Frequency Analysis of Discrete-Time Signals

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The Fourier Series for Discrete-Time Periodic Signals

Synthesis:

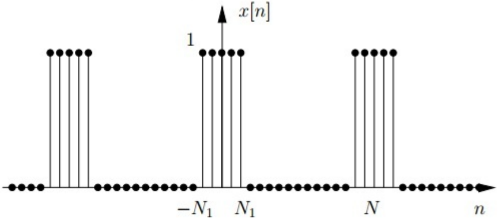
$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

Analysis:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

- Sequence $x(n)$ with period N, $x(n) = x(n+N)$
- $c_k = c_{k+N}$, c_k is a periodic sequence with fundamental period N
- For a sampling frequency F_s; range $0 \leq k \leq N-1$ corresponds to $0 \leq F \leq F_s$

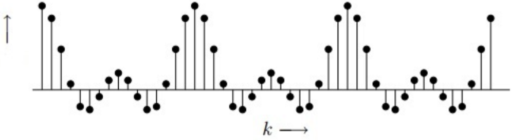
– Discrete–time periodic square wave with period N



– Fourier series coefficients

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\mathrm{e}^{-\mathrm{j}k(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} \mathrm{e}^{-\mathrm{j}k(2\pi/N)n}$$

$$= \begin{cases} \dfrac{1}{N}\dfrac{\sin(2\pi k(N_1+1/2)/N)}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ \dfrac{2N_1+1}{N}, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$



The Fourier Series for Discrete-Time Periodic Signals

Power Density Spectrum of Periodic Signals

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

in terms of Fourier coefficients $\{c_k\}$

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x^*(n) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \left(\sum_{k=0}^{N-1} c_k^* e^{-j2\pi kn/N} \right)$$

$$P_x = \sum_{k=0}^{N-1} c_k^* \left[\frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right]$$

$$P_x = \sum_{k=0}^{N-1} |c_k|^2$$

The Fourier Series for Discrete-Time Periodic Signals

Real signals

If $x(n)$ is a real signal, $x^*(n) = x(n)$, is shown that $c_k^* = c_{-k}$ i.e. spectrum $c_k, k = 1, 2, ..., N/2$ completely describes the signal in the frequency domain.

The Fourier Series for Discrete-Time Aperiodic Signals

Synthesis Equation:

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

Analysis Equation:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

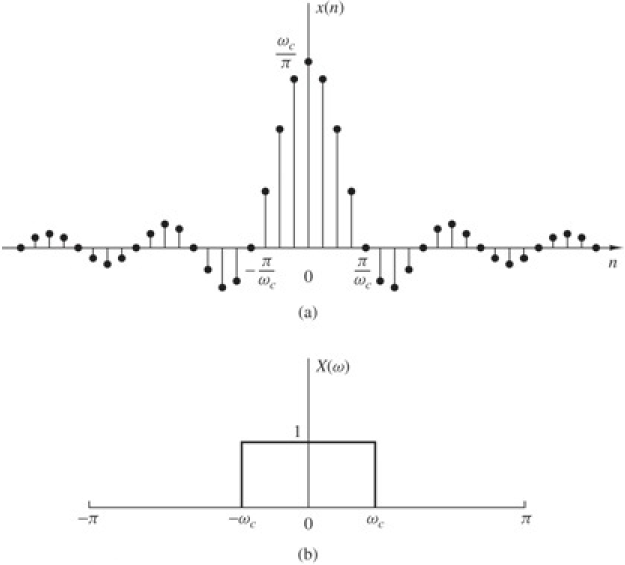


Figure 4.2.4 Fourier transform pair in (4.2.35) and (4.2.36).

The Fourier Series for Discrete-Time Aperiodic Signals

Energy Density Spectrum of Aperiodic Signals

Recall energy of a discrete-time signal $x(n)$

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E_x = \sum_{n=-\infty}^{\infty} x^*(n)x(n) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} d\omega \right]$$

$$E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right] d\omega$$

$$E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

- Energy Density Spectrum of $x(n)$: $S_{xx}(\omega) = |X(\omega)|^2$

The Fourier Series for Discrete-Time Aperiodic Signals

Real signals

Let $x(n)$ be a real signal:

- $|X(-\omega)| = |X(\omega)|$, even symmetry
- $S_{xx}(-\omega) = S_{xx}(\omega)$, even symmetry

$\Rightarrow$ Frequency range of real discrete-time signals can be limited to

$$0 \leq \omega \leq \pi \text{ or } 0 \leq F \leq F_s/2$$

Relationship of the Fourier Transform to the Z-Transform

Recall

z-Transform of sequence $x(n)$:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \text{ ROC: } r_2 \leq |z| \leq r_1$$

Express complex z in polar format:

$$z = re^{j\omega} \Rightarrow X(z)|_{z=re^{j\omega}} = \sum \left[x(n) r^{-n} \right] e^{-j\omega n}$$

If $X(z)$ converges for $|z| = 1$:

$$X(z)|_{z=e^{j\omega}} = X(\omega) = \sum \left[x(n) r^{-n} \right] e^{-j\omega n}$$

- Fourier Transform is viewed as the z-Transform of the sequence evaluated on the unit circle.

