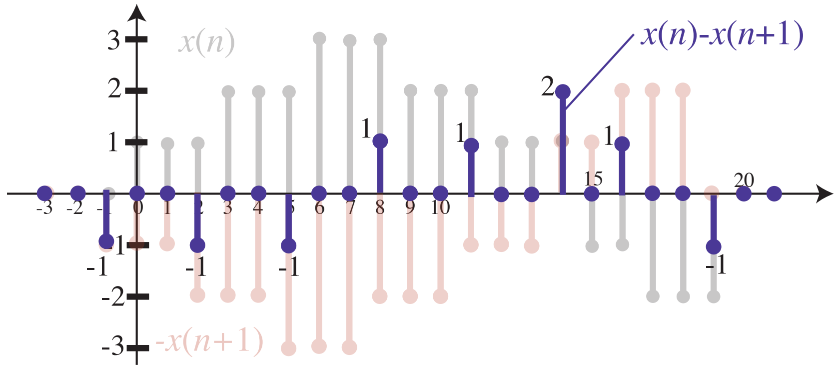
- Transformation of independent variable:
  - time shift: $n \to n - k$, $k \in \mathbb{Z}$
    - Question: what if $k \notin \mathbb{Z}$?
  - time scale: $n \to \alpha n$, $\alpha \in \mathbb{Z}$
    - Question: what if $\alpha \notin \mathbb{Z}$?
- Additional, multiplication and scaling:
  - amplitude scaling: $y(n) = Ax(n)$, $\quad -\infty < n < \infty$
  - sum: $y(n) = x_1(n) + x_2(n)$, $\quad -\infty < n < \infty$
  - product: $y(n) = x_1(n)x_2(n)$, $\quad -\infty < n < \infty$

Simple Manipulation of Discrete-Time Signals I

Find $x(n) - x(n+1)$.



Simple Manipulation of Discrete-Time Signals II



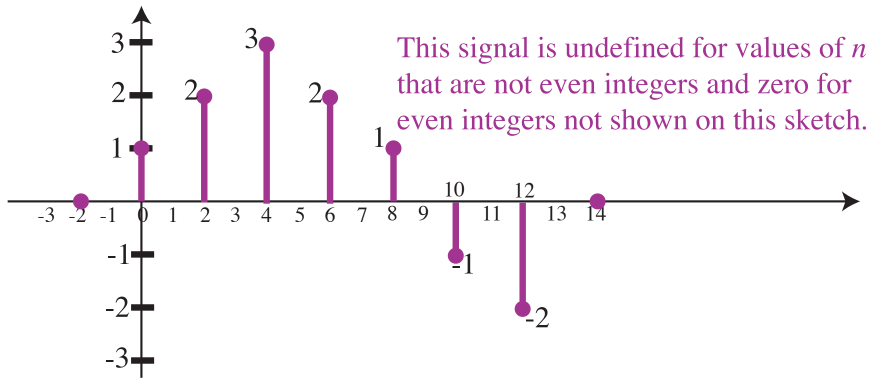
Simple Manipulation of Discrete-Time Signals I

Find $x(\frac{3}{2}n + 1)$.

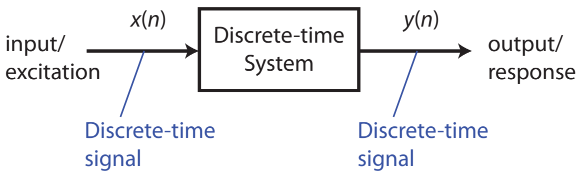
n	$\frac{3n}{2}+1$	$x(\frac{3n}{2}+1)$
< -1	$< -\frac{1}{2}$	0 if $\frac{3n}{2}+1$ is an integer; undefined otherwise
-1	$-\frac{1}{2}$	undefined
0	1	$x(1) = 1$
1	$\frac{5}{2}$	undefined
2	4	$x(4) = 2$
3	$\frac{11}{2}$	undefined
4	7	$x(7) = 3$
5	$\frac{17}{2}$	undefined
6	10	$x(10) = 2$
7	$\frac{23}{2}$	undefined
8	13	$x(13) = 1$
9	$\frac{29}{2}$	undefined
10	16	$x(16) = -1$
11	$\frac{35}{2}$	undefined
12	19	$x(19) = -2$
> 12	> 19	0 if $\frac{3n}{2}+1$ is an integer; undefined otherwise

Simple Manipulation of Discrete-Time Signals

Graph of $x(\frac{3}{2}n + 1)$.



