Continuous-Time Frequency Analysis

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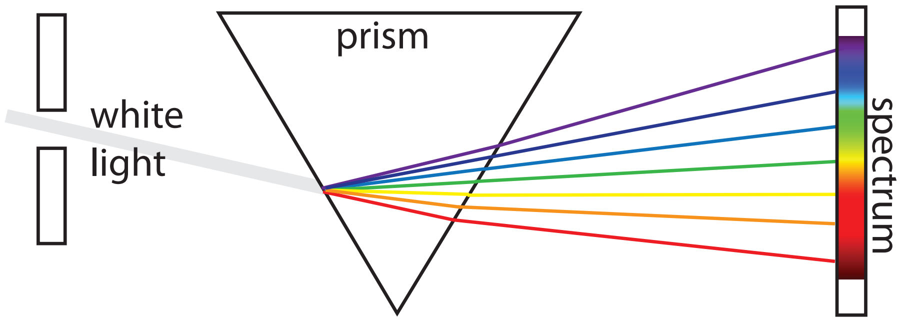
Update

Coverage:

Before Reading Week:	4.1
With Eman:	4.2, 4.3, 4.4, 5.1
Today and Wed:	Brief Rev of 4.1, 4.2, 4.3, 4.4, 5.1 and Sections 5.2, 5.4, 5.5

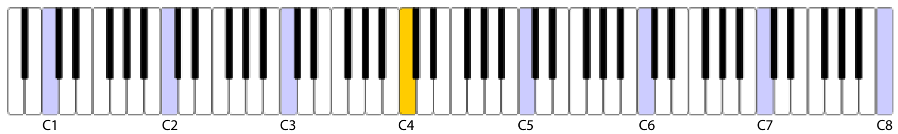
John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Frequency Analysis



Frequency Synthesis

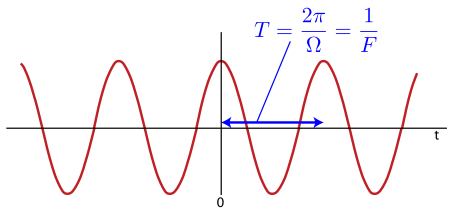
Scientific Designation	Frequency (Hz)	k for $F_0 = 8.176$
C1	32.703	4
C2	65.406	8
C3	130.813	16
C4 (middle C)	261.626	32
C5	523.251	64
C6	1046.502	128
C7	2093.005	256
C8	4186.009	512



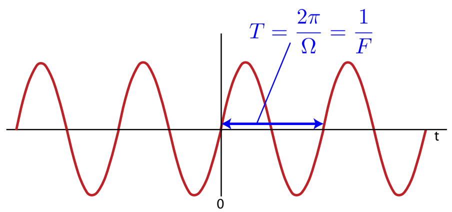
Complex Sinusoids: Continuous-Time

$e^{j\Omega t} = \cos(\Omega t) + j\sin(\Omega t) \quad \equiv \quad$ cts-time complex sinusoid

$\cos(2\pi F t)$



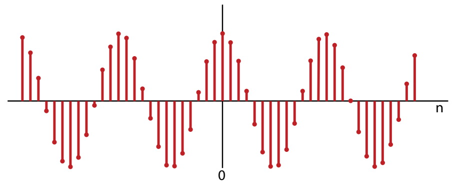
$\sin(2\pi F t)$



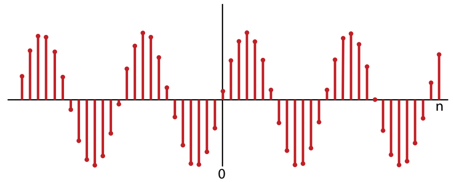
Complex Sinusoids: Discrete-Time

$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n) \quad \equiv \quad$ dst-time complex sinusoid

$\cos(2\pi f n)$



$\sin(2\pi f n)$



Classification of Fourier Pairs

	CTS-TIME	**DST-TIME**
PERIODIC	Continuous-Time Fourier Series (CTFS)	Discrete-Time Fourier Series (DTFS)
APERIODIC	Continuous-Time Fourier Transform (CTFT)	Discrete-Time Fourier Transform (DTFT)

