Frequency Domain Analysis of LTI Systems

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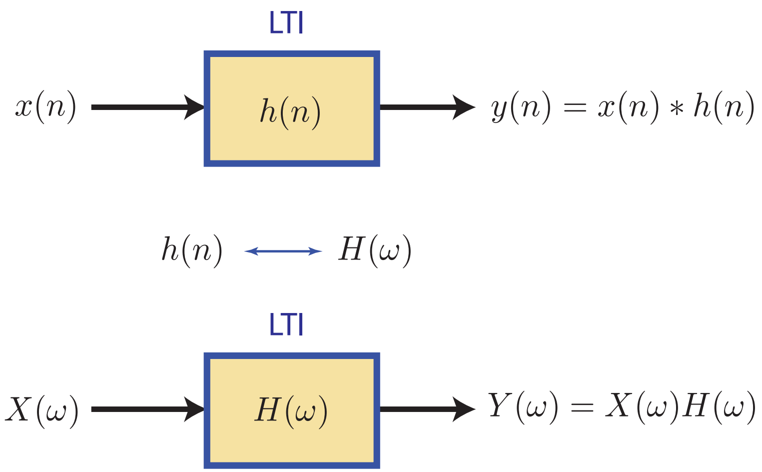
Reference:

Sections 5.1, 5.2, 5.4 and 5.5 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Linear Time-Invariant (LTI) Systems



The Frequency Response Function

- Recall for an LTI system: $y(n) = h(n) * x(n)$.
- Suppose we inject a complex exponential into the LTI system:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ x(n) &= Ae^{j\omega n} \end{aligned}$$

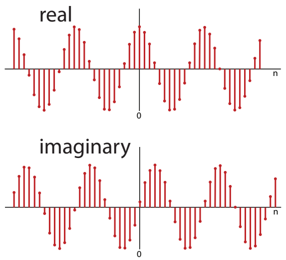
- Note: we consider $x(n)$ to be comprised of a pure frequency of ω rad/s

Linear Time-Invariant (LTI) Systems

LTI

$x(n) = Ae^{j\omega n} \longrightarrow \boxed{h(n)} \longrightarrow$???

$h(n) \longleftrightarrow H(\omega)$



The Frequency Response Function

$$\begin{aligned}
\therefore\ y(n) &= \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} \\
&= \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega n} \cdot e^{-j\omega k} \\
&= Ae^{j\omega n} \cdot \underbrace{\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]}_{\equiv H(\omega) = DTFT\{h(n)\}} \\
&= Ae^{j\omega n} H(\omega)
\end{aligned}$$

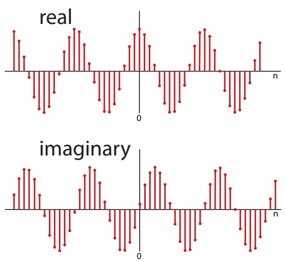
- Thus, $y(n) = H(\omega)x(n)$ when $x(n)$ is a pure frequency.

Linear Time-Invariant (LTI) Systems

LTI

$x(n) = Ae^{j\omega n} \longrightarrow \boxed{h(n)} \longrightarrow$???

$h(n) \longleftrightarrow H(\omega)$

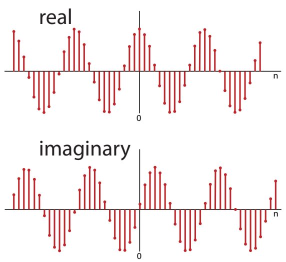


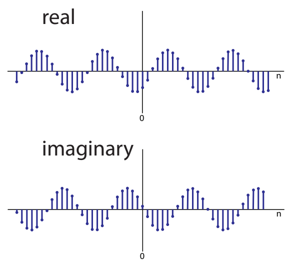
Linear Time-Invariant (LTI) Systems

LTI

$x(n) = Ae^{j\omega n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = Ae^{j\omega n}H(\omega)$

$h(n) \longleftrightarrow H(\omega)$





The Frequency Response Function

Thus, when $x(n)$ is a pure frequency,

$$\begin{aligned} y(n) &= H(\omega)x(n) \\ \text{output} &= \text{scaled input} \\ M \cdot v &= \lambda \cdot v \end{aligned}$$

LTI System Eigenfunction

$$M \cdot v = \lambda \cdot v$$

- Eigenfunction of a system:
  - an input signal that produces an output that differs from the input by a constant (possibly complex) multiplicative factor
  - multiplicative factor is called the eigenvalue

The Frequency Response Function

$$\underbrace{M \cdot v}_{\text{matrix-vector processing}} = \underbrace{\lambda \cdot v}_{\text{scaled input vector}}$$

$$y(n) = \underbrace{h(n) * Ae^{j\omega n}}_{\text{LTI system processing}} = \underbrace{H(\omega)Ae^{j\omega n}}_{\text{scaled input signal}}$$

- Therefore, a signal of the form $Ae^{j\omega n}$ is an eigenfunction of an LTI system.
- The function $H(\omega)$ represents the associated eigenvalue.

