DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Frequency Differentiation:	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \lvert X(\omega)\rvert^2$

DTFT Symmetry Properties

Time Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(-n)$	$X(-\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x(n)$ real	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
$x'_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x'_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$

DFT Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(k)$
Periodicity:	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift:	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift:	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate:	$x^*(n)$	$X^*(N-k)$
Circular convolution:	$x_1(n) \otimes x_2(n)$	$X_1(k)X_2(k)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \otimes X_2(k)$
Parseval's theorem:	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Note: The following tables are courtesy of Professors Ashish Khisti and Ravi Adve and were developed originally for ECE355. Please note that the notation used is *different* from that in ECE455.

Fourier Properties

Property	**DTFS**	**CTFS**	**DTFT**	**CTFT**
Synthesis	$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
Analysis	$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\Omega_0 n}$	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$	$X(e^{j\Omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\Omega n}$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
Linearity	$\alpha x[n] + \beta y[n] \leftrightarrow \alpha a_k + \beta b_k$	$\alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$	$\alpha x[n] + \beta y[n] \leftrightarrow \alpha X(e^{j\Omega}) + \beta Y(e^{j\Omega})$	$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(j\omega) + \beta Y(j\omega)$
Time Shifting	$x[n-n_0] \leftrightarrow a_k e^{-j2\pi n_0 k/N}$	$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$	$x[n-n_0] \leftrightarrow e^{-j\Omega n_0}X(e^{j\Omega})$	$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$
Frequency Shift	$x[n]e^{j2\pi mn/N} \leftrightarrow a_{k-m}$	$x(t)e^{jm\omega_0 t} \leftrightarrow a_{k-m}$	$x[n]e^{j\Omega_0 n} \leftrightarrow X(e^{j(\Omega-\Omega_0)n})$	$x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega-\omega_0))$
Conjugation	$x^*[n] \leftrightarrow a^*_{-k}$	$x^*(t) \leftrightarrow a^*_{-k}$	$x^*[n] \leftrightarrow X^*(e^{-j\Omega})$	$x^*(t) \leftrightarrow X^*(-j\omega)$
Time Reversal	$x[-n] \leftrightarrow a_{-k}$	$x(-t) \leftrightarrow a_{-k}$	$x[-n] \leftrightarrow X(e^{-j\Omega})$	$x(-t) \leftrightarrow X(-j\omega)$
Convolution	$\sum_{r=0}^{N-1} x[r]y[n-r] \leftrightarrow Na_k b_k$	$\int_T x(\tau)y(t-\tau)d\tau \leftrightarrow Ta_k b_k$	$x[n] * y[n] \leftrightarrow X(e^{j\Omega})Y(e^{j\Omega})$	$x(t) * y(t) \leftrightarrow X(j\omega)Y(j\omega)$
Multiplication	$x[n]y[n] \leftrightarrow \sum_{r=0}^{N-1} a_r b_{k-r}$	$x(t)y(t) \leftrightarrow a_k * b_k$	$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\Omega-\theta)})d\theta$	$x(t)y(t) \leftrightarrow \frac{1}{2\pi}X(j\omega) * Y(j\omega)$
First Difference/ Derivative	$x[n]-x[n-1] \leftrightarrow (1-e^{-j2\pi k/N})a_k$	$\frac{dx(t)}{dt} \leftrightarrow jk\omega_0 a_k$	$x[n]-x[n-1] \leftrightarrow (1-e^{-j\Omega})X(e^{j\Omega})$	$\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$
Running Sum/ Integration	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{a_k}{1-e^{-j2\pi k/N}}$	$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{a_k}{jk\omega_0}$	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{X(e^{j\Omega})}{1-e^{-j\Omega}} + \pi X(e^{j0})\delta(\Omega)$	$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$
Parseval's Relation	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert a_k\rvert^2$	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\Omega})\rvert^2 d\Omega$	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$
Real and even signals	Real and even in frequency domain			
Real and odd signals	Purely imaginary and odd in frequency domain			

Additional Property: A real-valued time-domain signal $x(t)$ or $x[n]$ will have a conjugate-symmetric Fourier representation.