Input-Output Description of Dst-Time Systems



- Input-output description (exact structure of system is unknown or ignored):

$$y(n) = \mathcal{T}[x(n)]$$

- "black box" representation:

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

Classification of Discrete-Time Systems

Common System Properties:

static	vs.	dynamic
time-invariant	vs.	time-variant
linear	vs.	nonlinear
causal	vs.	non-causal
stable	vs.	unstable systems
⋮		⋮

Static vs. Dynamic

- Static system (a.k.a. memoryless): the output at time n depends only on the input sample at time n; otherwise the system is said to be dynamic
- a system is static iff (if and only if)

$$y(n) = \mathcal{T}[x(n), n]$$

for every time instant n.

Static vs. Dynamic

- Consider the general system:

$$\begin{aligned} y(n) \quad = \quad & \mathcal{T}[x(n-N), x(n-N+1), \cdots, x(n-1), x(n), x(n+1), \\ & \cdots, x(n+M-1), x(n+M)], \qquad N, M > 0 \end{aligned}$$

  - For $N = M = 0$, $y(n) = \mathcal{T}[x(n)]$, the system is static.
  - For $0 < N, M < \infty$, the system is said to be dynamic with finite memory of duration $N + M + 1$.
  - For either N and/or M equal to infinite, the system is said to have infinite memory.

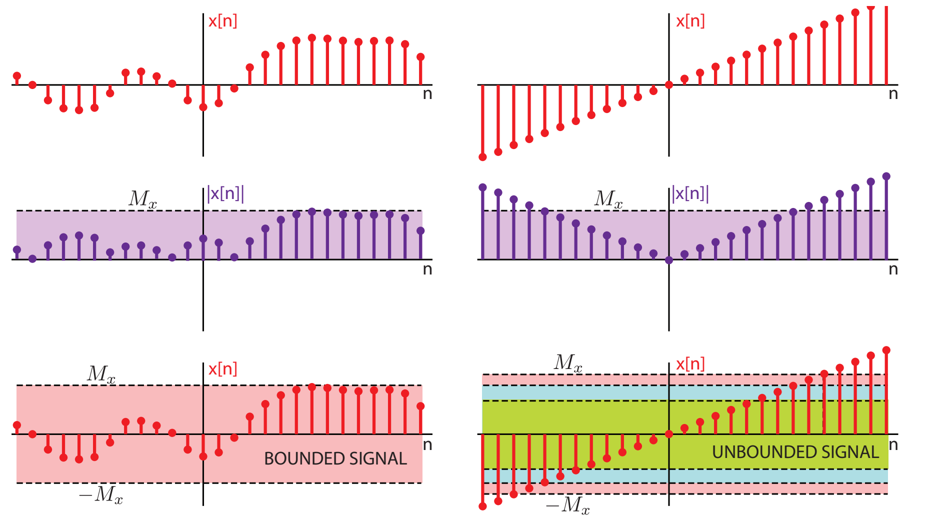
Static vs. Dynamic

Examples: memoryless or not?

- $y(n) = A\,x(n),\ A \neq 0$
- $y(n) = A\,x(n) + B,\ A, B, \neq 0$
- $y(n) = x(n)\cos(\frac{\pi}{25}(n-5))$
- $y(n) = x(-n)$
- $y(n) = x(n+1)$
- $y(n) = \frac{1}{1-x(n+2)}$
- $y(n) = e^{3x(n)}$
- $y(n) = \sum_{k=-\infty}^{n} x(k)$

Ans: Y, Y, Y, N, N, N, Y, N

Discrete-Time Bounded Signals



The Convolution Sum

Recall:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

The Convolution Sum

Let the response of a linear time-invariant (LTI) system to the unit sample input $\delta(n)$ be $h(n)$.

$$\begin{aligned}
\delta(n) &\xrightarrow{\mathcal{T}} h(n) \\
\delta(n-k) &\xrightarrow{\mathcal{T}} h(n-k) \\
\alpha\ \delta(n-k) &\xrightarrow{\mathcal{T}} \alpha\ h(n-k) \\
x(k)\ \delta(n-k) &\xrightarrow{\mathcal{T}} x(k)\ h(n-k) \\
\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) &\xrightarrow{\mathcal{T}} \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
x(n) &\xrightarrow{\mathcal{T}} y(n)
\end{aligned}$$

The Convolution Sum

Therefore,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

for any LTI system.