LTI System Eigenfunction

Implications:

- An LTI system can only change the amplitude and phase of a sinusoidal signal. It cannot change the frequency.
- An LTI system with inputs comprised of frequencies from set Ω_0 cannot produce an output signal with frequencies in the set Ω_0^c (i.e., the complement set of Ω_0).
- If you inject a signal comprised of frequencies 1 Hz, 4 Hz and 7Hz into a system and you get an output signal comprised of frequencies 1 Hz and 8 Hz, your system is not LTI.

Example: Nonlinear System

Suppose: $x(n) = \cos(2\pi f_1 n + \phi_1) + \cos(2\pi f_2 n + \phi_2)$ is injected into:

$$\begin{aligned}
y(n) &= x^2(n) \qquad \text{nonlinear system} \\
&= (\cos(2\pi f_1 n + \phi_1) + \cos(2\pi f_2 n + \phi_2))^2 \\
&= \cos^2(2\pi f_1 n + \phi_1) + \cos^2(2\pi f_2 n + \phi_2) + 2\cos(2\pi f_1 n + \phi_1)\cos(2\pi f_2 n + \phi_2) \\
&= \left[\frac{1 + \cos(2\pi(2f_1)n + 2\phi_1)}{2}\right] + \left[\frac{1 + \cos(2\pi(2f_2)n + 2\phi_2)}{2}\right] \\
&\quad + \left[\frac{\cos(2\pi(f_1 - f_2)n + (\phi_1 - \phi_2)) + \cos(2\pi(f_1 + f_2)n + (\phi_1 + \phi_2))}{2}\right] \\
&= \underbrace{1}_{\text{freq } 0} + \frac{1}{2}[\cos(2\pi(2f_1)n + 2\phi_1) + \cos(2\pi(2f_2)n + 2\phi_2) \\
&\quad + cos(2\pi(f_1 - f_2)n + (\phi_1 - \phi_2)) + \cos(2\pi(f_1 + f_2)n + (\phi_1 + \phi_2))]
\end{aligned}$$

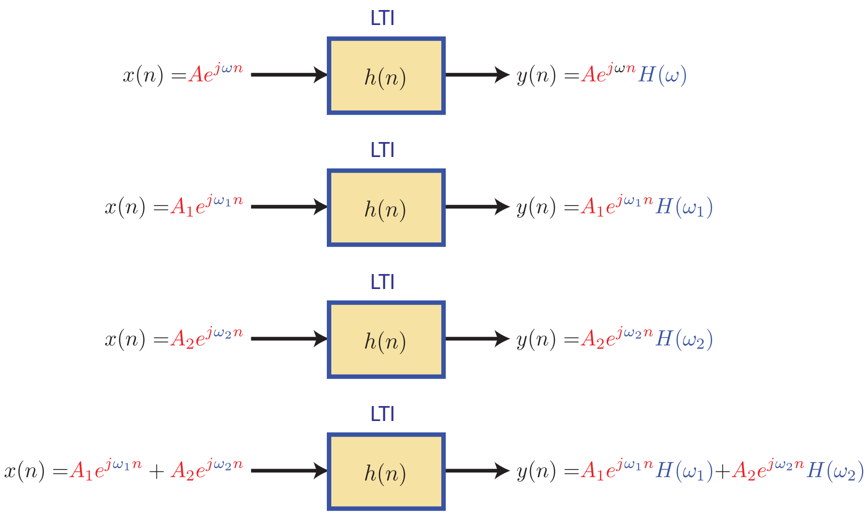
The output frequencies $(0, 2f_1, 2f_2, f_1 - f_2, f_1 + f_2)$ are different from the input frequencies (f_1, f_2).