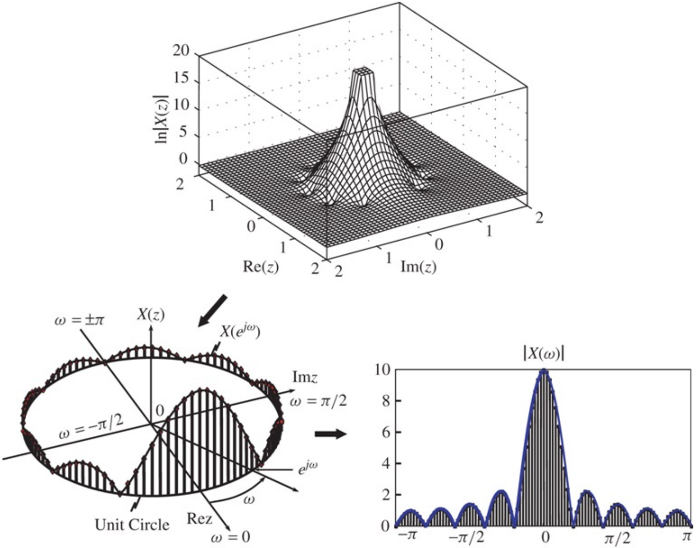


Figure 4.2.9 relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$

Figure courtesy of Proakis and Manolakis, Digital Signal Processing.

Frequency-Domain classification of Signals: Bandwidth

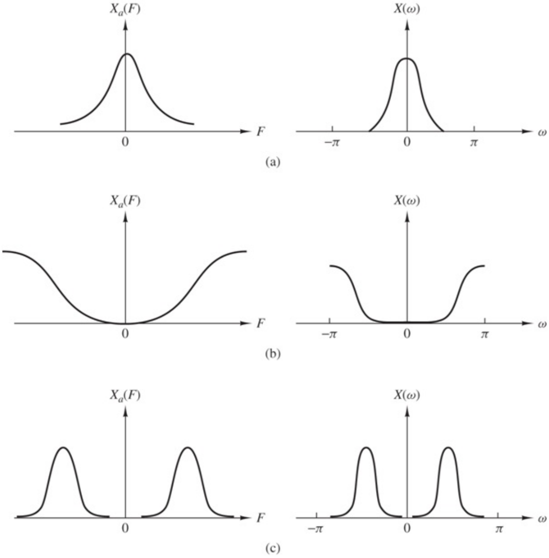


Figure 4.2.10 (a) Low-frequency, (b) high-frequency, and (c) medium-frequency signals.

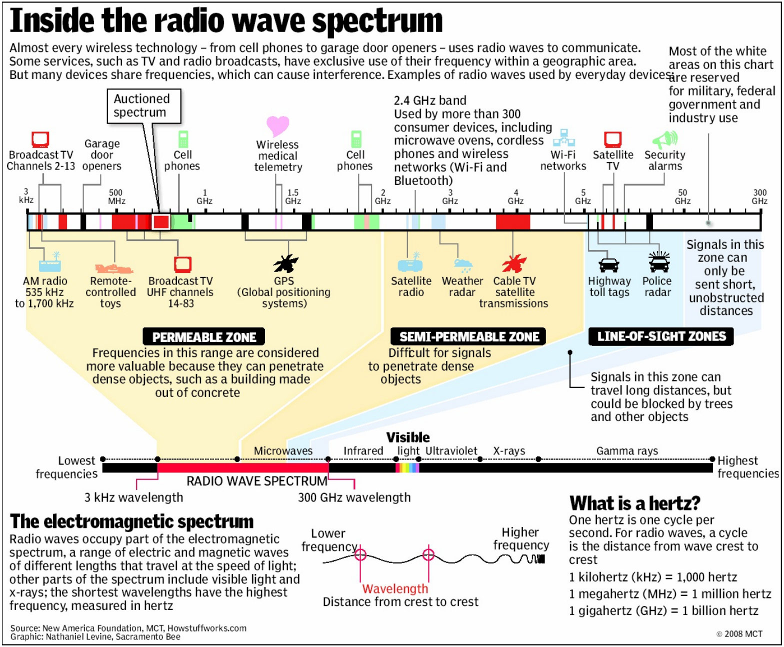
Frequency-Domain classification of Signals: Bandwidth

Broad frequency domain classification

- Low-frequency signal: power/energy density spectrum concentrated around zero.
- High-frequency signal.
- Medium-frequency/ bandpass signal.

