Frequency-Domain C/S of LTI Systems

- LTI: Linear Time-Invariant system
- $h(n)$, the impulse response of an LTI systems describes the time domain c/s.
- $H(\omega)$, the frequency response describes the frequency-domain c/s.
- $h(n) \xleftrightarrow{F} H(\omega)$
- study: system response to excitation signals that are a weighted linear combination of sinusoids or complex exponentials.

Frequency-Domain C/S of LTI Systems

- Recall the response of an LTI system to input signal $x(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} h(n)x(n-k) \tag{1}$$

Frequency-Domain C/S of LTI Systems

- Excite the system with a complex exponential, i.e. let $x(n) = Ae^{j\omega n}, -\infty < n < \infty, -\pi < \omega < \pi$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \left[Ae^{j\omega(n-k)} \right] = A \left[\sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \right] e^{j\omega n} = Ae^{j\omega n} H(\omega) \tag{2}$$

where,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \tag{3}$$

Frequency-Domain C/S of LTI Systems

Observations

- $y(n)$ is in the form of a complex exponential with same frequency as input, multiplied by a factor.
- The complex exponential signal $x(n)$ is called an eigenfunction of the system.
- $H(\omega)$ evaluated at the frequency of the input is the corresponding eigenvalue of the system.

Frequency-Domain C/S of LTI Systems

Observations, cont.

- As $H(\omega)$ is a Fourier transform, it is periodic with period 2π
- $H(\omega)$ is a complex-valued function, can be expressed in polar form

$$H(\omega) = |H(\omega)| e^{\phi\omega}$$

- $h(k)$ is related to $H(\omega)$ through

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega k} d\omega$$

Frequency-Domain C/S of LTI Systems

Real-valued impulse response

An LTI system with a real-valued impulse response, exhibits symmetry properties as derived in section 4.4.1

- $|H(\omega)| \Rightarrow$ even function of ω
- $\phi(\omega) \Rightarrow$ odd function of ω.
- Consequently, it is enough to know $|H(\omega)|$ and $\phi(\omega)$ for $-\pi \leq \omega \leq \pi$,

Example

- Determine the output sequence of the system with impulse response

$$h(n) = (\frac{1}{2})^n u(n)$$

when the input signal is

$$x(n) = Ae^{j\pi n/2}, -\infty < n < \infty$$

Example, cont.

- Solution: evaluate $H(\omega)$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$H(\omega = \frac{\pi}{2}) = \frac{1}{1+j\frac{1}{2}} = \frac{2}{\sqrt{5}}e^{-j26.6^o}$$

- Thus, output is

$$\begin{aligned} y(n) &= A\left(\frac{2}{\sqrt{5}}e^{-j26.6^o}\right)e^{j\pi n/2} \\ &= \frac{2}{\sqrt{5}}Ae^{j(\pi n/2 - 26.6^o)}, -\infty < n < \infty \end{aligned}$$

- system effect on the input: amplitude scale and phase shift
- change frequency, we change amount of scale change and phase shift.

Example: Moving Average Filter

- Determine the magnitude and phase of $H(\omega)$ for the three-point moving average (MA) system.

$$y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right]$$

and plot these functions for $0 \leq \omega \leq \pi$

Example: Moving Average Filter, cont.

- Solution:

$$h(n) = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$$

consequently,

$$H(\omega) = \sum_n h(k)e^{-j\omega k} = \frac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}(1 + 2\cos\omega)$$

Hence

$$|H(\omega)| = \frac{1}{3}|1 + \cos\omega|$$

$$\phi(\omega) = \begin{cases} 0, 0 \leq \omega \leq \frac{2\pi}{3} \\ \pi, \frac{2\pi}{3} \leq \omega < \pi \end{cases}$$

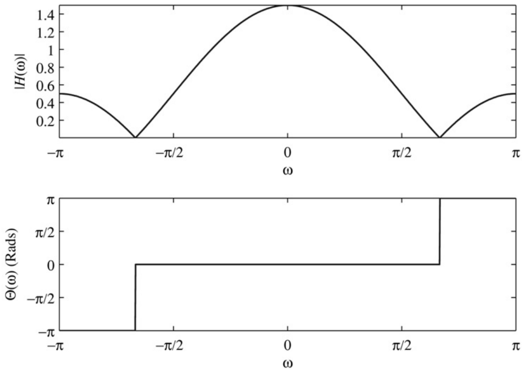

Figure 5.1.1 Magnitude and phase responses for the MA system in Example 5.1.2.

Example: system response to sinusoids

- Determine the response of the system in Example 5.1.1, for the input signal

$$x(n) = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n, -\infty < n < \infty$$

- Recall, the frequency response of the system is

$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Example, cont.

- Solution
- Idea Recognise the frequency of each part of the input signal, and find corresponding system response.
- First term 10, fixed signal $\Rightarrow \omega = 0$

$$H(0) = \frac{1}{1 - \frac{1}{2}} = 2$$

- $5 \sin \frac{\pi}{2} n$ has a frequency $\omega = \pi/2$, thus

$$H(\frac{\pi}{2}) = \frac{2}{\sqrt{5}} e^{-j26.6^o}$$

- $20 \cos \pi n$ has a frequency $\omega = \pi$

$$H(\pi) = \frac{2}{3}$$

Frequency Domain c/s of LTI systems

The General Case

- Most general case: input to the system is an arbitrary linear combination of sinusoids of the form

$$x(n) = \sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i), -\infty < n < \infty$$

Where $\{A_i\}$ and $\{\phi_i\}$ amplitude and phase of corresponding sinusoidal component i .