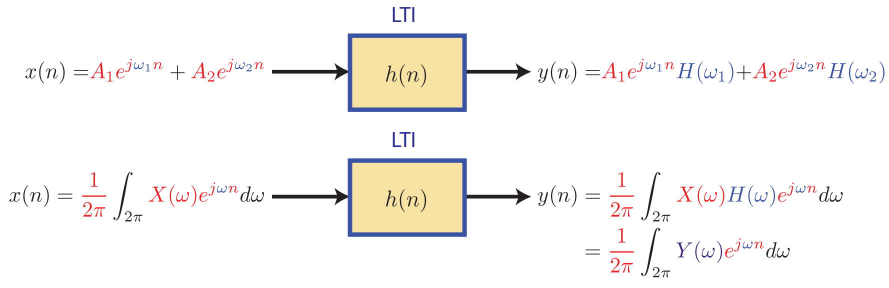
Magnitude and Phase of $H(\omega)$

$$H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$$

$$\begin{aligned}
|H(\omega)| &\equiv \text{system gain for freq } \omega \\
\angle H(\omega) = \Theta(\omega) &\equiv \text{phase shift for freq } \omega
\end{aligned}$$

$$\begin{aligned}
y(n) &= H(\omega)Ae^{j\omega n} \\
&= |H(\omega)|e^{j\Theta(\omega)}Ae^{j\omega n} \\
&= A|H(\omega)|e^{j(\omega n + \Theta(\omega))}
\end{aligned}$$





- An LTI system changes the amplitudes and phase shifts of the individual frequency components within $x(n)$ to produce $y(n)$.
- $H(\omega)$ dictates how frequency ω is changed in the signal.

Example:

Determine the magnitude and phase of $H(\omega)$ for the three-point moving average (MA) system

$$y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right]$$

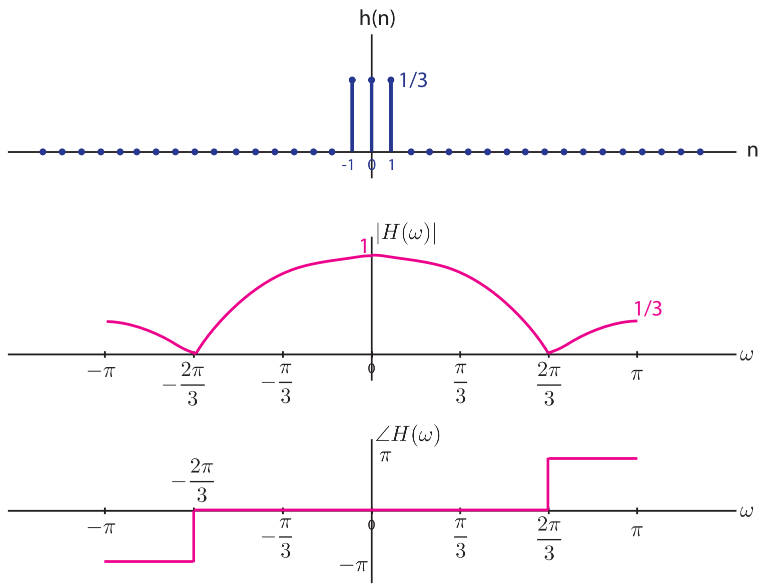
By inspection, $h(n) = \frac{1}{3}\delta(n+1) + \frac{1}{3}\delta(n) + \frac{1}{3}\delta(n-1)$.

$$\begin{aligned} \therefore H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-1}^{1} \frac{1}{3} e^{-j\omega n} \\ &= \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos(\omega)) \end{aligned}$$

Example:

What is the phase of $H(\omega) = \frac{1}{3}(1 + 2\cos(\omega))$?

$$\begin{aligned} |H(\omega)| &= \frac{1}{3}|1 + 2\cos(\omega)| \\ \Theta(\omega) &= \begin{cases} 0 & 0 \le \omega \le \frac{2\pi}{3} \\ \pi & \frac{2\pi}{3} \le \omega < \pi \end{cases} \end{aligned}$$



