Continuous-Time Frequency Analysis

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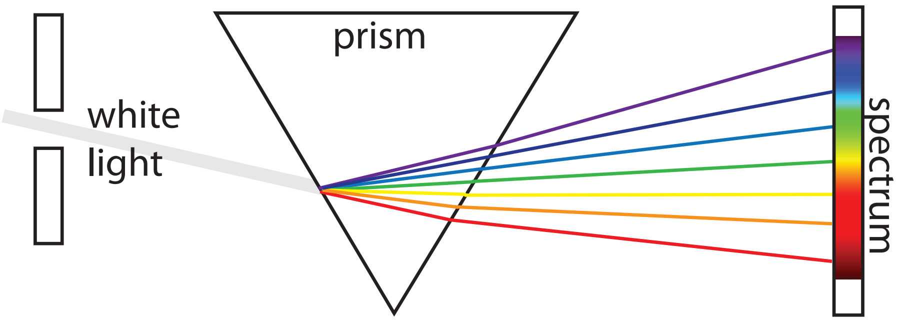
Reference

Reference:

Section 4.1 of

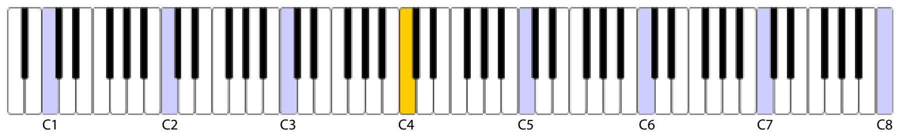
John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Frequency Analysis



Frequency Synthesis

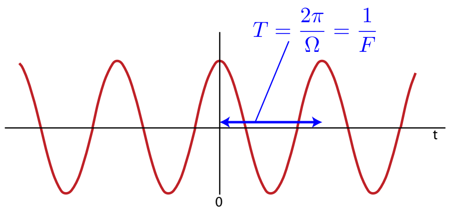
Scientific Designation	Frequency (Hz)	k for $F_0 = 8.176$
C1	32.703	4
C2	65.406	8
C3	130.813	16
C4 (middle C)	261.626	32
C5	523.251	64
C6	1046.502	128
C7	2093.005	256
C8	4186.009	512



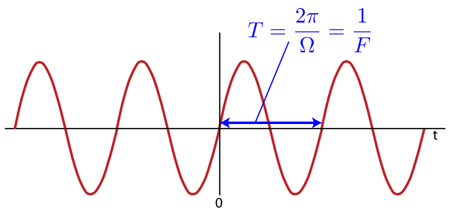
Complex Sinusoids

$e^{j\Omega t} = \cos(\Omega t) + j\sin(\Omega t) \quad \equiv \quad$ complex sinusoid

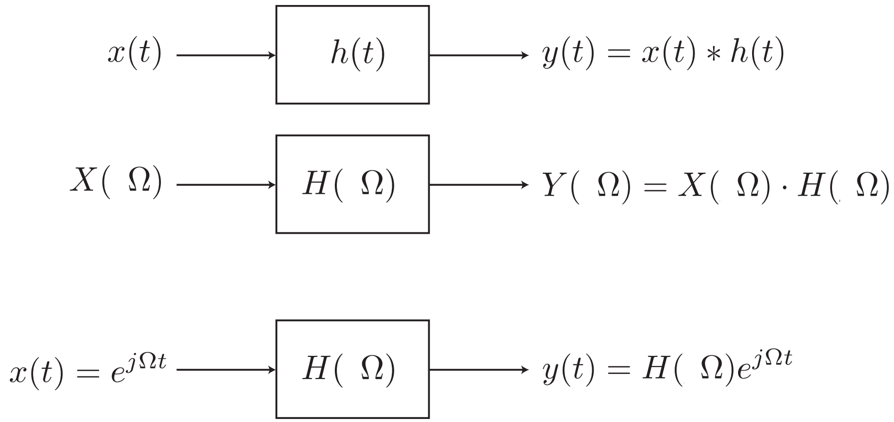
$\cos(2\pi ft)$



$\sin(2\pi ft)$



Complex Sinusoids: as Eigenfunctions



Complex Sinusoids: as Eigenfunctions

$$y(t) = h(t) * e^{j\Omega t} \quad = \quad H(\Omega)e^{j\Omega t}$$

$$\mathbf{A}v \quad = \quad \lambda \mathbf{v}$$

Therefore, the set of functions $\{e^{j\Omega t}\}$ for $\Omega \in \mathbb{R}$ represent eigenfunctions of LTI systems.

