The Discrete Fourier Transform

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Overlap-Save and Overlap-Add for Real-time Processing

Reference:

Section 7.1 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

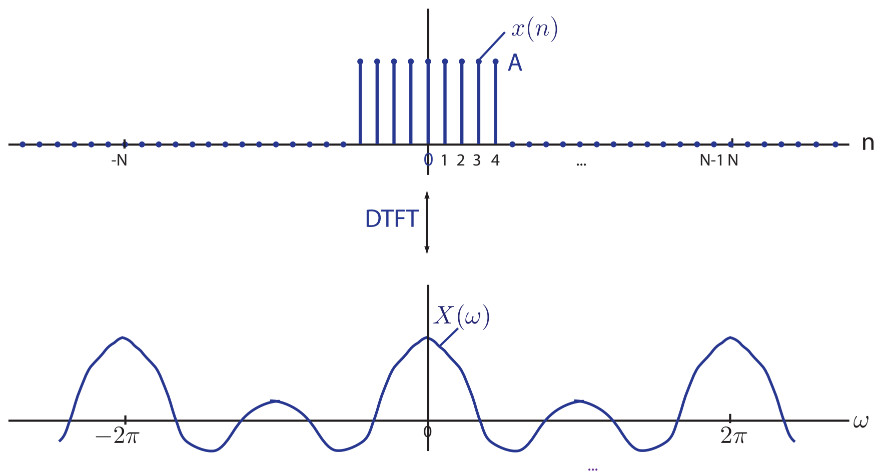
Intuition

aperiodic + dst in time	$\xleftrightarrow{\text{DTFT}}$	cts + periodic in freq
$\downarrow$ periodic repetition		$\downarrow$ sampling
periodic + dst in time	$\xleftrightarrow{\text{DTFS}}$	dst + periodic in freq
one period of dst-time samples $n = 0, 1, \ldots, N-1$	$\xleftrightarrow{\text{DFT}}$	one period of dst-freq samples $k = 0, 1, \ldots, N-1$

Therefore, we define the Discrete Fourier Transform (DFT) as being a computable transform that approximates the DTFT.