Frequency Response of LTI Systems

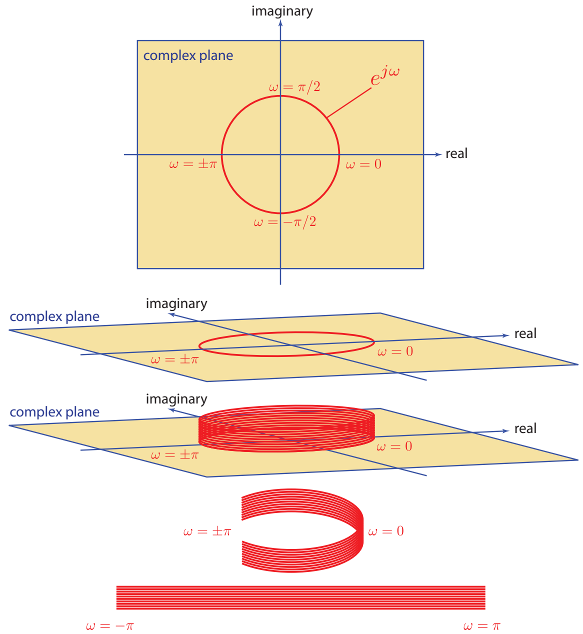
z-Domain		ω-Domain
$H(z)$	$\overset{z=e^{j\omega}}{\Longrightarrow}$	$H(\omega)$
system function	$\overset{z=e^{j\omega}}{\Longrightarrow}$	frequency response
$Y(z) = X(z)H(z)$	$\overset{z=e^{j\omega}}{\Longrightarrow}$	$Y(\omega) = X(\omega)H(\omega)$

Frequency Response of LTI Systems

- If $H(z)$ converges on the unit circle, then we can obtain the frequency response by letting $z = e^{j\omega}$:

$$H(\omega) = H(z)|_{z=e^{j\omega n}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$
$$= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

for rational system functions.



LTI Systems as Frequency-Selective Filters

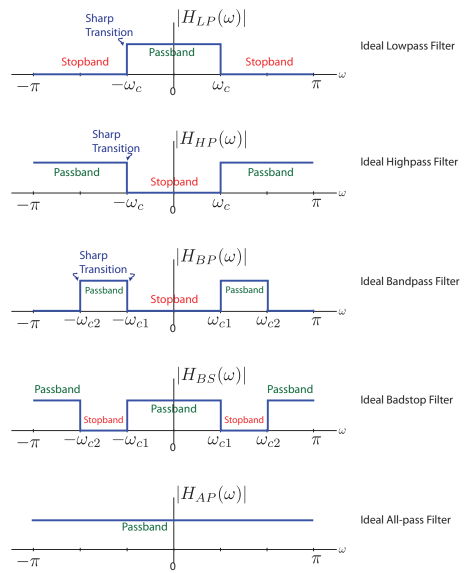
- Filter: device that discriminates, according to some attribute of the input, what passes through it
- For LTI systems, given $Y(\omega) = X(\omega)H(\omega)$
  - $H(\omega)$ acts as a kind of weighting function or spectral shaping function of the different frequency components of the signal

LTI system $\Longleftrightarrow$ Filter

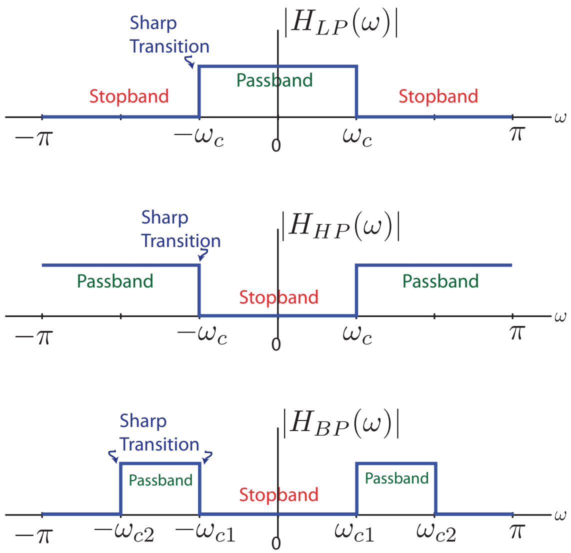
Ideal Filters

Classification:

- lowpass
- highpass
- bandpass
- bandstop
- all-pass



Ideal Filters

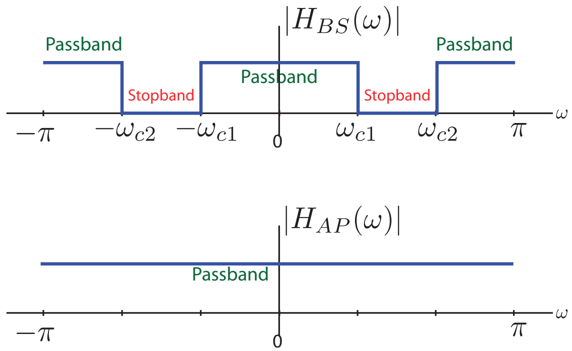


Ideal Lowpass Filter

Ideal Highpass Filter

Ideal Bandpass Filter

Ideal Filters



Ideal Badstop Filter

Ideal All-pass Filter

Ideal Filters

- Common characteristics:
  - flat (typically unity for $C = 1$) frequency response magnitude in passband and zero frequency response in stopband
  - linear phase; for constants C and n_0