Bandwidth

- A quantitative measure that refers to the range of frequencies over which the power/energy density spectrum is concentrated.
- Narrowband, wideband, bandlimited.


Inside the radio wave spectrum
Almost every wireless technology – from cell phones to garage door openers – uses radio waves to communicate. Some services, such as TV and radio broadcasts, have exclusive use of their frequency within a geographic area. But many devices share frequencies, which can cause interference. Examples of radio waves used by everyday devices:
Most of the white areas on this chart are reserved for military, federal government and industry use
Auctioned spectrum
Broadcast TV Channels 2-13
Garage door openers
Cell phones
Wireless medical telemetry
Cell phones
2.4 GHz band Used by more than 300 consumer devices, including microwave ovens, cordless phones and wireless networks (Wi-Fi and Bluetooth)
Wi-Fi networks
Satellite TV
Security alarms
3 kHz
500 MHz
1 GHz
1.5 GHz
2 GHz
3 GHz
4 GHz
5 GHz
50 GHz
300 GHz
AM radio 535 kHz to 1,700 kHz
Remote-controlled toys
Broadcast TV UHF channels 14-83
GPS (Global positioning systems)
Satellite radio
Weather radar
Cable TV satellite transmissions
Highway toll tags
Police radar
Signals in this zone can only be sent short, unobstructed distances
PERMEABLE ZONE
Frequencies in this range are considered more valuable because they can penetrate dense objects, such as a building made out of concrete
SEMI-PERMEABLE ZONE
Difficult for signals to penetrate dense objects
LINE-OF-SIGHT ZONES
Signals in this zone can travel long distances, but could be blocked by trees and other objects
Visible
Microwaves
Infrared
light
Ultraviolet
X-rays
Gamma rays
Lowest frequencies
Highest frequencies
RADIO WAVE SPECTRUM
3 kHz wavelength
300 GHz wavelength
The electromagnetic spectrum
Radio waves occupy part of the electromagnetic spectrum, a range of electric and magnetic waves of different lengths that travel at the speed of light; other parts of the spectrum include visible light and x-rays; the shortest wavelengths have the highest frequency, measured in hertz
Lower frequency
Higher frequency
Wavelength
Distance from crest to crest
What is a hertz?
One hertz is one cycle per second. For radio waves, a cycle is the distance from wave crest to crest
1 kilohertz (kHz) = 1,000 hertz
1 megahertz (MHz) = 1 million hertz
1 gigahertz (GHz) = 1 billion hertz
Source: New America Foundation, MCT, Howstuffworks.com
Graphic: Nathaniel Levine, Sacramento Bee
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The frequency Ranges of Some Natural Signals

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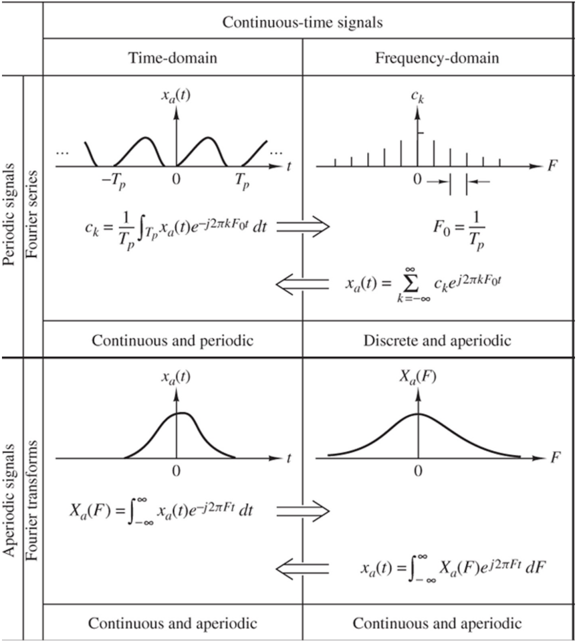
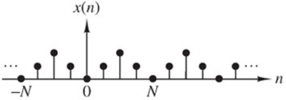
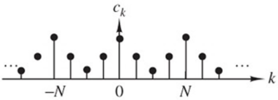
- Biological signals: Speech $100 - 4000Hz$, Sphygmomanogram $0 - 200Hz$
- Seismic Signals: Seismic exploration signals $10 - 100Hz$, Earthquakes $0.01 - 10Hz$
- Electromagnetic Signals: Infrared $3x10^11 - 3x10^14$, Bluetooth $2,4002,483.5MHz$