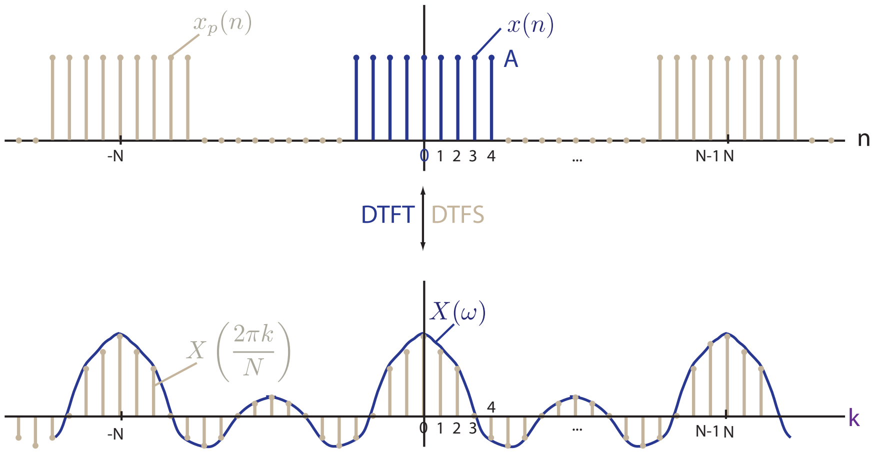
Intuition

Example



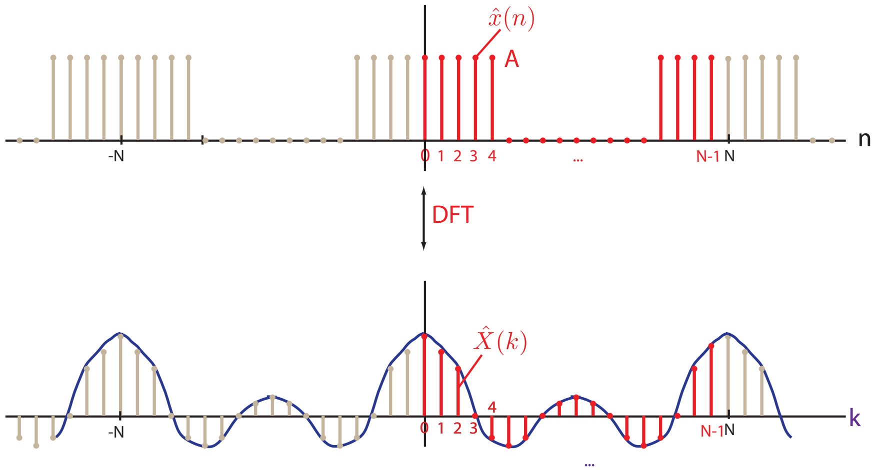
Intuition

Example



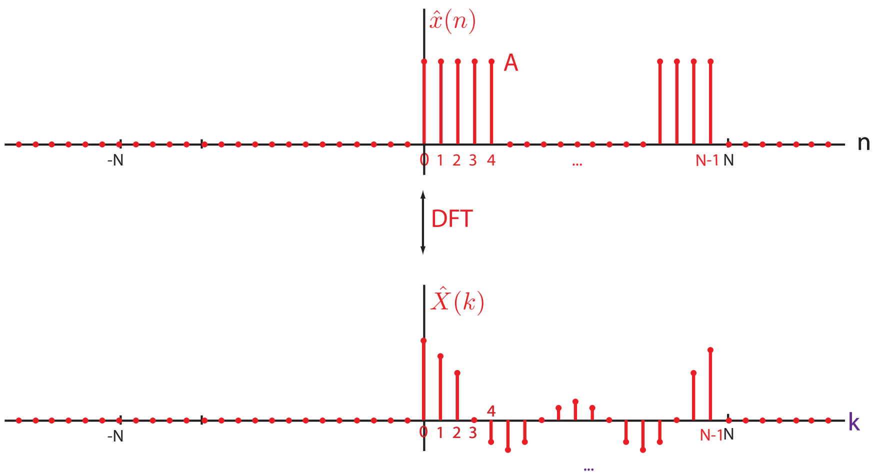
Intuition

Example



Intuition

Example



DTFT, DTFS and DFT

$$x(n) \text{ for all } n \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad X(\omega) \text{ for all } \omega$$

$$\downarrow \text{ periodic repetition} \qquad \downarrow \text{ sampling}$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n+lN) \text{ for all } n \quad \overset{\text{DTFS}}{\longleftrightarrow} \quad X(k) = X(\omega)|_{\omega=\frac{2\pi}{N}k} \text{ for all } k$$

$$\hat{x}(n) \quad \overset{\text{DFT}}{\longleftrightarrow} \quad \hat{X}(k)$$

where

$$\hat{x}(n) = \begin{cases} x_p(n) & \text{for } n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{X}(k) = \begin{cases} X(k) & \text{for } k = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Frequency Domain Sampling

- Recall, sampling in time results in a periodic repetition in frequency.

$$x(n) = x_a(t)|_{t=nT} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(\omega + \frac{2\pi}{T}k)$$

- Similarly, sampling in frequency results in periodic repetition in time.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n + lN) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(k) = X(\omega)|_{\omega=\frac{2\pi}{N}k}$$

Frequency Domain Sampling and Reconstruction

- Therefore,

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$
$$x_p(n) \overset{\mathcal{F}}{\longleftrightarrow} X(k)$$

- Implications:
  - The samples of $X(\omega)$ can be used to reconstruct $x_p(n)$.

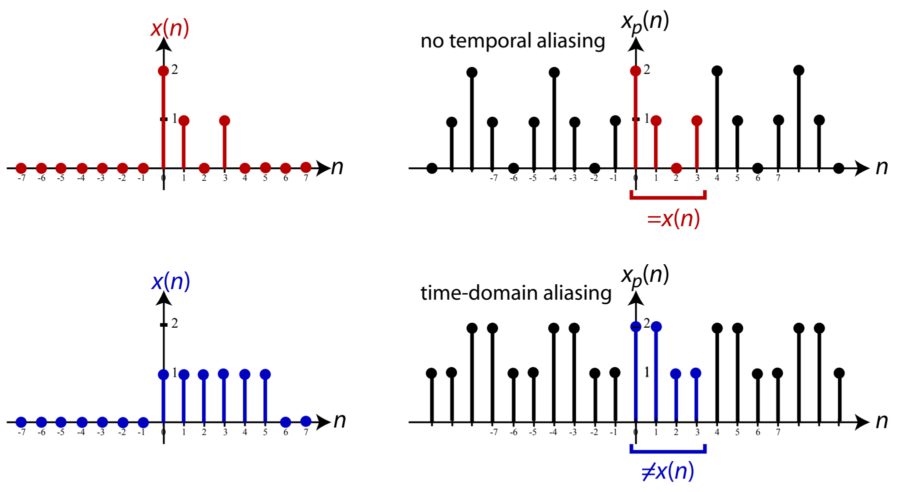
Frequency Domain Sampling and Reconstruction