- System response will be of the form:

$$y(n) = \sum_{i=1}^{L} A_i |H(\omega_i)| \cos\left[\omega_i n + \phi_i + \Theta(\omega_i)\right]$$

Where $|H(\omega_i)|$ and $\Theta(\omega_i)$ are the magnitude and phase imparted by the system to the individual frequency components of the input signal.

Steady-State and Transient Response to Sinusoidal Input Signals

- If excitation signal (exponential or sinusoidal) applied at some finite time instant,e.g. $n = 0$

$$\text{response} = \text{steady-state} + \text{transient}$$

- Example: let

$$y(n) = ay(n-1) + x(n)$$

system response to any $x(n)$ applied at $n = 0$

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k), n \geq 0, y(-1) \text{ initial condition}$$

- Let $x(n)$ be a complex exponential

$$x(n) = Ae^{j\omega n}, n \geq 0$$

Steady-State and Transient Response to Sinusoidal Input Signals, cont.

- We get,

$$
\begin{aligned}
y(n) &= a^{n+1}y(-1) + A\sum_{k=0}^{n} a^k e^{j\omega(n-k)} \\
&= a^{n+1}y(-1) + A\left[\sum_{k=0}^{n}(ae^{-j\omega})^k\right] e^{j\omega n} \\
&= a^{n+1}y(-1) + A\frac{1-a^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}} e^{j\omega n}, n \geq 0 \\
&= a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}} e^{j\omega n} + \frac{A}{1-ae^{-j\omega}} e^{j\omega n}
\end{aligned}
$$

Steady-State and Transient Response to Sinusoidal Input Signals, cont.

- BIBO stable, if $|a| < 1$
- Since $|a| < 1$, the terms containing $a^{n+1} \to 0$, as $n \to \infty$.
- Steady-state response:

$$\begin{aligned} y_{ss}(n) &= \lim_{n\to\infty} y(n) = \frac{A}{1 - ae^{-j\omega}} e^{j\omega n} \\ &= AH(\omega)e^{j\omega n} \end{aligned}$$

- Transient response of the system:

$$y_{tr}(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1 - ae^{-j\omega}} e^{j\omega n}, n \geq 0$$

Steady-State Response to Periodic Input Signals

- Let the input signal $x(n)$ to a stable LTI be periodic with fundamental period N.
- Periodic $\rightarrow -\infty < n < \infty \rightarrow$ total response of the system is the steady-state response.
- Using the Fourier series representation of a periodic signal

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, \quad k = 0, 0, ..., N-1$$

Steady-State Response to Periodic Input Signals, cont.

- Evaluating the system response for each complex exponential

$$x_k(n) = c_k e^{j2\pi kn/N}, \quad k = 0, 1, ..., N-1$$

$$y_k(n) = c_k H\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N}, \quad k = 0, 1, ..., N-1$$

where

$$H(\frac{2\pi k}{N}) = H(\omega)|_{\omega=2\pi kn/N}, \quad k = 0, 1, ..., N-1$$

Steady-State Response to Periodic Input Signals, cont.

- Superposition principle for linear systems:

$$y(n) = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N}, \quad -\infty < n < \infty$$

- LTI system response to a periodic input signal is also periodic with the same period N, with coefficients related by

$$d_k = c_k H\left(\frac{2\pi k}{N}\right), \quad k = 0, 1, ..., N-1$$

Response to Aperiodic Input Signals

- Let $\{x(n)\}$ ne the aperiodic input sequence, $\{y(n)\}$ output sequence, and $\{\mathrm{h(n)}\}$ unit sample response.
- By Convolution theorem

  $Y(\omega) = H(\omega)X(\omega)$

  In polar form, magnitude and phase of the output signal:

  $|Y(\omega)| = |H(\omega)||X(\omega)|$

  $\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$

- Output of an LTI system can NOT contain frequency components that are not contained in the input signal.

Response to Aperiodic Input Signals, cont.

- Energy density spectra of input and output

$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$
$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

- Energy of the output signal

$$E_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{yy}(\omega) d\omega$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 S_{xx}(\omega) d\omega$$

Example

- A linear time-invariant system is characterized by its impulse response

$$h(n) = \left(\frac{1}{2}\right)^2 u(n)$$

- Determine the spectrum and energy density spectrum of the output signal when the system is excited by the signal

$$x(n) = \left(\frac{1}{4}\right)^2 u(n)$$

Example: Solution

- Frequency response of the system

$$H(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^2 e^{-j\omega n}$$
$$= \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

- Similiarly, Fourier transform of the input sequence

$$X(\omega) = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

- Spectrum of the output signal

$$Y(\omega) = H(\omega)X(\omega)$$
$$= \frac{1}{(1 - \frac{1}{2} e^{-j\omega})(1 - \frac{1}{4} e^{-j\omega})}$$

Example, cont.

- energy density spectrum

$$
\begin{aligned}
S_{xx}(\omega) &= |Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2 \\
&= \frac{1}{(\frac{5}{4} - \cos\omega)(\frac{17}{16} - \frac{1}{2}\cos\omega)}
\end{aligned}
$$