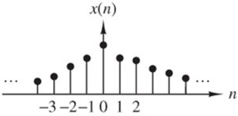
Frequency-Domain and Time-Domain Signal Properties

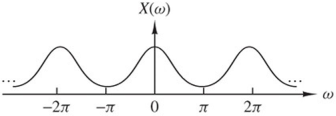
- Continuous-time $\Rightarrow$ aperiodic spectra
- Discrete-time $\Rightarrow$ periodic spectra
- Periodic signals $\Rightarrow$ discrete spectra
- Aperiodic (finite energy) $\Rightarrow$ continuous spectra


Continuous-time signals
Time-domain
Frequency-domain
Periodic signals
Fourier series
$x_a(t)$
$-T_p$ 0 T_p t
c_k
0 F
$c_k = \frac{1}{T_p}\int_{T_p} x_a(t)e^{-j2\pi kF_0 t}\,dt$
$F_0 = \frac{1}{T_p}$
$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$
Continuous and periodic
Discrete and aperiodic
Aperiodic signals
Fourier transforms
$x_a(t)$
0 t
$X_a(F)$
0 F
$X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}\,dt$
$x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}\,dF$
Continuous and aperiodic
Continuous and aperiodic

<table>
<tr><th></th><th></th><th colspan="2">Discrete-time signals</th></tr>
<tr><th></th><th></th><th>Time-domain</th><th>Frequency-domain</th></tr>
<tr><td rowspan="2">Periodic signals</td><td rowspan="2">Fourier series</td><td>

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}$$</td><td>

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$</td></tr>
<tr><td>Discrete and periodic</td><td>Discrete and periodic</td></tr>
<tr><td rowspan="2">Aperiodic signals</td><td rowspan="2">Fourier transforms</td><td>

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$</td><td>

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\, d\omega$$</td></tr>
<tr><td>Discrete and aperiodic</td><td>Continuous and periodic</td></tr>
</table>

Properties of the Fourier Transform for Discrete-Time Signals

Notation

Direct Transform (Analysis)

$$X(\omega) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Inverse Transform (Synthesis)

$$x(n) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

Fourier Transform pair

$$x(n) \xleftrightarrow{\mathrm{F}} X(\omega)$$

Symmetry Properties of the Fourier Transform

Real Signals

- symmetry leads to simpler formulas for direct and inverse Fourier transform.
- If $x(n)$ is real, $x_I(n) = 0$ and using $e^{-j\omega} = \cos\omega - j\sin\omega$
- Spectrum of a Real signal has Hermitian symmetry:

$$X^*(\omega) = X(-\omega) \; X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

- Magnitude has even symmetry $|X(\omega)| = |X(-\omega)|$
- Phase has odd symmetry $\angle X(-\omega) = -\angle X(\omega)$
- Table 4.4 for a summery of symmetry properties.

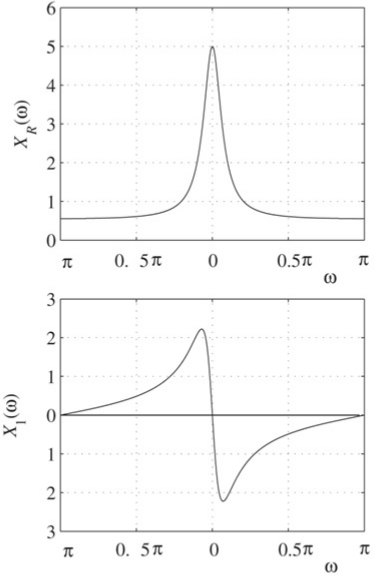


Figure 4.4.3 Graph of $X_R(\omega)$ and $X_I(\omega)$ for the transform in Example 4.4.1.

Fourier Transform Theorems and Properties

Periodicity

- Discrete-time Fourier Transform is periodic with period 2π

$$X(\omega + 2\pi) = X(\omega)$$

Fourier Transform Theorems and Properties

Linearity

If

$$x_1(n) \xleftrightarrow{\mathrm{F}} X_1(\omega)$$

$$x_2(n) \xleftrightarrow{\mathrm{F}} X_2(\omega)$$

Then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{\mathrm{F}} a_1 X_1(\omega) + a_2 X_2(\omega)$$

Fourier Transform Theorems and Properties

Time Shifting

If

$$x(n) \xleftrightarrow{\text{F}} X(\omega)$$

Then

$$x(n-k) \xleftrightarrow{\text{F}} e^{-j\omega k} X(\omega)$$

$$F\{x(n-k)\} = |X(\omega)| e^{j[\angle X(\omega) - \omega k]}$$

- A shift in time domain by k samples, affects only the phase of the signal.