- **Q:** Can we reconstruct $x(n)$ from the samples of $X(\omega)$?
  - Can we reconstruct $x(n)$ from $x_p(n)$?
- **A:** Maybe.

$$x_p(n) = \left[\sum_{l=-\infty}^{\infty} x(n + lN)\right]$$

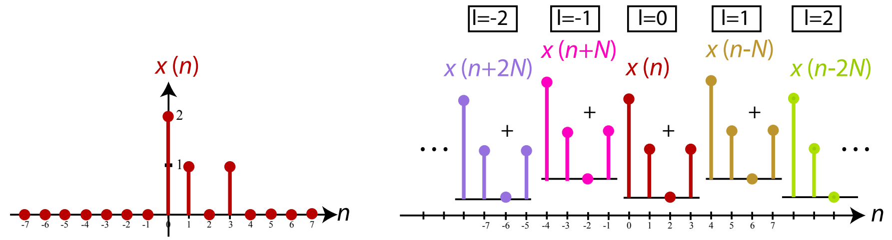
Frequency Domain Sampling and Reconstruction

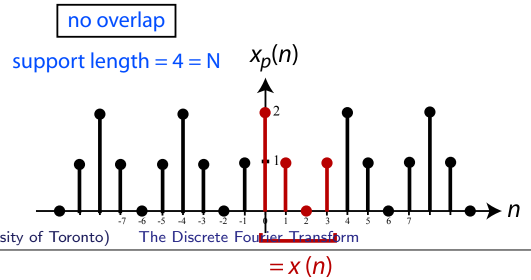
$N = 4$



Frequency Domain Sampling and Reconstruction

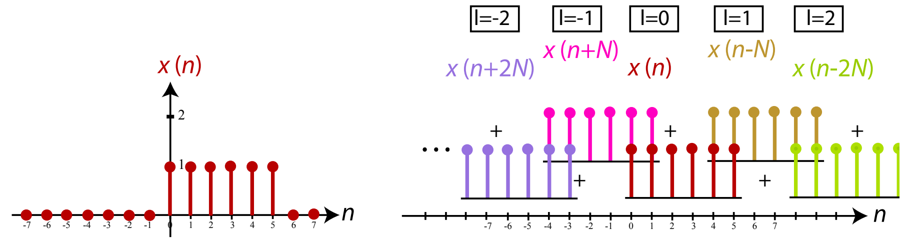
$N = 4$

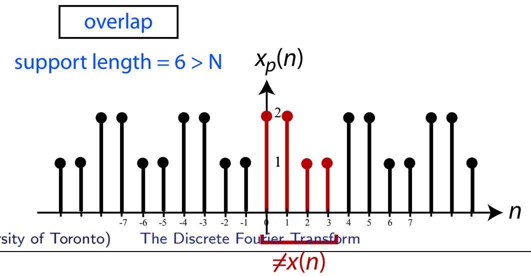




Frequency Domain Sampling and Reconstruction

$N = 4$





Frequency Domain Sampling and Reconstruction

- $x(n)$ can be recovered from $x_p(n)$ if there is no overlap when taking the periodic repetition.
- If $x(n)$ is finite duration and non-zero in the interval $0 \le n \le L-1$, then

$$x(n) = x_p(n), \quad 0 \le n \le N-1 \quad \text{when } N \ge L$$

- If $N < L$ then, $x(n)$ cannot be recovered from $x_p(n)$.
  - or equivalently $X(\omega)$ cannot be recovered from its samples $X\left(\frac{2\pi}{N}k\right)$ due to time-domain aliasing

The Discrete Fourier Transform Pair

- DFT and inverse-DFT (IDFT):

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \quad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k \frac{n}{N}}, \quad n = 0, 1, \ldots, N-1$$

Note: we drop the $\hat{\cdot}$ notation from now on.

■