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0} & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Ideal Filters

- Suppose $H(\omega) = Ce^{-j\omega n_0}$ for all ω:

$$\begin{aligned} \delta(n) & \stackrel{\mathcal{F}}{\longleftrightarrow} 1 \\ \delta(n - n_0) & \stackrel{\mathcal{F}}{\longleftrightarrow} 1 \cdot e^{-j\omega n_0} \\ C \cdot \delta(n - n_0) & \stackrel{\mathcal{F}}{\longleftrightarrow} C \cdot 1 \cdot e^{-j\omega n_0} = Ce^{-j\omega n_0} \end{aligned}$$

- Therefore, $h(n) = C\delta(n - n_0)$ and:

$$y(n) = x(n) * h(n) = x(n) * C\delta(n - n_0) = Cx(n - n_0)$$

Ideal Filters

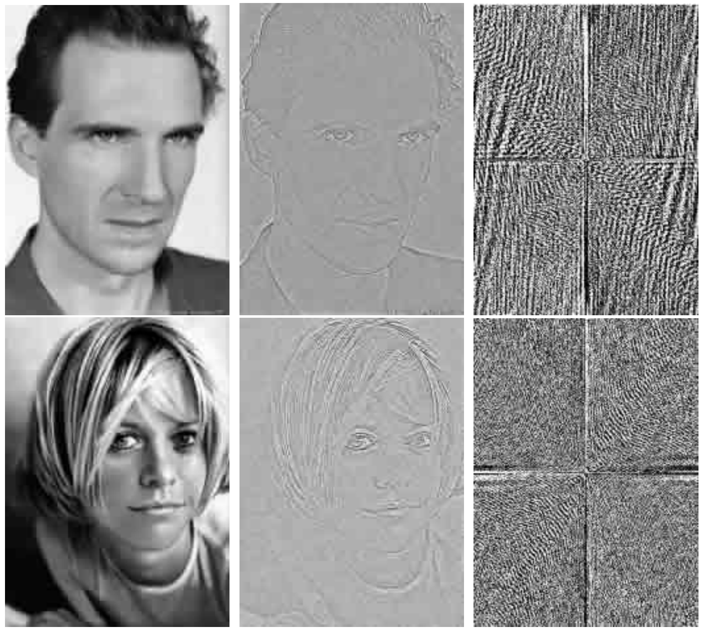
- Therefore for ideal linear phase filters:

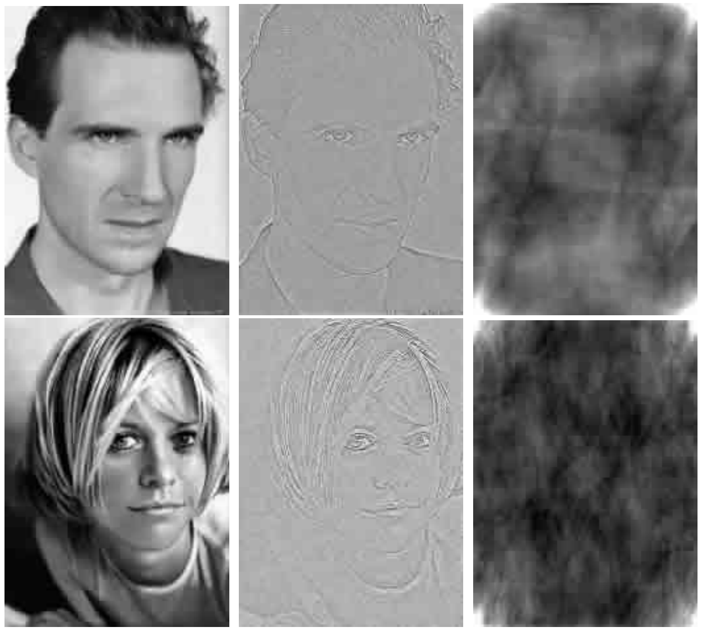
$$H(\omega) = \begin{cases} Ce^{-j\omega n_0} & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

  - signal components in stopband are annihilated
  - signal components in passband are shifted (and scaled by passband gain which is unity (for $C = 1$))

Phase versus Magnitude

What's more important?