Notes:

1. For the CTFS, the signal $x(t)$ has a period of T, fundamental frequency $\omega_0 = 2\pi/T$; for the DTFS, the signal $x[n]$ has a period of N, fundamental frequency $\Omega_0 = 2\pi/N$. a_k and b_k denote the Fourier coefficients of $x(t)$ (or $x[n]$) and $y(t)$ (or $y[n]$) respectively.
2. Periodic convolutions can be evaluated by summing or integrating over *any* single period, not just those indicated above.
3. The "Running Sum" formula for the DTFT above is valid for Ω in the range $-\pi < \Omega \leq \pi$.

Fourier Pairs

Fourier Series Coefficients of Periodic Signals*			
Continuous-Time		**Discrete-Time****	
Time Domain – $x(t)$	Frequency Domain – a_k	Time Domain – $x[n]$	Frequency Domain – a_k
$Ae^{j\omega_0 t}$	$a_1 = A$ $a_k = 0,\ k \neq 1$	$Ae^{j\Omega_0 n}$	$a_1 = A,$ $a_k = 0, k \neq 1$
$A\cos(\omega_0 t)$	$a_1 = a_{-1} = A/2$ $a_k = 0, k \neq 1$	$A\cos(\Omega_0 n)$	$a_1 = a_{-1} = A/2$ $a_k = 0, k \neq 1$
$A\sin(\omega_0 t)$	$a_1 = a_{-1}^* = \frac{A}{2j}$ $a_k = 0, k \neq 1$	$A\sin(\Omega_0 n)$	$a_1 = a_{-1}^* = \frac{A}{2j}$ $a_k = 0, k \neq 1$
$x(t) = A$	$a_0 = A, a_k = 0$ otherwise	$x[n] = A$	$a_0 = A, a_k = 0$ otherwise
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$a_k = \frac{1}{T}$	$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$a_k = \frac{1}{N}$
Periodic square wave $x(t) = \begin{cases} 1 & \lvert t\rvert < T_1 \\ 0 & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t) = x(t+T)$	$a_0 = \dfrac{2T_1}{T}$ $a_k = \dfrac{\sin(k\omega_0 T_1)}{k\pi},\ k \neq 0$		

Fourier Transform Pairs			
Continuous-Time		**Discrete-Time****	
Time Domain – $x(t)$	Frequency Domain – $X(j\omega)$	Time Domain – $x[n]$	Frequency Domain – $X(e^{j\Omega})$
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\dfrac{2\sin(\omega T_1)}{\omega}$	$x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\dfrac{\sin(\Omega(N_1 + 1/2))}{\sin(\Omega/2)}$
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin Wn}{\pi n}$	$X(e^{j\Omega}) = \begin{cases} 1, & \lvert\Omega\rvert \leq W \\ 0, & \text{otherwise} \end{cases}$
$\delta(t)$	1	$\delta[n]$	1
1	$2\pi\delta(\omega)$	1	$2\pi\delta(\Omega)$
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$	$u[n]$	$\dfrac{1}{1 - e^{-j\Omega}} + \pi\delta(\Omega)$
$e^{-at}u(t), \mathrm{Re}(a) > 0$	$\dfrac{1}{a + j\omega}$	$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1 - ae^{-j\Omega}}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t), \mathrm{Re}(a) > 0$	$\dfrac{1}{(a + j\omega)^n}$	$\dfrac{(n+r-1)!}{n!(r-1)!}a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{(1 - ae^{-j\Omega})^r}$

*In the Fourier series table, $\omega_0 = \frac{2\pi}{T}$ and $\Omega_0 = \frac{2\pi}{N}$, where T and N are the periods of $x(t)$ and $x[n]$ respectively.
**For the DTFS, a_k is given only for k in the range $-N/2+1 \leq k \leq N/2$ for even N, $-(N-1)/2 \leq k \leq (N-1)/2$ for odd N, and $a_k = a_{k+N}$; for the DTFT $X(e^{j\Omega})$ is given only for Ω in the range $-\pi < \Omega \leq \pi$, and $X(e^{j\Omega}) = X(e^{j(\Omega+2\pi)})$.

Fourier Transform for Periodic Signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \leftrightarrow X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$a^n \cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$a^n \sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	At least ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\ x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $ROC_1 \cap ROC_2$