Classification of Fourier Pairs

	CTS-TIME	**DST-TIME**
PER	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$ $c_k = \frac{1}{T_p}\int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$ $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$
APER	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$ $X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{j\omega n} d\omega$ $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Duality

time domain	$\overset{\mathcal{F}}{\longleftrightarrow}$	frequency domain
rectangle	$\overset{\mathcal{F}}{\longleftrightarrow}$	sinc
sinc	$\overset{\mathcal{F}}{\longleftrightarrow}$	rectangle
convolution	$\overset{\mathcal{F}}{\longleftrightarrow}$	multiplication
multiplication	$\overset{\mathcal{F}}{\longleftrightarrow}$	convolution
periodic	$\overset{\mathcal{F}}{\longleftrightarrow}$	discrete
discrete	$\overset{\mathcal{F}}{\longleftrightarrow}$	periodic
aperiodic	$\overset{\mathcal{F}}{\longleftrightarrow}$	continuous
continuous	$\overset{\mathcal{F}}{\longleftrightarrow}$	aperiodic

Duality

	CTS-TIME	DST-TIME
PER	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$ $c_k = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t}dt$	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$ $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$
APER	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$ $X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$ $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

periodic	$\overset{\mathcal{F}}{\longleftrightarrow}$	discrete
discrete	$\overset{\mathcal{F}}{\longleftrightarrow}$	periodic
aperiodic	$\overset{\mathcal{F}}{\longleftrightarrow}$	continuous
continuous	$\overset{\mathcal{F}}{\longleftrightarrow}$	aperiodic

Convergence

	CTS-TIME	DST-TIME
PER	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$ $c_k = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t}dt$	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$ $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$
APER	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$ $X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$ $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

- Convergence issues are prevalent when you have infinite sums and integration.
- Dirichlet conditions provide sufficient conditions for convergence of the Fourier pair at continuous points of the signal.

The Continuous-Time Fourier Series

(CTFS)

Continuous-Time Fourier Series (CTFS)

For continuous-time periodic signals with period $T_p = \frac{1}{F_0}$:

- Synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- Analysis equation:

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

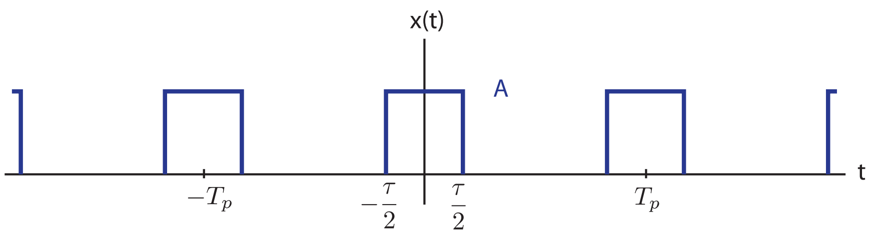
CTFS: Dirichlet Conditions

1. $x(t)$ has a finite number of discontinuities in any period.
2. $x(t)$ contains a finite number of maxima and minima during any period.
3. $x(t)$ is absolutely integrable in any period:

$$\int_{T_p} |x(t)| dt < \infty$$

CTFS: Example

Find the CTFS of the following periodic square wave:



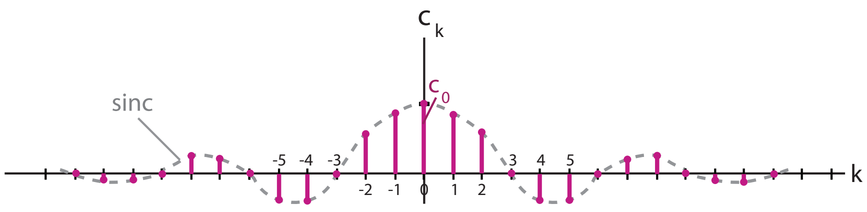
CTFS: Example

$$\begin{aligned}
c_k &= \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi k F_0 t} dt \\
&= \frac{1}{T_p} \int_{-\tau/2}^{\tau/2} A \cdot e^{-j2\pi k F_0 t} dt = \frac{A}{T_p} \left. \frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0} \right|_{-\tau/2}^{\tau/2} \\
&= \frac{A}{\pi k T_p \cdot F_0} \left[\frac{e^{-j2\pi k F_0 \tau/2} - e^{+j2\pi k F_0 \tau/2}}{-2j} \right] \\
&= \frac{A}{\pi k \cdot 1} \left[\frac{e^{j2\pi k F_0 \tau/2} - e^{-j2\pi k F_0 \tau/2}}{2j} \right] \\
&= \frac{A \sin(\pi k F_0 \tau)}{\pi k}
\end{aligned}$$