Fourier Transform Theorems and Properties

Frequency Shifting

If

$$x(n) \xleftrightarrow{\text{F}} X(\omega)$$

Then

$$e^{j\omega_0 n}x(n) \xleftrightarrow{\text{F}} X(\omega - \omega_0)$$

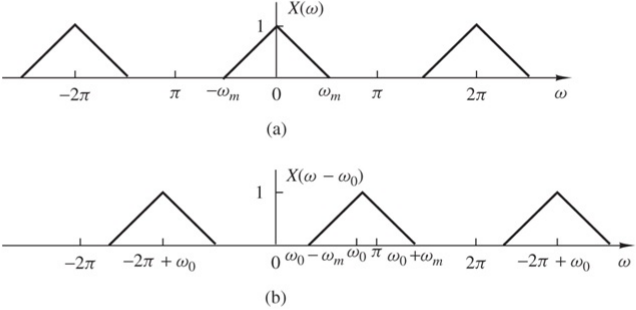


Figure 4.4.8 Illustration of the frequency-shifting property of the Fourier transform ($\omega_0 \leq 2\pi - \omega_m$).

Fourier Transform Theorems and Properties

Convolution

If

$$x_1(n) \xleftrightarrow{\mathsf{F}} X_1(\omega)$$

$$x_2(n) \xleftrightarrow{\mathsf{F}} X_2(\omega)$$

Then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{\mathsf{F}} X(\omega) = X_1(\omega)X_2(\omega)$$

Fourier Transform Theorems and Properties

Modulation

If

$$x(n) \xleftrightarrow{\mathsf{F}} X(\omega)$$

Then

$$x(n) \cos w_0 n \xleftrightarrow{\mathsf{F}} \frac{1}{2} \left[X(\omega + \omega_0) + X(\omega - \omega_0) \right]$$

Proof uses $cos \omega_0 n = \frac{1}{2} \left(e^{j\omega_0 n} + e^{-j\omega_0 n} \right)$

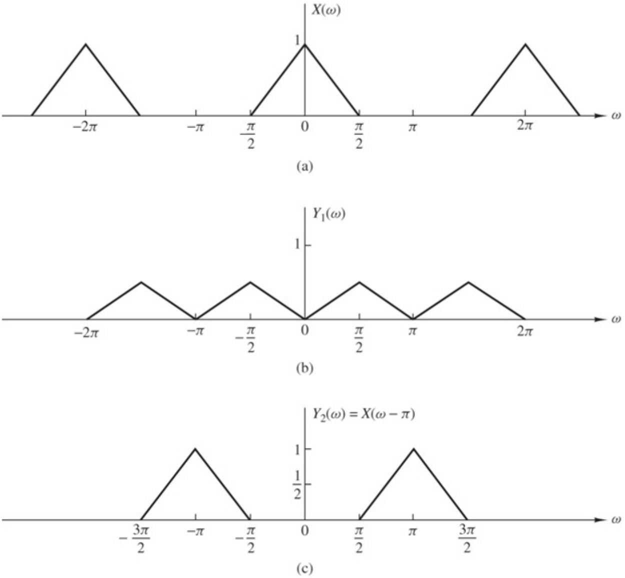


Figure 4.4.9 Graphical representation of the modulation theorem.

Fourier Transform Theorems and Properties

Correlation Theorem

If

$$x_1(n) \xleftrightarrow{\mathsf{F}} X_1(\omega)$$

Then

$$x_2(n) \xleftrightarrow{\mathsf{F}} X_2(\omega)$$

Then

$$r_{x_1 x_2}(m) \xleftrightarrow{\mathsf{F}} S_{x_1 x_2}(\omega) = X_1(\omega) X_2(-\omega)$$

Fourier Transform Theorems and Properties

Correlation Theorem: proof

cross correlation

$$r_{x_1x_2}(n) = \sum_{-\infty}^{\infty} x_1(k)x_2(k-n)$$

multiply by $e^{-j\omega n}$ and then $\sum_{n=-\infty}^{\infty}$

$$S_{x_1x_2}(\omega) = \sum_{n=-\infty}^{\infty} r_{x_1x_2}(n)e^{-j\omega n} =$$

$$\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(k-n)\right]e^{-j\omega n}$$