The Continuous-Time Fourier Series

(CTFS)

Continuous-Time Fourier Series (CTFS)

For continuous-time periodic signals with period $T_p = \frac{1}{F_0}$

- Synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- Analysis equation:

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

CTFS: Intuition

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- $\{e^{j2\pi k F_0 t}\}$ forms an orthogonal set for $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\begin{aligned} < e^{j2\pi k F_0 t}, e^{j2\pi m F_0 t} > &= \int_{T_p} e^{j2\pi k F_0 t} (e^{j2\pi m F_0 t})^* dt \\ &= \int_{T_p} e^{j2\pi k F_0 t} e^{-j2\pi m F_0 t} dt = \int_0^{T_p} e^{j2\pi (k-m) F_0 t} dt \\ &= \begin{cases} t\big|_0^{T_p} & k = m \\ \frac{e^{j2\pi(k-m)F_0 t}}{j2\pi(k-m)F_0}\Big|_0^{T_p} & k \neq m \end{cases} = \begin{cases} T_p & k = m \\ 0 & k \neq m \end{cases} \end{aligned}$$

CTFS: Intuition

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- Thus, $x(t)$ is being broken down into a series of orthogonal basis functions that are sinusoidal in nature.
- c_k are the coefficients needed to represent $x(t)$ in the basis set $\{e^{j2\pi k F_0 t}\}$.
- There is a decoupling that takes place such that modifying the frequency components of $x(t)$ related to $2\pi k F_0$ will not affect those related to $2\pi m F_0$ for $m \neq k$.

CTFS: Dirichlet Conditions

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

Q: For what conditions is $\tilde{x}(t)$ equal to $x(t)$?

CTFS: Dirichlet Conditions

- **A:** Sufficient conditions are given by Dirichlet conditions:
  1. $x(t)$ has a finite number of discontinuities in any period.
  2. $x(t)$ contains a finite number of maxima and minima during any period.
  3. $x(t)$ is absolutely integrable in any period:

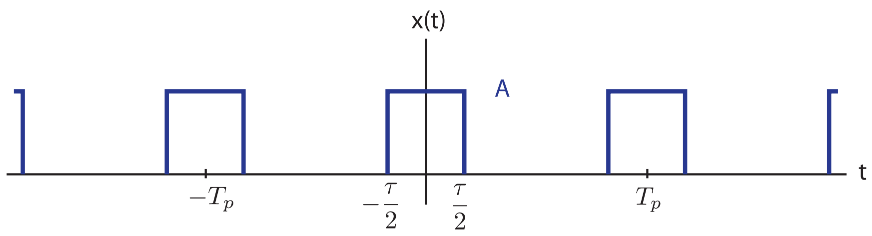
$$\int_{T_p} |x(t)| dt < \infty$$

CTFS: Dirichlet Conditions

- Note: the Dirichlet conditions guarantee equality except at values of t for which $x(t)$ is discontinuous.
- At discontinuities, $\sum_{\forall k} c_k e^{j2\pi k F_0 t}$ convergences to the midpoint of the discontinuity.