CTFS: Example

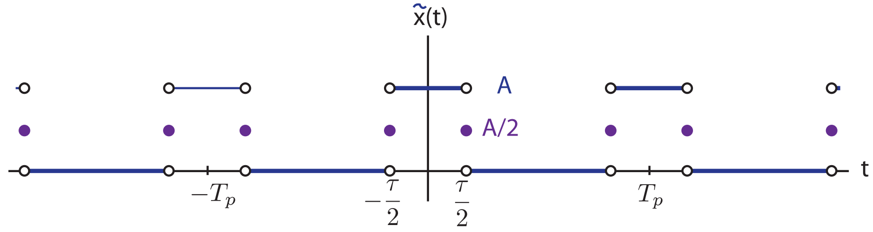
For $\tau = \frac{T_p}{3} = \frac{1}{3F_0}$:

$$c_k = \frac{A \sin(\pi k/3)}{\pi k}$$



CTFS: Example

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} = \sum_{k=-\infty}^{\infty} \frac{A \sin(\pi k/3)}{\pi k} e^{j2\pi k F_0 t}$$



Note: At square wave discontinuities (e.g., $t = \tau/2$),

$$x(\tau/2) = \sum_{k=-\infty}^{\infty} \frac{A \sin(\pi k/3)}{\pi k} e^{j2\pi k F_0 (\tau/2)} = \frac{A}{2}$$

The Continuous-Time Fourier Transform

(CTFT)

Continuous-Time Fourier Transform (CTFT)

For continuous-time aperiodic signals:

- Synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

- Analysis equation:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Continuous-Time Fourier Transform (CTFT)

Cyclic frequency can also be used.

- Synthesis equation:

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF$$

- Analysis equation:

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt$$

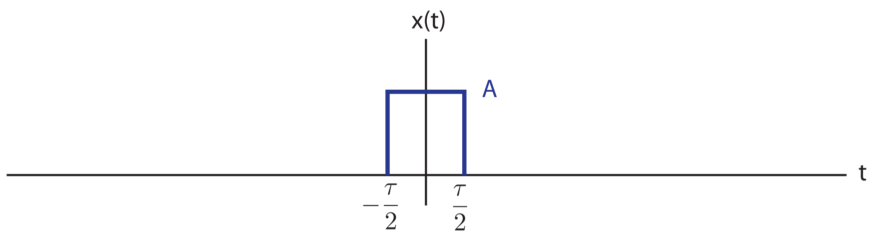
CTFT: Dirichlet Conditions

- Allowing $T_p \to \infty$ in CTFS Dirichlet conditions:
  1. $x(t)$ has a finite number of finite discontinuities.
  2. $x(t)$ has a finite number of maxima and minima.
  3. $x(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

CTFT: Example

Find the CTFS of the following periodic square wave:

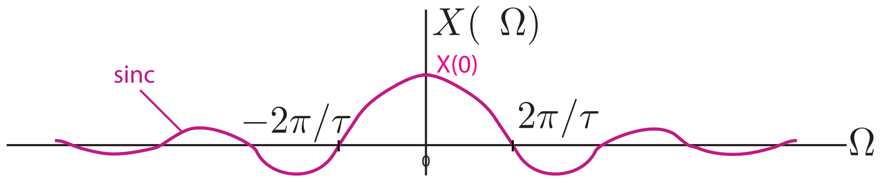


CTFT: Example

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\tau/2}^{\tau/2} A e^{-j\Omega t} dt \\ &= A \left. \frac{e^{-j\Omega t}}{-j\Omega} \right|_{-\tau/2}^{\tau/2} = 2A \frac{\sin(\Omega\tau/2)}{\Omega} \end{aligned}$$

CTFT: Example

$$X(\Omega) = 2A\frac{\sin(\Omega\tau/2)}{\Omega}$$



The Discrete-Time Fourier Series

(DTFS)

Discrete-Time Fourier Series (DTFS)

For discrete-time periodic signals with period N:

- Synthesis equation:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

- Analysis equation:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

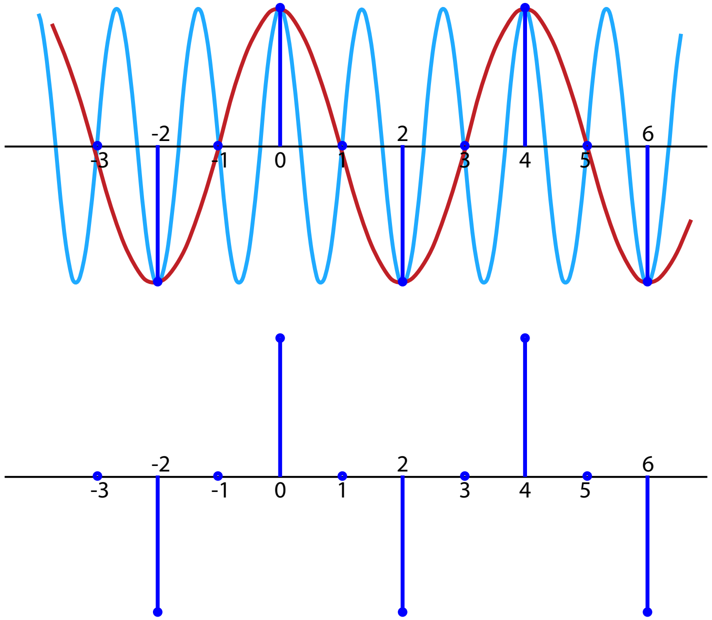
DTFS: Convergence Conditions

None due to finite sums.