Finite vs. Infinite Impulse Response

Implementation: Two classes

Finite impulse response (FIR):

$$\left. y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \quad \right\} \text{nonrecursive systems}$$

Infinite impulse response (IIR):

$$\left. y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \quad \right\} \text{recursive systems}$$

System Realization

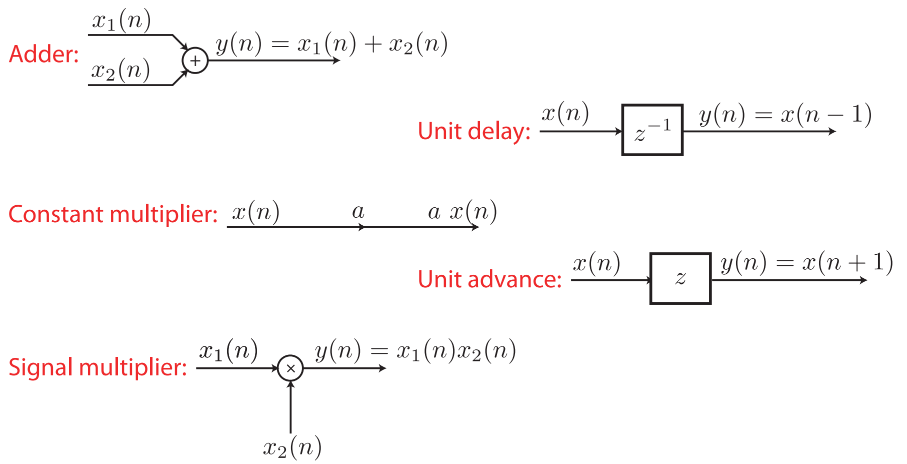
General expression for Nth-order LCCDE:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 \triangleq 1$$

Initial conditions: $y(-1), y(-2), y(-3), \ldots, y(-N)$.

Need: (1) constant scale, (2) addition, (3) delay elements.

Building Block Elements



Direct Form I vs. Direct Form II Realizations

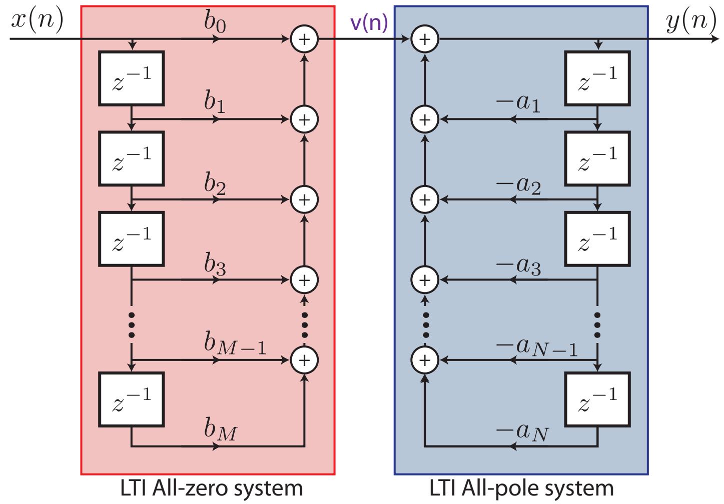
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

is equivalent to the cascade of the following systems:

$$\underbrace{v(n)}_{\text{output 1}} = \sum_{k=0}^{M} b_k \underbrace{x(n-k)}_{\text{input 1}} \qquad \underline{\text{nonrecursive}}$$

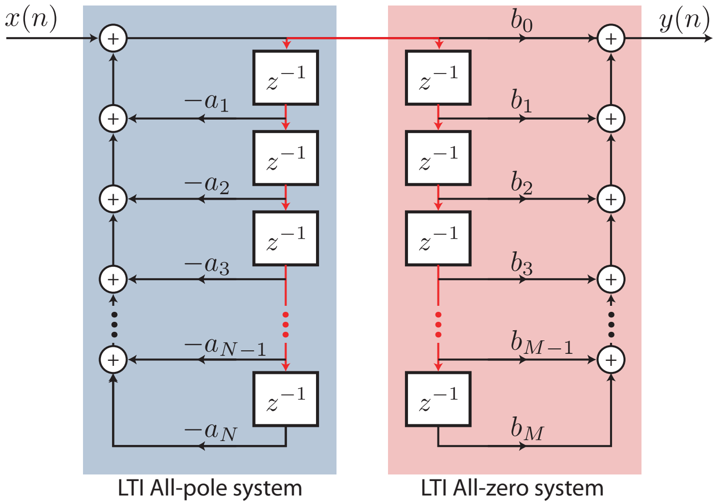
$$\underbrace{y(n)}_{\text{output 2}} = -\sum_{k=1}^{N} a_k y(n-k) + \underbrace{v(n)}_{\text{input 2}} \qquad \underline{\text{recursive}}$$