CTFS: Example

Find the CTFS of the following periodic square wave:



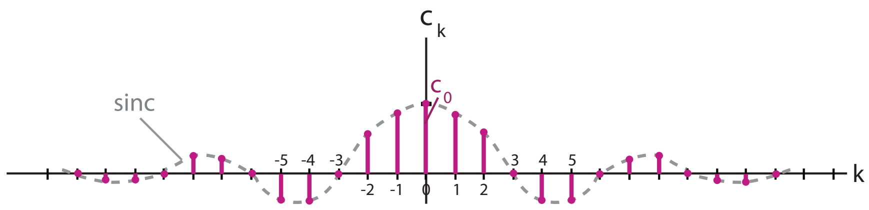
CTFS: Example

$$\begin{aligned}
c_k &= \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi k F_0 t}dt = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)e^{-j2\pi k F_0 t}dt \\
&= \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A \cdot e^{-j2\pi k F_0 t}dt = \frac{A}{T_p}\left.\frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0}\right|_{-\tau/2}^{\tau/2} \\
&= \frac{A}{\pi k T_p \cdot F_0}\left[\frac{e^{-j2\pi k F_0 \tau/2} - e^{+j2\pi k F_0 \tau/2}}{-2j}\right] \\
&= \frac{A}{\pi k \cdot 1}\left[\frac{e^{j2\pi k F_0 \tau/2} - e^{-j2\pi k F_0 \tau/2}}{2j}\right] \\
&= \frac{A\sin(\pi k F_0 \tau)}{\pi k}
\end{aligned}$$

CTFS: Example

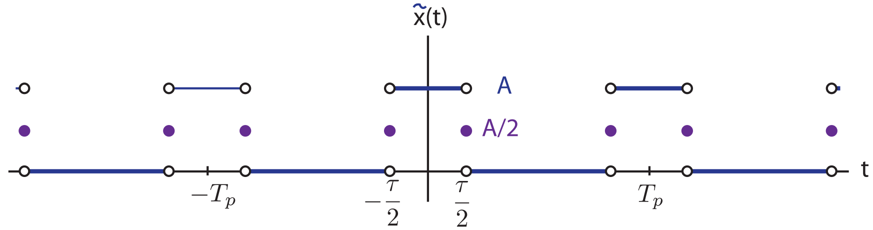
For $\tau = \frac{T_p}{3} = \frac{1}{3F_0}$:

$$c_k = \frac{A\sin(\pi k/3)}{\pi k}$$



CTFS: Example

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} = \sum_{k=-\infty}^{\infty} \frac{A\sin(\pi k/3)}{\pi k} e^{j2\pi k F_0 t}$$



Note: At square wave discontinuities (e.g., $t = \tau/2$),

$$x(\tau/2) = \sum_{k=-\infty}^{\infty} \frac{A\sin(\pi k/3)}{\pi k} e^{j2\pi k F_0(\tau/2)} = \frac{A}{2}$$

The Continuous-Time Fourier Transform

(CTFT)

Continuous-Time Fourier Transform (CTFT)

For continuous-time aperiodic signals

- Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

- Analysis equation:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Continuous-Time Fourier Transform (CTFT)

Cyclic frequency can also be used.

- Synthesis equation:

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

- Analysis equation:

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

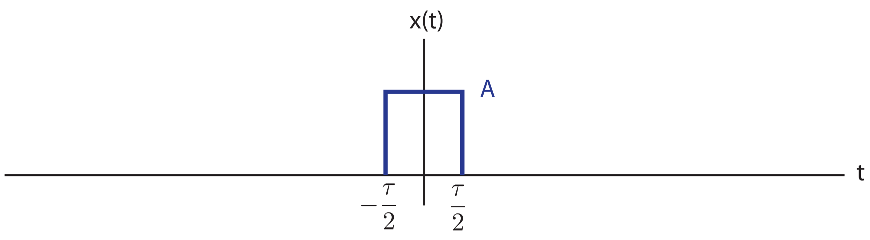
CTFT: Dirichlet Conditions

- Allowing $T_p \to \infty$ in CTFS Dirichlet conditions:
  1. $x(t)$ has a finite number of finite discontinuities.
  2. $x(t)$ has a finite number of maxima and minima.
  3. $x(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

CTFT: Example

Find the CTFS of the following periodic square wave:

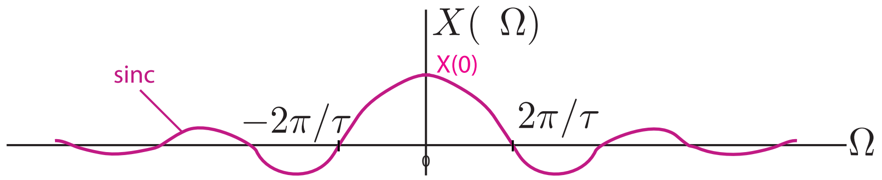


CTFT: Example

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\tau/2}^{\tau/2} A e^{-j\Omega t} dt \\ &= A \left. \frac{e^{-j\Omega t}}{-j\Omega} \right|_{-\tau/2}^{\tau/2} = 2A \frac{\sin(\Omega\tau/2)}{\Omega} \end{aligned}$$

CTFT: Example

$$X(\Omega) = 2A\frac{\sin(\Omega\tau/2)}{\Omega}$$



CTFT: Intuition

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

- Suppose $a(t)$ and $b(t)$ are continuous-time aperiodic signals. We define:

$$< a(t), b(t) > \;=\; \int_{-\infty}^{\infty} a(t)b^*(t)dt$$

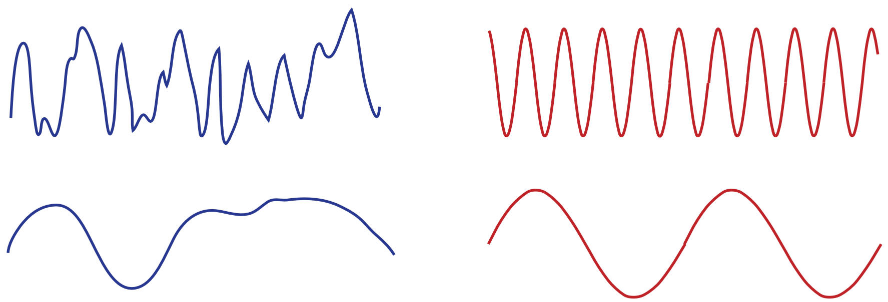
- Therefore, we can interpret $X(\Omega)$ as follows:

$$X(\Omega) = < x(t), e^{j\Omega t} >$$

CTFT: Intuition

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

- We may consider $x(t)$ as a linear combination of $e^{j\Omega t}$ for $\Omega \in \mathbb{R}$.
- The larger $|X(\Omega)|$, the more $x(t)$ will look like a sinusoid with Ω.



CTFT: Duality

$$x(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$
$$X(\Omega) \;=\; \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$\begin{array}{rcl}
x(t) & \stackrel{\mathcal{F}}{\longleftrightarrow} & X(\Omega) \\
\text{rectangle} & \stackrel{\mathcal{F}}{\longleftrightarrow} & \text{sinc} \\
\text{sinc} & \stackrel{\mathcal{F}}{\longleftrightarrow} & \text{rectangle} \\
\text{convolution} & \stackrel{\mathcal{F}}{\longleftrightarrow} & \text{multiplication} \\
\text{multiplication} & \stackrel{\mathcal{F}}{\longleftrightarrow} & \text{convolution}
\end{array}$$

CTFT: Duality

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \\
X(\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt
\end{aligned}
$$

$$
\begin{aligned}
\text{Shape A} &\overset{\mathcal{F}}{\longleftrightarrow} \text{Shape B} \\
\text{Shape B} &\overset{\mathcal{F}}{\longleftrightarrow} \text{Shape A} \\
\text{Operation A} &\overset{\mathcal{F}}{\longleftrightarrow} \text{Operation B} \\
\text{Operation B} &\overset{\mathcal{F}}{\longleftrightarrow} \text{Operation A}
\end{aligned}
$$

CTFT: Magnitude and Phase

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|e^{j\angle X(\Omega)}e^{j\Omega t}d\Omega \\
&= \int_{\infty}^{\infty} |X(\Omega)|e^{j(\Omega t+\angle X(\Omega))}df
\end{aligned}
$$

- $|X(\Omega)|$ dictates the relative presence of the sinusoid of frequency Ω in $x(t)$.
- $\angle X(\Omega)$ dictates the relative alignment of the sinusoid of frequency Ω in $x(t)$.