DTFS vs. CTFS: Why a finite sum?

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad \text{vs.} \qquad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

- Continuous-time sinusoids are unique for distinct frequencies; $e^{j\frac{2}{3}\pi t} \neq e^{-j\frac{16}{3}\pi t}$.
- Discrete-time sinusoids with cyclic frequencies an integer number apart are the same; $e^{j\frac{2}{3}\pi n} = e^{-j\frac{16}{3}\pi n}$.



DTFS vs. CTFS: Why a finite sum?

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad \text{vs.} \qquad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

Consider

$$s_k(n) = e^{j2\pi kn/N}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- $s_k(n)$ is periodic since $e^{j2\pi kn/N} = e^{j2\pi f_0 n}$ where $f_0 = \frac{k}{N} \equiv$ rational.
- There are only N distinct dst-time harmonics $s_k(n)$: $k = 0, 1, 2, \ldots, N-1$.

Harmonically Related Dst-Time Sinusoids

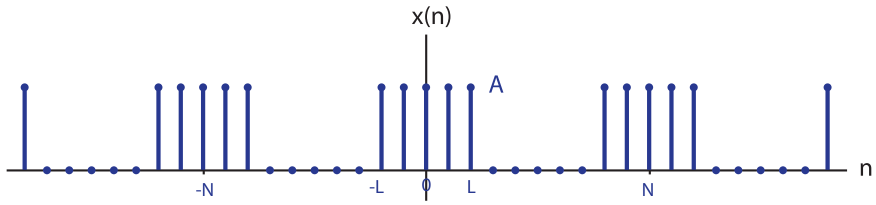
- There are only N distinct dst-time harmonics $s_k(n)$.

k	f_0
$\vdots$	$\vdots$
-2	$\frac{N-2}{N} - 1$
-1	$\frac{N-1}{N} - 1$
0	0
1	$\frac{1}{N}$
2	$\frac{2}{N}$
$\vdots$	$\vdots$
$N-2$	$\frac{N-2}{N}$
$N-1$	$\frac{N-1}{N}$
N	1
$N+1$	$1 + \frac{1}{N}$
$N+2$	$1 + \frac{2}{N}$
$\vdots$	$\vdots$

- $s_k(n)$ harmonics are unique for $k = 0, 1, 2, \ldots, N-1$.
- Outside this range of k, the cyclic frequencies are integers apart thus resulting in the same sinusoids as for $k = 0, 1, 2, \ldots, N-1$.

DTFS: Example

Find the DTFS of the following periodic square wave:



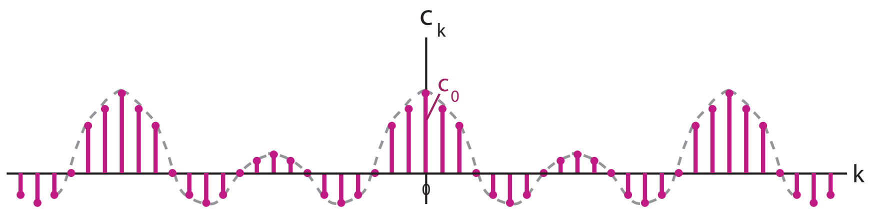
DTFS: Example

$$
\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=\lfloor -N/2\rfloor}^{\lfloor N/2\rfloor} x(n)e^{-j2\pi kn/N} \\
&= \frac{1}{N}\sum_{n=-L}^{L} Ae^{-j2\pi kn/N} = \frac{1}{N}\sum_{m=0}^{2L} Ae^{-j2\pi k(m-L)/N} \\
&= \frac{Ae^{j2\pi kL/N}}{N}\sum_{m=0}^{2L} e^{-j2\pi km/N} = \frac{Ae^{j2\pi kL/N}}{N}\sum_{m=0}^{2L}(e^{-j2\pi k/N})^m \\
&= \frac{Ae^{j2\pi kL/N}}{N}\frac{1-(e^{-j2\pi k/N})^{2L+1}}{1-(e^{-j2\pi k/N})} = \frac{A}{N}\frac{e^{j2\pi kL/N}-e^{-j2\pi kL/N}}{e^{j\pi k/N}-e^{-j\pi k/N}}\frac{2j}{2j} \\
&= \frac{A}{N}\frac{\sin(2\pi kL/N)}{\sin(\pi k/N)}
\end{aligned}
$$

DTFS: Example

For $L = 3$ and $N = 18$:

$$c_k = \frac{A}{18}\frac{\sin(\pi k/3)}{\sin(\pi k/18)}$$



Note: c_k is periodic with period N.