Direct Form I IIR Filter Implementation



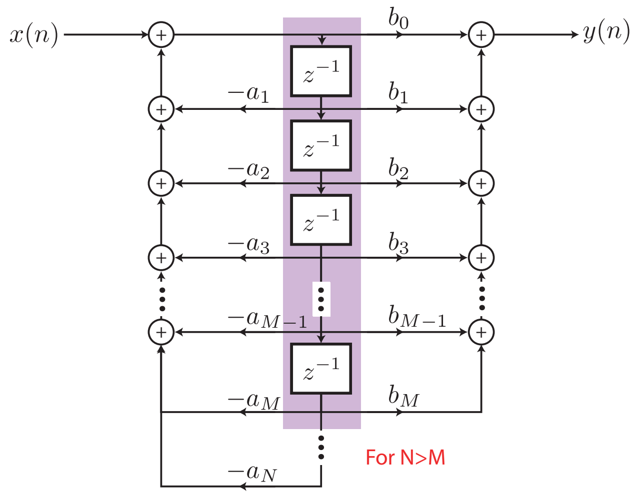
Requires: $M+N+1$ multiplications, $M+N$ additions, $M+N$ memory locations

Direct Form II IIR Filter Implementation



Requires: $M+N+1$ multiplications, $M+N$ additions, $M+N$ memory locations

Direct Form II IIR Filter Implementation



Requires: $M+N+1$ multiplications, $M+N$ additions, $\max(M,N)$ memory locations

The Direct z-Transform

- Direct z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- Notation:

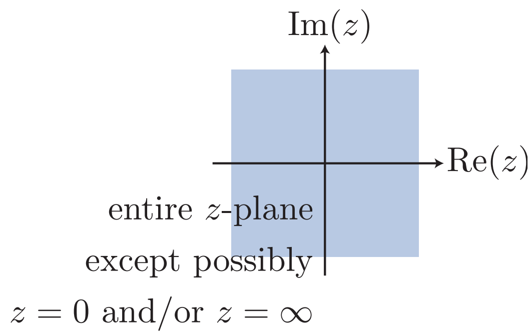
$$X(z) \equiv \mathcal{Z}\{x(n)\}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

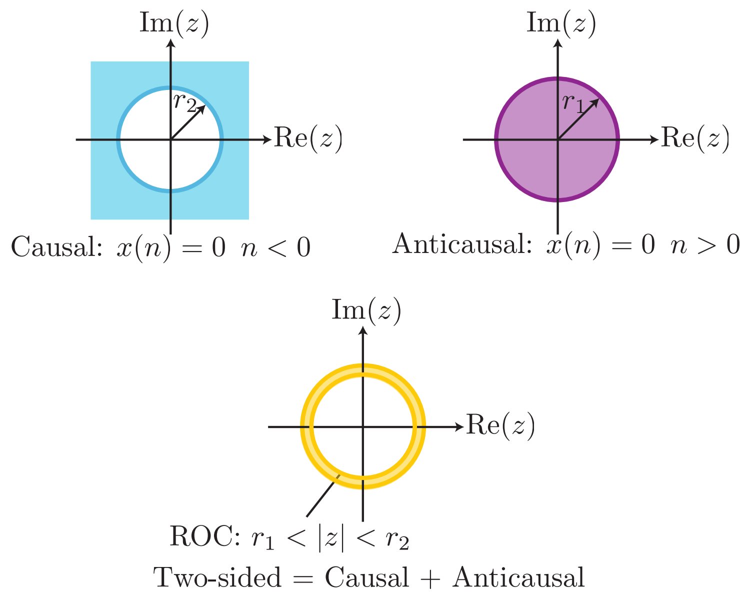
Region of Convergence

- the region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value
- The z-Transform is, therefore, uniquely characterized by:
  1. expression for $X(z)$
  2. ROC of $X(z)$

ROC Families: Finite Duration Signals



ROC Families: Infinite Duration Signals



z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\ x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$

among others ...

Common Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$