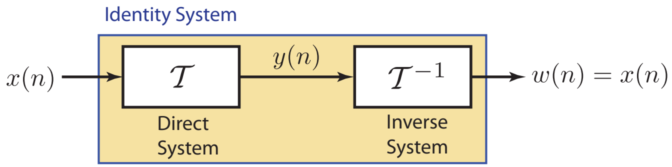
Why Invert?

- There is a fundamental necessity in engineering applications to undo the unwanted processing of a signal.
  - reverse intersymbol interference in data symbols in telecommunications applications to improve error rate; called equalization
  - correct blurring effects in biomedical imaging applications for more accurate diagnosis; called restoration/enhancement
  - increase signal resolution in reflection seismology for improved geologic interpretation; called deconvolution

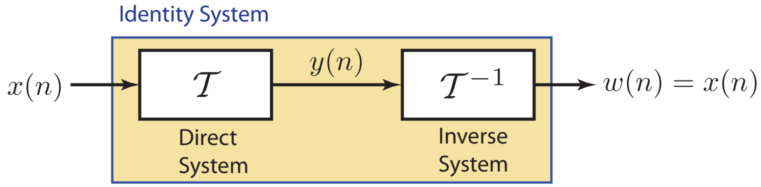
Invertibility of Systems

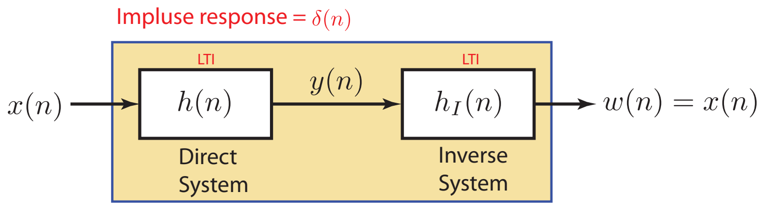
- Invertible system: there is a one-to-one correspondence between its input and output signals
- the one-to-one nature allows the process of reversing the transformation between input and output; suppose

$$
\begin{aligned}
y(n) &= \mathcal{T}[x(n)] \quad \text{where } \mathcal{T} \text{ is one-to-one} \\
w(n) &= \mathcal{T}^{-1}[y(n)] = \mathcal{T}^{-1}[\mathcal{T}[x(n)]] = x(n)
\end{aligned}
$$



Invertibility of LTI Systems





Invertibility of LTI Systems

- Therefore,

$$h(n) * h_I(n) = \delta(n)$$

- For a given $h(n)$, how do we find $h_I(n)$?
- Consider the $z-$domain

$$
\begin{aligned}
H(z)H_I(z) &= 1 \\
H_I(z) &= \frac{1}{H(z)}
\end{aligned}
$$

Invertibility of Rational LTI Systems

- Suppose, $H(z)$ is rational:

$$
\begin{aligned}
H(z) &= \frac{A(z)}{B(z)} \\
H_I(z) &= \frac{B(z)}{A(z)} \\
\text{poles of } H(z) &= \text{zeros of } H_I(z) \\
\text{zeros of } H(z) &= \text{poles of } H_I(z)
\end{aligned}
$$

Example

Determine the inverse system of the system with impulse response $h(n) = (\frac{1}{2})^n u(n)$.

Common Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos(\omega_0 n))u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin(\omega_0 n))u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Example

Determine the inverse system of the system with impulse response $h(n) = (\frac{1}{2})^n u(n)$.

- $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$, ROC: $|z| > \frac{1}{2}$.
- Therefore,

$$H_I(z) = \frac{1}{H(z)} = 1 - \frac{1}{2}z^{-1}$$

- By inspection,

$$h_I(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Common Transform Pairs

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1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
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10	$(a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\, x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\text{ROC}_1 \cap \text{ROC}_2$

among others ...

Another Example

Determine the inverse system of the system with impulse response $h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$.

$$
\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=-\infty}^{\infty} [\delta(n) - \frac{1}{2}\delta(n-1)]z^{-n} \\
&= 1 - \frac{1}{2}z^{-1} \\
H_I(z) &= \frac{1}{1 - \frac{1}{2}z^{-1}}
\end{aligned}
$$

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10	$(a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Another Example

There are two possibilities for inverses:

- Causal + stable inverse ($|z| > \frac{1}{2}$ includes unit circle):

$$h_I(n) = \left(\frac{1}{2}\right)^n u(n)$$

- Anticausal + unstable inverse ($|z| < \frac{1}{2}$ does not include unit circle):

$$h_I(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$$

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