The Discrete-Time Fourier Transform

(DTFT)

Discrete-Time Fourier Transform (DTFT)

For discrete-time aperiodic signals:

- Synthesis equation:

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

- Analysis equation:

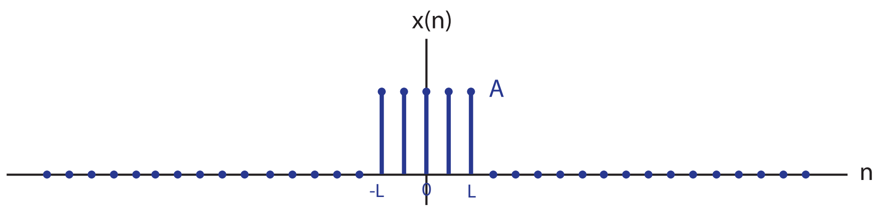
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

DTFT: Convergence Conditions

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

DTFT: Example

Find the DTFT of the following rectangle function:



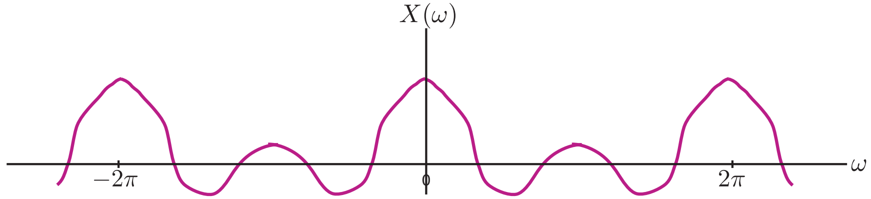
DTFT: Example

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-L}^{L} Ae^{-j\omega n} = A\sum_{n=-L}^{L}(e^{-j\omega})^n \\
&= A\sum_{m=0}^{2L}(e^{-j\omega})^{(m-L)} = Ae^{j\omega L}\sum_{m=0}^{2L}(e^{-j\omega})^m \\
&= Ae^{j\omega L}\frac{1-e^{-j\omega(2L+1)}}{1-e^{-j\omega}}\frac{2j}{2j} \\
&= A\frac{\sin(\omega L)}{\sin(\omega/2)}
\end{aligned}$$

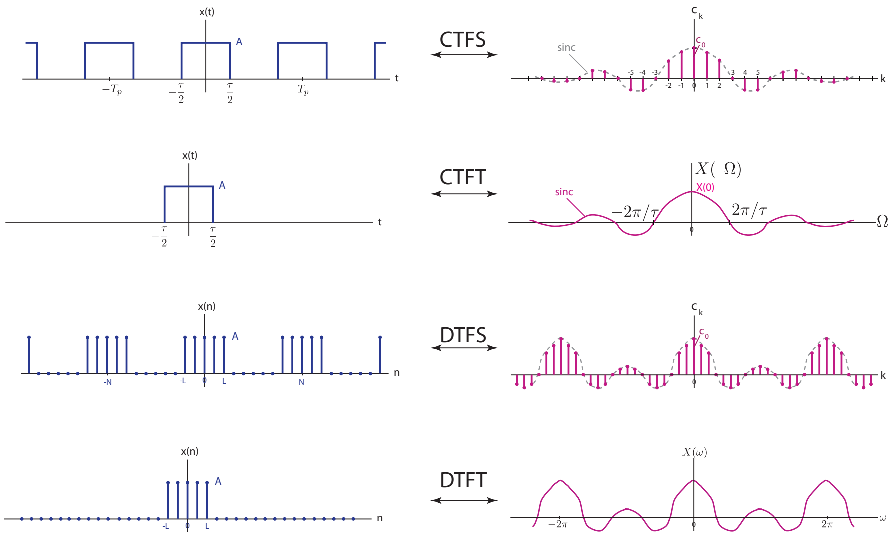
DTFT: Example

For $L = 3$:

$$X(\omega) = A\frac{\sin(\omega L)}{\sin(\omega/2)}$$



Note: $X(\omega)$ is periodic with period 2π.



DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \lvert X(\omega)\rvert^2$

among others ...

DTFT Symmetry Properties

Time Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(-n)$	$X(-\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x(n)$ real	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
$x_e'(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o'(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$