CTFT: Magnitude and Phase

Q: Which is more important for a given signal?

- Does one component (magnitude or phase) contain more information than another?
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be?

CTFT: Audio Example

- An audio signal is represented by a real function $x(t)$.
- The function $x(-t)$ represents playing the audio signal backwards.
- Since $x(t)$ is real:

$$
\begin{aligned}
X(\Omega) &= X^*(-\Omega) \\
|X(\Omega)| &= |X^*(-\Omega)| = |X(-\Omega)| \quad \text{since } |c| = |c^*| \text{ for } c \in \mathbb{C}
\end{aligned}
$$

- Therefore,

$$|X(\Omega)| = |X(-\Omega)|$$

That is, the CTFT magnitude is even for $x(t)$ real.

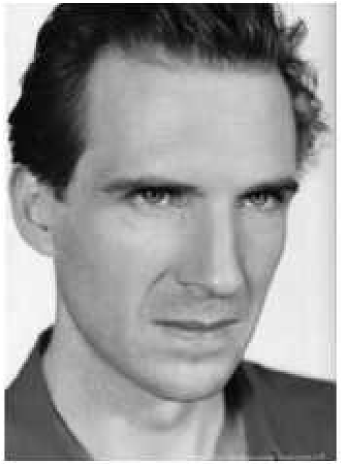
CTFT: Audio Example

- Recall, $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$ $\quad x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-\Omega)$
- Therefore,

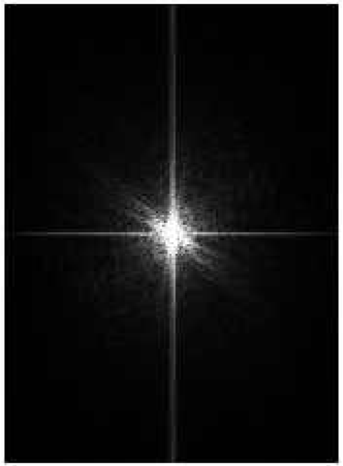
$$\underbrace{|X(\Omega)|}_{\text{spectrum magnitude of } x(t)} = \overbrace{|X(-\Omega)|}^{\text{spectrum magnitude of } x(-t)}$$

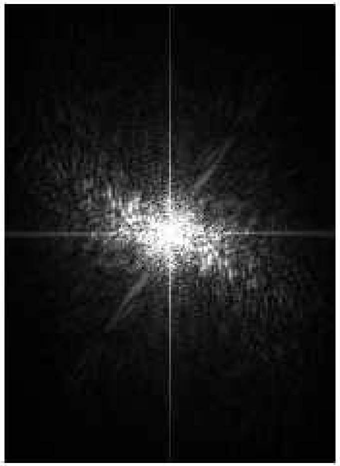
Therefore, the magnitude of the FT of an audio signal played forward and backward is the same!

Example: Still Image $x(t_1, t_2)$

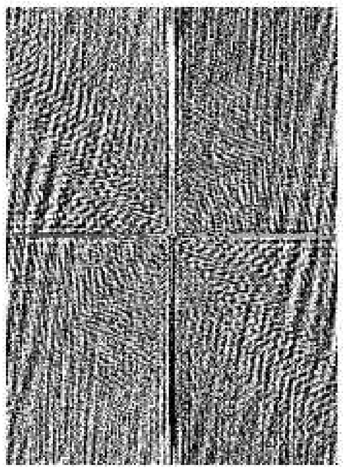


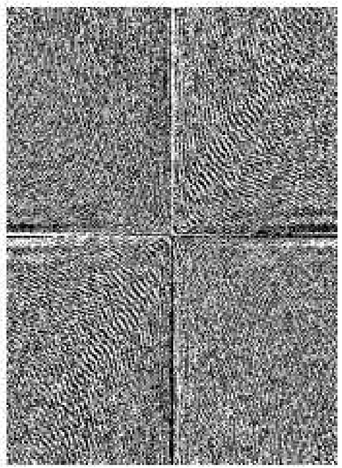
Example: $|X(\Omega_1, \Omega_2)|$

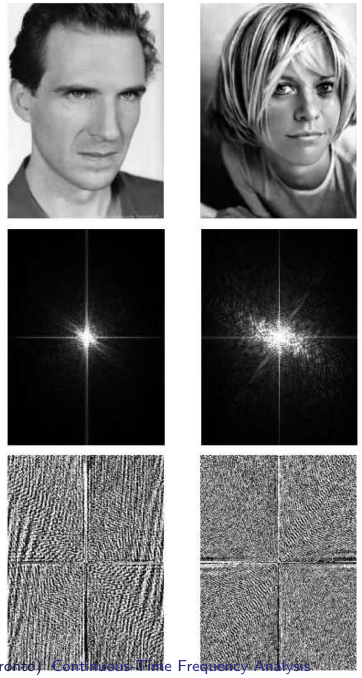




Example: $\angle X(\Omega_1, \Omega_2)$



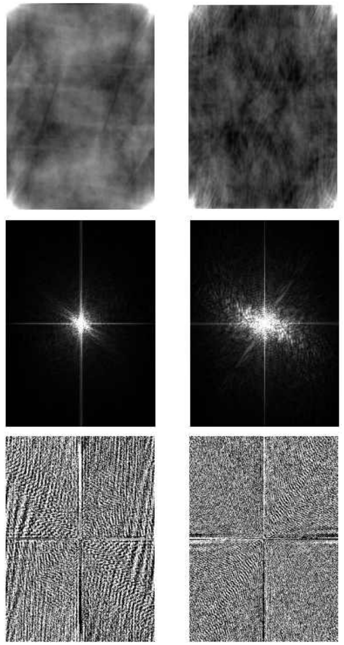




Reconstruction using magnitude only

Top Left Photo: Ralph's magnitude is the same, Phase = 0

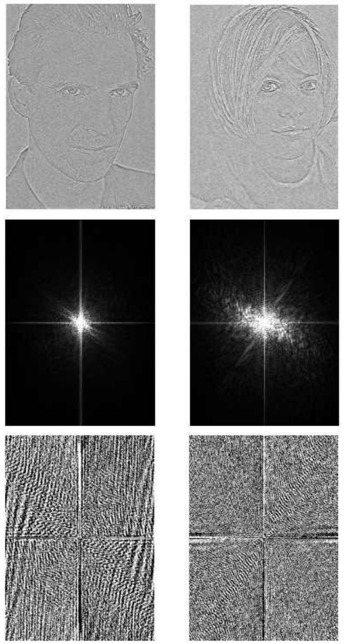
Top Right Photo: Meg's magnitude is the same, Phase = 0



Reconstruction using phase only

Top Left Photo: Ralph's magnitude normalized to one, Phase is the same

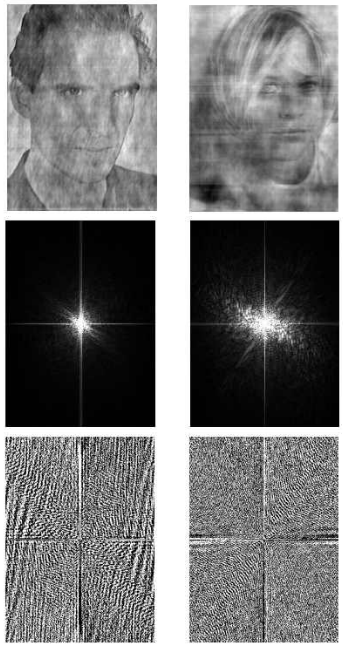
Top Right Photo: Meg's magnitude normalized to one, Phase is the same



Reconstruction swapping magnitude and phase of the images.

Top Left Photo: Ralph's phase + Meg's magnitude

Top Right Photo: Meg's phase + Ralph's magnitude



CTFT: Magnitude and Phase

Q: Which is more important for a given signal? **A: Phase**

- Does one component (magnitude or phase) contain more information than another? **A: Yes, phase**
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be?
  A: Phase

■